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On the Asymptotic Expansion for the Trace of the Heat Kernel on Locally Symmetric Einstein Spaces and its Application

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ON THE ASYMPTOTIC EXPANSION FOR THE TRACE OF THE HEAT KERNEL ON LOCALLY SYMMETRIC EINSTEIN SPACES AND ITS APPLICATION

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0. Introduction. Let (M,g) be an n-dimensional closed and connected Riemannian manifold and $\triangle$ be the Laplacian for functions defined by

$$\triangle f = -g^{ij}\nabla_i\nabla_j f. \tag{0.1}$$

Let $\mathrm{Spec}(M,g) = \{\lambda_i\}_{i=0}^{\infty}$ be the spectrum of the Laplacian, that is, the set of eigenvalues of $\triangle$ counting with multiplicities. It is well-known that the coefficients a_i of Minakshisundaram-Pleijel's asymptotic expansion

$$\sum_{i=0}^{\infty} e^{-\lambda_i t} \sim (4\pi t)^{-\frac{n}{2}} \sum_{i=0}^{\infty} a_i t^i, \quad t \to +0, \tag{0.2}$$

are determined by the spectrum. a_0, a_1 and a_2 are easily calculated by Taylor asymptotic expansion of the metric tensor g_{ij}. a_3 was calculated by Sakai [12] and Gilkey [6]. Similarly we can treat the spectrum $\mathrm{Spec}^1(M,g)$ of the Laplacian for 1-forms. In this paper we calculate a_4 for a locally symmetric Einstein space and give some geometric applications. The main result is the following:

Proposition. *For two oriented closed Riemannian manifolds (M,g) and (M',g') assume that one of them is an 8-dimensional locally symmetric Einstein space. If (M,g) and (M',g') have the same spectra for functions and for 1-forms, respectively, i.e.,*

$$\mathrm{Spec}(M,g) = \mathrm{Spec}(M',g') \quad \textit{and} \quad \mathrm{Spec}^1(M,g) = \mathrm{Spec}^1(M',g'),$$

then (1) $\chi(M) = \chi(M')$ and (2) $|\sigma(M)| = |\sigma(M')|$ are equivalent, where $\chi(M)$, $\sigma(M)$ denote the Euler characteristic and signature of M, respectively.

1. Preliminaries. We assume that (M, g) is an n-dimensional closed locally symmetric Einstein space. We define the curvature tensor as

$$R_{ijk}{}^{l}\partial_l = R(\partial_i, \partial_j)\partial_k, \tag{1.1}$$

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{1.2}$$

Then the contracted values of curvature tensors given in the following (1.3) are constant on (M, g).

$$\begin{aligned} \tau &:= \sum_{ij=1}^{n} R_{ij}{}^{ji}: \text{ scalar curvature,} \\ R^2 &:= \sum R_{ijkl}R^{ijkl}: \\ &\qquad \text{the square of the norm of the curvature tensor,} \\ R^3 &:= \sum R_{ijkl}R^{klmn}R_{mn}{}^{ij}, \\ \tilde{R}^3 &:= \sum R_{ikjl}R^{kmln}R_{m}{}^{i}{}_{n}{}^{j}, \\ R^4 &:= \sum R_{ijkl}R^{kl}{}_{mn}R^{mn}{}_{ab}R^{abij}, \\ \tilde{R}^4 &:= \sum R_{ikjl}R^{k}{}_{m}{}^{l}{}_{n}R^{m}{}_{a}{}^{n}{}_{b}R^{aibj}, \end{aligned} \tag{1.3}$$

and so on. We take the local expression for tensors in a normal coordinate at the center. In the following we follow the Einstein's convension on summation.

