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ON GENERALIZATIONS OF P_1-RINGS AND *gsr*-RINGS

Dedicated to Professor Yukiyosi Kawada on his 60th birthday

KATSUO CHIBA and HISAO TOMINAGA

In his papers [5], [6] and [7], F. Szász considered several generalizations of strongly regular rings: A ring R is called a *P_1-ring* (resp. *gsr-ring*) if $aR=aRa$ (resp. $aRa=a^2Ra^2$) for any $a\in R$. As was shown in [5, Theorem 6], [6, Theorem 9] and [7, Theorem 9], there holds the following:

(I) A ring R is strongly regular if and only if one of the following equivalent conditions is satisfied:

1) R is a semi-prime P_1-ring.
2) R is semi-prime and $aR=a^2R$ for any $a\in R$.
3) R is a semi-prime *gsr*-ring.

More recently, S. Ligh and Y. Utumi [4, Theorems 1, 3] and K. Chiba and H. Tominaga [2, Theorem 2] have characterized a P_1-ring R as follows:

(II) R is a P_1-ring if and only if one of the following equivalent conditions is satisfied:

1) R is a direct sum of a strongly regular ring and a zero ring.
2) $aR\subseteq Ra^2$ for any $a\in R$.
3) $l(R)=r(R)$ and $R/l(R)$ is strongly regular.
4) $aR=a^2R$ and $Ra=Ra^2$ for any $a\in R$.
5) $R/l(R)$ and $R/r(R)$ are strongly regular.

Concening *gsr*-rings, by the aid of Nagata-Higman theorem, F. Szász [7, Theorem 7] proved also the following:

(III) If a *gsr*-ring R with prime radical N is an algebra over a field whose characteristic is 0 or a prime $p\geq 5$, then $N^7=0$.

In this note, we shall introduce the notion of a P_n'-ring (resp. Q_n-ring) as a generalization of that of a P_1-ring (resp. *gsr*-ring), and prove three theorems which contain (I), (II) and (III) respectively.

Throughout R will represent an associative ring (with or without 1), and n a positive integer. As for notations and terminologies used here, we follow [2].

If $aR^n=aR^na$ (resp. $aR^na=a^2R^na^2$) for any $a\in R$, R is called a *P_n'-ring*

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(resp. Q_n-*ring*). Obviously, the notion of a P_1'-ring (resp. Q_1-ring) coincides with that of a P_1-ring (resp. *gsr*-ring). If R is a P_n'-ring or if $aR^n = a^2R^n$ for any $a \in R$, then it is easy to see that the set N of all nilpotent elements of R equals $l(R^n)$. While, if $aR^n \subseteq R^n a^2$ for any $a \in R$, then one can prove $N = l(R^{2n})$ (cf. the proof of [4, Theorem 3]). Finally, in case R is a Q_n-ring, for any $a \in N$ we have $aR^n a = 0$, so that $(aR)^{2n} \subseteq (aR^n)^2 = 0$, which means that N coincides with the prime radical of R.

Our first theorem is a generalization of (I).

Theorem 1. *The following are equivalent*:

1) *R is strongly regular.*
2) *R is a semi-prime P_n'-ring.*
3) *R is semi-prime and $aR^n = a^2R^n$ for any $a \in R$.*
4) *R is semi-prime and $aR^n \subseteq R^n a^2$ for any $a \in R$.*
5) *R is a semi-prime Q_n-ring.*

Proof. It is easy to see that 1) implies 2)—5). Suppose 2). In any rate, R is a reduced ring. Given $a \in R$, $a^{n+1} - ab_0 a = 0$ for some $b_0 \in R^n$. Since $(a^n - ab_0)^2 = 0$, it follows $a^n - ab_0 = 0$ and $a^n - ab_1 a = 0$ for some $b_1 \in R^n$. Repeating the same procedure, we obtain eventually $a - ab_n a = 0$ for some $b_n \in R^n$, proving 1). Next, noting that in a reduced ring $xy = 0$ if and only if $yx = 0$, the implications 3) $\Longrightarrow$ 1) and 4) $\Longrightarrow$ 1) will be shown quite similarly. Finally, suppose 5). If a is an arbitrary element of R then

$$a^{n+2} \in aR^n a = a^2 R^n a^2 = \cdots = a^{2(n+2)} R^n a^{2(n+2)}.$$

Hence, by [1, Lemma 1] there exists an element $b \in R$ such that $a^{n+2} = a^{2(n+2)} b$ and $ab = ba$. Then it is easy to see that $(a - a^{n+3} b)^{n+2} = 0$. Since R is seen to be a reduced ring, we obtain $a = a^2(a^{n+1} b)$, which proves 1) (cf. also [1, Lemma 4]).

If e is an idempotent of R such that $ea = ae$ for any $a \in R$ wth $a^2 = 0$ then e is central. To see this, it is enough to recall that $(ex - exe)^2 = 0 = (xe - exe)^2$ for any $x \in R$. Following A. Kertész and O. Steinfeld [3], R is called a *Z-ring* if every idempotent of R is central. For instance, any ring whose nilpotent elements are central is a Z-ring (cf. [4, Theorem 1]).

Lemma 1. (1) *If R is a P_n'-ring then it is a Z-ring.*

(2) *If $aR^n \subseteq R^n a^2$ for any $a \in R$ then R is a Z-ring.*

(3) *If $l(R^n) = r(R^n)$ and $R/l(R^n)$ is strongly regular, then R is a Z-ring.*

Proof. (1) Let $e=e^2$ and $a^2=0$ in R. Obviously, $ae=ae^n=0$ and $ea=e\cdot e^{n-1}a\in eR^ne$, whence it follows $ea=eae=0$. Hence, R is a Z-ring. Similarly, we can prove (2).