The relations characterizing a locally symmetric Einstein space are

$$\nabla_i R_{jklm} = 0, \tag{1.4}$$

$$\rho_{ij} = \frac{\tau}{n}\delta_{ij}, \tag{1.5}$$

where ρ_{ij} is the Ricci curvature tensor. We use notations a_i and u_i in the sense of [2],[12];

$$a_i = \int_M u_i \, dv. \tag{1.6}$$

In the following we compute a_4 for a locally symmetric Einstein space (M, g). Since (M, g) is locally symmetric, all we have to do is to compute u_4. In computing u_4 we shall find what kind of terms appear in u_4. Taylor asymptotic expansion of g_{ij} tells that (see e.g. [2],[12])

(1) each term consists of a function which is obtained by contracting four curvature tensors,

(2) the coefficient of each term is independent of the shape of the Riemannian manifold (however depends on the dimension n because of (1.5)).

We use a notation $(abcd)$ for the curvature tensor instead of R_{abcd}, that is, discribe only indices. We try to classify the contraction of four curvature tensors. We begin with the classification in the case of two curvature tensors. We obtain

$$(1.7)\qquad \begin{aligned} &(abcd)(abcd) = R^2, \\ -R^2 &= (abcd)(abdc) = (abcd)(bacd) = (abcd)(dcab) \\ &= (abcd)(cdba), \\ R^2 &= (abcd)(badc) = (abcd)(cdab) = (abcd)(dcba). \end{aligned}$$

By the Bianchi identity

$$(1.8)\qquad (abcd)(abcd) = 2(abcd)(acbd) = -2(abcd)(adbc),$$

we obtain

$$(1.9)\qquad \begin{aligned} -\frac{1}{2}R^2 &= (abcd)(acdb) = (abcd)(cabd) = (abcd)(bdca) \\ &= (abcd)(dbac) = (abcd)(dacb) = (abcd)(bcad) \\ &= (abcd)(cbda), \\ \frac{1}{2}R^2 &= (abcd)(cadb) = (abcd)(bdac) = (abcd)(dbca) \\ &= (abcd)(dabc) = (abcd)(adcb) = (abcd)(bcda) \\ &= (abcd)(cbad). \end{aligned}$$

So in this case the independent factor is only R^2. As for the case containing τ, we obtain

$$(1.10)\qquad (abca)(dbcd) = \frac{\tau^2}{n}, \qquad (abba)(dccd) = \tau^2.$$

In the case of three curvature tensors, we obtain the following relations by the Bianchi identity,

$$(1.11)\qquad \begin{aligned} R^3 &= (abcd)(cdef)(efab) = 2(abcd)(cdef)(eafb) \\ &= 4(abcd)(cedf)(eafb), \\ \tilde{R}^3 &= (acbd)(cedf)(eafb) = (adbc)(cedf)(eafb) + \frac{1}{4}R^3. \end{aligned}$$

By the Ricci identity we obtain

$$(1.12)\qquad \begin{aligned} 0 &= \nabla_u \nabla_v R_{uabc} - \nabla_v \nabla_u R_{uabc} \\ &= R_{uvlu}R_{labc} + R_{uvla}R_{ulbc} + R_{uvlb}R_{ualc} + R_{uvlc}R_{uabl}, \end{aligned}$$

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and by multiplying R_{vabc} to the both side of (1.12) and contracting, we obtain

$$\widetilde{R}^3 = -\frac{1}{2}\frac{\tau}{n}R^2 - \frac{1}{4}R^3. \tag{1.13}$$

In the case of four curvature tensors, we classify them into three types except the cases containing τ and R^2. In each type we neglect the difference of the arrangement of the indices in R_{abcd}. They are

(1.14) 3-type: $(uabc)(udef)(vabc)(vdef)$,

(1.15) (2, 2)-type: $(abcd)(cdef)(efgh)(ghab)$,

(1.16) (1, 1, 2)-type: $(abcd)(abkl)(uvck)(uvdl)$.

Remark. The cases containing τ and R^2 are

$$\tau^4, \quad \tau^2R^2, \quad \tau R^3 \quad \text{and} \quad R^2R^2. \tag{1.17}$$

2. The classification in the case of four curvature tensors. In the following we obtain the relations among the changes of the arrangement of indices.

Step 1: The relations obtained by using the Bianchi identity. As for 3-type:

$$\begin{aligned}(uabc)(udef)(vabc)(vdef) &= 2(uabc)(uedf)(vabc)(vdef)\\ &= 4(ubac)(uedf)(vabc)(vdef)\\ &= 4(ubac)(udef)(vabc)(vedf).\end{aligned} \tag{2.1}$$

As for (2, 2)-type: We denote

$$R^4 = (abcd)(cdef)(efgh)(ghab), \quad \widetilde{R}^4 = (acbd)(cedf)(egfh)(gahb), \tag{2.2}$$

and get

$$\begin{aligned}R^4 &= 2(abcd)(cdef)(efgh)(gahb) = 4(abcd)(cdef)(egfh)(gahb)\\ &= 4(abcd)(cedf)(efgh)(gahb) = 8(abcd)(cedf)(egfh)(gahb),\end{aligned} \tag{2.3}$$

$$\begin{aligned}\widetilde{R}^4 &= (adbc)(cfde)(egfh)(gahb) = (adbc)(cedf)(ehfg)(gahb)\\ &= (adbc)(cedf)(egfh)(gahb) + \frac{1}{8}R^4.\end{aligned} \tag{2.4}$$

As for $(1,1,2)$-type:

$$(2.5)\quad \begin{aligned}&(abcd)(abkl)(uvck)(uvdl)\\ &\quad = 2(abcd)(abkl)(uvck)(udvl) = 4(acbd)(abkl)(ucvk)(uvdl)\\ &\quad = 4(abcd)(akbl)(ucvk)(uvdl),\end{aligned}$$

$$(2.6)\quad \begin{aligned}&(acbd)(akbl)(uvck)(uvdl)\\ &\quad = \frac{1}{2}(abcd)(abkl)(uvck)(uvdl) + (adbc)(akbl)(uvck)(uvdl),\end{aligned}$$

$$(2.7)\quad \begin{aligned}&(acbd)(akbl)(ucvk)(uvdl\\ &\quad = (acbd)(akbl)(uvck)(uvdl) + (acbd)(akbl)(ukvc)(uvdl)\\ &\quad = -(acbd)(akbl)(ukvc)(uvdl),\\ &2(acbd)(akbl)(ucvk)(uvdl)\\ &\quad = (acbd)(akbl)(uvck)(uvdl),\end{aligned}$$

$$(2.8)\quad \begin{aligned}&(adbc)(akbl)(ucvk)(uvdl)\\ &\quad = -\frac{1}{4}(abcd)(abkl)(uvck)(uvdl) + \frac{1}{2}(acbd)(akbl)(uvck)(uvdl),\end{aligned}$$

$$(2.9)\quad \begin{aligned}&(acbd)(akbl)(ucvk)(udvl)\\ &\quad = \frac{1}{2}(acbd)(akbl)(uvck)(uvdl) + (acbd)(akbl)(ucvk)(ulvd),\\ &(adbc)(akbl)(ukvc)(udvl)\\ &\quad = (acbd)(akbl)(ucvk)(udvl) - (acbd)(akbl)(uvck)(uvdl)\\ &\qquad + \frac{1}{4}(abcd)(abkl)(uvck)(uvdl),\\ &(acbd)(akbl)(ucvk)(ulvd)\\ &\quad = (acbd)(albk)(ukvc)(ulvd),\\ &(acbd)(akbl)(ucvk)(udvl)\\ &\quad = (acbd)(akbl)(ukvc)(ulvd) = (adbc)(albk)(ukvc)(udvl).\end{aligned}$$

Then we may choose independent factors in each type, and use the following notations:

$$\begin{aligned}&\text{3-type} && (t) := (uabc)(udef)(vabc)(vdef),\\ &(2,2)\text{-type} && R^4 := (abcd)(cdef)(efgh)(ghab),\\ &&& \widetilde{R}^4 := (acbd)(cedf)(egfh)(gahb),\\ &(1,1,2)\text{-type} && (a) := (abcd)(abkl)(uvck)(uvdl),\\ &&& (b) := (acbd)(akbl)(uvck)(uvdl),\\ &&& (c) := (acbd)(akbl)(ucvk)(udvl).\end{aligned}$$

Step 2: The relations obtained by using the Ricci identity.