(3) Let $e=e^2$ in R. Since $R/l(R^n)$ is strongly regular, $ae-ea\in l(R^n)$ for any $a\in R$, so that $ae-eae=(ae-ea)e^n=0$, and similarly $ea-eae=0$. Hence $ae=eae=ea$.

Now, we are at the position to prove our principal theorem, which contains (II).

Theorem 2. *The following are equivalent:*

1) *R is a direct sum of a strongly regular ring and a nilpotent ring of nilpotency index at most $n+1$.*

2) *R is a P_n'-ring.*

3) *$aR^n\subseteq R^na^2$ for any $a\in R$.*

4) *R is a Z-ring and $R/l(R^n)$ is regular.*

5) *$l(R^n)=r(R^n)$ and $R/l(R^n)$ is strongly regular.*

6) *$aR^n=a^2R^n$ and $R^na=R^na^2$ for any $a\in R$.*

7) *$R/l(R^n)$ and $R/r(R^n)$ are strongly regular.*

2′)—4′) *The left-right analogues of* 2)—4).

Proof. It is easy to see that 1) implies 2)—7).

2) $\Longrightarrow$ 4) By Lemma 1 (1) and Theorem 1.

6) $\Longrightarrow$ 7) $\Longrightarrow$ 5) $\Longrightarrow$ 4) By Theorem 1 and Lemma 1 (3).

4) $\Longrightarrow$ 1) By the regularity of $\bar{R}=R/l(R^n)$ we have $R=R^{n+1}+l(R^n)$. Now, let $x=\sum x_i^{(1)}x_i^{(2)}\cdots x_i^{(n+1)}$ be an arbitrary element of $R^{n+1}\cap l(R^n)$. Then, by the regularity of $\bar{R}$ and the nilpotency of $l(R^n)$ we can find a central idempotent e such that $x_i^{(1)}-x_i^{(1)}e\in l(R^n)$ for all i. Hence,

$$x=\sum (x_i^{(1)}-x_i^{(1)}e)x_i^{(2)}\cdots x_i^{(n+1)}+\sum x_i^{(1)}ex_i^{(2)}\cdots x_i^{(n+1)}=xe=xe^n=0,$$

whence it follows $R=R^{n+1}\oplus l(R^n)$.

3) $\Longrightarrow$ 1) By Lemma 1 (2) and Theorem 1, R is a Z-ring such that $R/l(R^{2n})$ is strongly regular. Hence, the proof of 4) $\Longrightarrow$ 1) shows that $R=R^{2n+1}\oplus l(R^{2n})$. Since $R^{n+1}=R^{n+2}$, we readily see $R=R^{n+1}\oplus l(R^{2n})$. If $aR^{2n}=0$ then $aR^n\subseteq R^{n+1}\cap l(R^{2n})=0$, which means $l(R^{2n})=l(R^n)$. Thus $R=R^{n+1}\oplus l(R^n)$, proving 1).

Remark. By Theorem 2, any P_n'-ring is a Q_n-ring. However, the converse is not true in general: Let R be the subring of $(\mathrm{GF}(2))_2$ consisting of 0, e_{11}, e_{12} and $e_{11}+e_{12}$, where e_{ij}'s are matrix units of $(\mathrm{GF}(2))_2$. Then, $aR=a^2R$ and $aRa=a^2Ra^2$ for any $a\in R$, $Re_{12}\neq 0=Re_{12}^2$, $l(R)\neq 0=r(R)$,

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and $R/l(R)$ is isomorphic to GF(2).

Finally, we shall present a sharpening of (III).

Lemma 2. *Let R be a Q_n-ring. If a_1, a_2, a_4, $a_5 \in N$ and $a_3 \in R^n$ then* $2a_1a_2a_3a_4a_5=0$.

Proof. If a, b are in N then $aR^na=0$, $bR^nb=0$ and $(a+b)R^n(a+b)=0$. Hence, one will easily see that $axb=-bxa$ for any $x \in R^n$. By making use of this relation, we obtain

$$a_1a_2a_3a_4a_5=-a_4a_5a_3a_1a_2=a_4a_2a_3a_1a_5=-a_1a_2a_3a_4a_5,$$

namely, $2a_1a_2a_3a_4a_5=0$.

By Lemma 2, without making use of Nagata-Higman theorem, we readily obtain the following:

Theorem 3. *Let R be an algebra over a field of characteristic $\neq 2$. If R is a Q_n-ring then $N^{n+4}=0$. Especially, if R is a gsr-ring then $N^5=0$.*

REFERENCES

[1] G. AZUMAYA: Strongly π-regular rings, J. Fac. Sci. Hokkaido Univ., Ser. I, **13** (1954), 34—39.

[2] K. CHIBA and H. TOMINAGA: Note on strongly regular rings and P_1-rings, Proc. Japan Acad. **51** (1975), 259—261.

[3] A. KERTÉSZ and O. STEINFELD: Über reguläre Z-ringe, Beiträge zur Algebra und Geometrie **3** (1974), 7—10.

[4] S. LIGH and Y. UTUMI: Direct sum of strongly regular rings and zero rings, Proc. Japan Acad. **50** (1974), 589—592.

[5] F. SZÁSZ: Some generalizations of strongly regular rings I, Math. Japonicae **17** (1972), 115—118.

[6] F. SZÁSZ: Some generalizations of strongly regular rings II, Math. Japonicae **18** (1973), 87—90.

[7] F. SZÁSZ: On generalized subcommutative regular rings, Monatshefte für Math. **77** (1973), 67—71.

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