Proposition 1. *We obtain the following relations*:

$$(a)+2(b)=(t), \tag{2.10}$$

$$\frac{\tau}{n}R^3+\frac{1}{2}R^4=-2(b), \tag{2.11}$$

$$\frac{1}{2}\frac{\tau^2}{n^2}R^2+\frac{1}{4}\frac{\tau}{n}R^3=\tilde{R}^4+\frac{3}{2}(b)-(c)-\frac{1}{4}(a). \tag{2.12}$$

Proof. By the Ricci identity and $\nabla_i R_{abcd}=0$, we obtain

$$\begin{aligned}(2.13)\quad &(uval)(lbcd)(kbcd)(uvka)+(uvbl)(alcd)(kbcd)(uvka)\\ &\quad+(uvcl)(abld)(kbcd)(uvka)+(uvdl)(abcl)(kbcd)(uvka)=0,\\ &(uvul)(lbcd)(cdgh)(ghvb)+(uvbl)(ulcd)(cdgh)(ghvb)\\ &\quad+(uvcl)(ubld)(cdgh)(ghvb)+(uvdl)(ubcl)(cdgh)(ghvb)=0,\\ &(uvul)(lbcd)(vgch)(gbhd)+(uvbl)(ulcd)(vgch)(gbhd)\\ &\quad+(uvcl)(ubld)(vgch)(gbhd)+(uvdl)(ubcl)(vgch)(gbhd)=0.\end{aligned}$$

Then we apply (2.1), ..., (2.9) to the above.

Therefore we can choose the following sets of independent factors to discribe the terms of u_4 as

(2.14a) $(a), (b), (c)$, or (2.14b) (t), R^4, (c),

and can set u_4 as

$$\begin{aligned}(2.15)\quad u_4=&\left(c_1+\frac{c_2}{n}+\frac{c_3}{n^2}+\frac{c_4}{n^3}\right)\tau^4+\left(c_5+\frac{c_6}{n}+\frac{c_7}{n^2}\right)\tau^2R^2\\ &+\left(c_8+\frac{c_9}{n}\right)\tau\tilde{R}^3+c_{10}R^2R^2+c_{11}(a)+c_{12}(b)+c_{13}(c),\end{aligned}$$

where $c_1, c_2, \ldots, c_{13}$ are constants.

3. The calculation of u_4. We use the following data;

(1) Sphere $S^n(1)$ of radius 1:

$$\begin{aligned}(3.1)\quad &\tau=n(n-1), && R^2=2n(n-1),\\ &R^3=-4n(n-1), && R^4=8n(n-1),\\ &\tilde{R}^3=-n(n-1)(n-2), && \tilde{R}^4=n(n-1)(n^2-3n+4),\\ &(a)=4n(n-1), && (b)=2n(n-1)(n-2),\\ &(c)=n(n-1)(3n-5), && (t)=4n(n-1)^2,\end{aligned}$$

(2) Complex projective space $CP^n(4)$ $(m = 2n)$ of constant holomorphic sectional curvature 4:

$$
\begin{aligned}
&\tau = m(m+2), && R^2 = 8m(m+2),\\
&R^3 = -8m(m+2)(m+6), && R^4 = 16m(m+2)(m^2+6m+16),\\
(3.2)\quad &\tilde{R}^3 = -2m(m+2)(m-2), && \tilde{R}^4 = 2m(m+2)(m^2+6m+48),\\
&(a) = 16m(m+2)(3m+10), && (b) = 8m(m+2)(m-2),\\
&(c) = 4m(m+2)(3m+10), && (t) = 64m(m+2)^2.
\end{aligned}
$$

Computation of u_4 is divided into the following steps.
Step 1: We express u_4 for $S^n(1)$ as a polynomial of n.
Step 2: We express u_4 for $S^n(1)\times S^n(1)$, $S^n(1)\times S^n(1)\times S^n(1)$ and $\overset{4}{\times} S^n(1)$.
Step 3: We express the curvature data as polynomials of n.
Step 4: We make up the system of equations for $c_1, c_2, \dots, c_{13}$.
Step 5: We carry out the same procedure for $S^2(1) \times S^6(\sqrt{5})$, etc.
Step 6: We carry out the same procedure for $CP^4(4)$, etc.

Step 1: Take a normal coordinate system and r denotes the distance to the center y of the normal coordinate neighbourhood. Then on $S^n(1)$ or $CP^n(4)$

$$(3.3)\qquad v(r) = (\det(g_{ij}(y)))^{-\frac{1}{4}},$$

depends only on r. In fact, on $S^n(1)$ we obtain $v(r) = (\sin r/r)^{(1-n)/2}$ and on $CP^n(4)$ we have $v(r) = (\cos r)^{-1/2}(\sin r/r)^{(1-2n)/2}$. By Taylor asymptotic expansion of $v(r)$ (see e.g. [1]), on $S^n(1)$ we obtain

$$
\begin{aligned}
&u_0 = 1,\\
&u_1 = \frac{1}{6}n(n-1),\\
(3.4)\quad &u_2 = \frac{1}{360}n(n-1)(5n^2-7n+6),\\
&u_3 = \frac{1}{45360}n(n-1)(35n^4-112n^3+187n^2-110n+96),\\
&u_4 = \frac{1}{5443200}n(n-1)(175n^6-945n^5+2389n^4\\
&\qquad\qquad -3111n^3+3304n^2-516n+2160).
\end{aligned}
$$

Step 2: We use the following formula (see [2])

$$u_i(M\times N) = \sum_{i_1+i_2=i} u_{i_1}(M)u_{i_2}(N).$$

Then on $S^n(1)\times S^n(1)$, $\overset{3}{\times} S^n(1)$ and $\overset{4}{\times} S^n(1)$, we obtain the following, respectively;

$$(3.5)\quad u_4=\frac{1}{340200}n(n-1)(175n^6-735n^5+1516n^4-1638n^3+1243n^2-399n+270),$$

$$u_4=\frac{1}{604800}n(n-1)(1575n^6-5985n^5+10789n^4-10359n^3+6368n^2-1956n+720),$$

$$u_4=\frac{1}{85050}n(n-1)(700n^6-2502n^5+4141n^4-3630n^3+1924n^2-534n+135).$$

Step 3 and 4: By putting (3.1) into (2.15), we obtain the following formulas for u_4 on $S^n(1)$, $S^n(1)\times S^n(1)$, $\overset{3}{\times} S^n(1)$ and $\overset{4}{\times} S^n(1)$, respectively;

$$\begin{aligned}(3.6)\quad u_4 = n(n-1)\{&c_1n^6+(c_2-3c_1)n^5+(3c_1-3c_2+c_3+2c_5)n^4\\ &-(c_1-3c_2+3c_3-c_4+4c_5-2c_6+c_8)n^3\\ &-(c_2-3c_3+3c_4-2c_5+4c_6-2c_7-3c_8+c_9-4c_{10})n^2\\ &-(c_3-3c_4-2c_6+4c_7+2c_8-3c_9+4c_{10}-2c_{12}-3c_{13})n\\ &-c_4+2c_7-2c_9+4c_{11}-4c_{12}-5c_{13}\},\end{aligned}$$

$$\begin{aligned}u_4 = n(n-1)\{&16c_1n^6+(8c_2-48c_1)n^5+(48c_1-24c_2+4c_3+16c_5)n^4\\ &-(16c_1-24c_2+12c_3-2c_4+32c_5-8c_6+4c_8)n^3\\ &-(8c_2-12c_3+6c_4-16c_5+16c_6-4c_7-12c_8+2c_9-16c_{10})n^2\\ &-(4c_3-6c_4-8c_6+8c_7+8c_8-6c_9+16c_{10}-4c_{12}-6c_{13})n\\ &+(\text{lower order terms})\},\end{aligned}$$

$$\begin{aligned}u_4 = n(n-1)\{&(\text{higher ordre terms})\\ &-(27c_2-27c_3+9c_4-54c_5+36c_6-6c_7-27c_8+3c_9-36c_{10})n^2\\ &-(9c_3-9c_4-18c_6+12c_7+18c_8-9c_9+36c_{10}-6c_{12}-9c_{13})n\\ &+(\text{lower order terms})\},\end{aligned}$$

$$\begin{aligned}u_4 = n(n-1)\{&(\text{higher order terms})\\ &-(64c_2-48c_3+12c_4-128c_5+64c_6-8c_7-48c_8+4c_9-64c_{10})n^2\\ &-(16c_3-12c_4-32c_6+16c_7+32c_8-12c_9+64c_{10}-8c_{12}-12c_{13})n\\ &+(\text{lower order terms})\}.\end{aligned}$$

We can compare the coefficients of the polynomials of n for u_4 in (3.4), (3.5) and (3.6), then we obtain

$$
\begin{aligned}
&c_1 = \frac{1}{31104}, \qquad c_2 = -\frac{1}{12960}, \qquad c_3 = -\frac{59}{1360800},\\
&c_4 = -\frac{1}{113400}, \qquad c_5 = \frac{1}{12960}, \qquad c_6 = (\text{parameter}),\\
&c_7 = (\text{parameter}), \qquad c_8 = \frac{41}{340200} + 2c_6, \qquad c_9 = \frac{1}{5670} + 2c_7,\\
&c_{10} = \frac{1}{13608} - \frac{c_6}{2}, \qquad c_{11} = (\text{parameter}), \qquad c_{12} = 2c_7 + 6c_{11} - \frac{4}{4725},\\
&c_{13} = -2c_7 - 4c_{11} + \frac{1}{1890},
\end{aligned}
\tag{3.7}
$$

where c_6, c_7 and c_{11} are undetermined. So we express them as parameters.

Step 5: We have to treat the product of spheres of distinct dimensions.

Proposition 2. *$S^k(\sqrt{k-1}) \times S^l(\sqrt{l-1})$ is a locally symmetric Einstein space, where $S^n(r)$ denotes the sphere of radius r.*

We obtain the following table.

Table 1

	$S^2(1)\times S^6(\sqrt{5})$	$S^3(\sqrt{2})\times S^5(2)$	$\overset{2}{\times} S^3(\sqrt{2})\times S^2(1)$	$\overset{2}{\times} S^2(1)\times S^4(\sqrt{3})$
τ^4	4096	4096	4096	4096
$\tau^2 R^2$	$\frac{2048}{5}$	352	640	$\frac{2048}{3}$
$\tau\widetilde{R}^3$	$-\frac{192}{25}$	$-\frac{27}{2}$	-12	$-\frac{64}{9}$
R^2R^2	$\frac{1024}{25}$	$\frac{121}{4}$	100	$\frac{1024}{9}$
(a)	$\frac{1024}{125}$	$\frac{29}{16}$	11	$\frac{448}{27}$
(b)	$\frac{48}{125}$	$\frac{39}{32}$	$\frac{3}{2}$	$\frac{16}{27}$
(c)	$\frac{328}{125}$	$\frac{73}{32}$	5	$\frac{136}{27}$
u_4	$\frac{74243}{590625}$	$\frac{15}{128}$	$\frac{41}{280}$	$\frac{19541}{127575}$

Now we restrict the dimension to $n = 8$. Then we obtain

$$u_4 = d_1\tau^4 + d_2\tau^2R^2 + d_3\tau\widetilde{R}^3 + d_4R^2R^2 + d_5(a) + d_6(b) + d_7(c), \tag{3.8}$$

where

$$
\begin{aligned}
d_1 &= c_1 + \frac{c_2}{8} + \frac{c_3}{64} + \frac{c_4}{512} = \frac{3799}{174182400},\\
d_2 &= c_5 + \frac{c_6}{8} + \frac{c_7}{64} = \frac{1}{12960} + \frac{c_6}{8} + \frac{c_7}{64},\\
d_3 &= c_8 + \frac{c_9}{8} = \frac{97}{680400} + 2c_6 + \frac{c_7}{4},\\
d_4 &= c_{10} = \frac{1}{13608} - \frac{c_6}{2}, \qquad d_5 = c_{11},\\
d_6 &= c_{12} = 2c_7 + 6c_{11} - \frac{4}{4725},\\
d_7 &= c_{13} = -2c_7 - 4c_{11} + \frac{1}{1890}.
\end{aligned}
$$

By putting the data in Table 1 into (3.8), we obtain

$$
(3.9)\quad
\begin{aligned}
d_1 &= \frac{3799}{174182400}, & d_2 &= (\text{parameter}),\\
d_3 &= 16d_2 - \frac{743}{680400}, & d_4 &= \frac{1}{64800},\\
d_5 &= (\text{parameter}), & d_6 &= 128d_2 + 6d_5 - \frac{107}{8505},\\
d_7 &= -128d_2 - 4d_5 + \frac{149}{12150},
\end{aligned}
$$

where d_2 and d_5 are undetermined. So we express them as parameters.

Step 6: We carry out the following calculation. On $CP^n(4)$ we obtain $v(r) = (\cos r)^{-1/2}(\sin r/r)^{(1-2n)/2}$. By Taylor asymptotic expansion we obtain on $CP^2(4)$ and $CP^4(4)$, respectively;

$$
(3.10)\quad
\begin{aligned}
&u_0 = 1, \quad u_1 = 4, \quad u_2 = \frac{124}{15}, \quad u_3 = \frac{3856}{315}, \quad u_4 = \frac{5008}{315},\\
&u_0 = 1, \quad u_1 = \frac{40}{3}, \quad u_2 = 88, \quad u_3 = 384, \quad u_4 = \frac{1184368}{945}.
\end{aligned}
$$

Then we obtain $u_4 = 103984/525$ on $CP^2(4) \times CP^2(4)$. By putting (3.10) and (3.2) into (3.9), we obtain (as for the curvature data refer to Table 2 below)

$$d_2 = \frac{101}{1088640}, \qquad d_5 = \frac{11}{113400}, \tag{3.11}$$

$$\begin{pmatrix} c_6 = \frac{79}{680400}, & c_7 = \frac{1}{14175}, & c_8 = \frac{1}{2835}, & c_9 = \frac{1}{3150} \\ c_{10} = \frac{1}{64800}, & c_{11} = \frac{11}{113400}, & c_{12} = -\frac{1}{8100}, & c_{13} = 0 \end{pmatrix}.$$

By the Ricci identity (1.13), (2.10) and (2.11), we have

$$\begin{aligned} \tau\tilde{R}^3 &= -\frac{1}{2}\frac{\tau^2}{n}R^2 - \frac{1}{4}\tau R^3, \\ (a) &= (t) + \frac{\tau}{n}R^3 + \frac{1}{2}R^4, \\ (b) &= -\frac{1}{4}R^4 - \frac{1}{2}\frac{\tau}{n}R^3. \end{aligned} \tag{3.12}$$

Then we obtain

$$\begin{aligned} u_4 = &\Big(\frac{1}{31104} - \frac{1}{12960}\frac{1}{n} - \frac{59}{1360800}\frac{1}{n^2} - \frac{1}{113400}\frac{1}{n^3}\Big)\tau^4 \\ &+ \Big(\frac{1}{12960} - \frac{41}{680400}\frac{1}{n} - \frac{1}{11340}\frac{1}{n^2}\Big)\tau^2 R^2 \\ &+ \Big(-\frac{1}{11340} + \frac{1}{12600}\frac{1}{n}\Big)\tau R^3 + \frac{1}{64800}R^2R^2 \\ &+ \frac{11}{113400}(t) + \frac{1}{12600}R^4 + 0(c), \end{aligned} \tag{3.13}$$

for an n-dimensional locally symmetric Einstein space. Especially in the case of $n = 8$, we obtain

$$\begin{aligned} u_4 = &\frac{3799}{174182400}\tau^4 + \frac{743}{10886400}\tau^2R^2 - \frac{71}{907200}\tau R^3 + \frac{1}{64800}R^2R^2 \\ &+ \frac{11}{113400}(t) + \frac{1}{12600}R^4 + 0(c). \end{aligned} \tag{3.14}$$

Remark. Avramidi [1] also gave the explicit expression of u_4. But it is too complicated to apply for geometry.

Remark. For an n-dimensionl locally symmetric Einstein space

$$\begin{aligned} u_3 = &\Big(\frac{1}{1296} - \frac{1}{1080}\frac{1}{n} - \frac{1}{2835}\frac{1}{n^2}\Big)\tau^3 \\ &+ \Big(\frac{1}{1080} - \frac{1}{5670}\frac{1}{n}\Big)\tau R^2 - \frac{1}{1890}R^3, \end{aligned} \tag{3.15}$$

was given by Sakai [12].

4. The calculation of $\chi(M)$. We calculate the Euler characteristic $\chi(M)$ of an 8-dimensional locally symmetric Einstein space. We know that $\chi(M^8)$ is given by the following

$$(4.1)\qquad \chi(M^8) = \frac{1}{2^{12}\pi^4 4!}\int_M \varepsilon_{i_1\ldots i_8}\varepsilon_{j_1\ldots j_8} R_{i_1 i_2 j_1 j_2}\ldots R_{i_7 i_8 j_7 j_8}\,dv$$
$$= \frac{1}{\pi^4}\int_M (e_1\tau^4 + e_2\tau^2 R^2 + e_3\tau\widetilde{R}^3 + e_4 R^2 R^2 + e_5(a) + e_6(b) + e_7(c))\,dv,$$

where $e_1,\ldots,e_7$ are constants. We take the following models

(1) $S^8(1)$, (2) $S^4(1)\times S^4(1)$, (3) $\overset{4}{\times} S^2(1)$,
(4) $S^2(1)\times S^6(\sqrt{5})$, (5) $S^3(\sqrt{2})\times S^5(2)$, (6) $\overset{2}{\times} S^3(\sqrt{2})\times S^2(1)$,
(7) $\overset{2}{\times} S^2(1)\times S^4(\sqrt{3})$, (8) $CP^4(4)$, (9) $CP^2(4)\times CP^2(4)$.

By (3.1) and (3.2) we obtain the following table.

Table 2

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
τ^4	9834496	331776	4096	4096	4096	4096	4096	40960000	5308416
τ^2R^2	351232	27648	1024	$\frac{2048}{5}$	352	640	$\frac{2048}{3}$	4096000	884736
$\tau\widetilde{R}^3$	−18816	−1152	0	$-\frac{192}{25}$	$-\frac{27}{2}$	−12	$-\frac{64}{9}$	−76800	−9216
R^2R^2	12544	2304	256	$\frac{1024}{25}$	$\frac{121}{4}$	100	$\frac{1024}{9}$	409600	147456
(a)	224	96	32	$\frac{1024}{125}$	$\frac{29}{16}$	11	$\frac{448}{27}$	43520	16896
(b)	672	96	0	$\frac{48}{125}$	$\frac{39}{32}$	$\frac{3}{2}$	$\frac{16}{27}$	3840	768
(c)	1064	168	8	$\frac{328}{125}$	$\frac{73}{32}$	5	$\frac{136}{27}$	10880	4224
$\frac{\chi(M)}{\mathrm{Vol}}$	$\frac{105}{16\pi^4}$	$\frac{9}{16\pi^4}$	$\frac{1}{16\pi^4}$	$\frac{3}{400\pi^4}$	0	0	$\frac{1}{48\pi^4}$	$\frac{120}{\pi^4}$	$\frac{36}{\pi^4}$

We put the data in Table 2 into (4.1). For the spherical data of (1), ..., (7) we have

$$\begin{aligned}
&e_1 = -\frac{1}{98304}, \ e_2 = (\text{parameter}), \qquad e_3 = 16e_2 - \frac{23}{3072}, \\
(4.2)\ &e_4 = \frac{1}{2048}, \quad e_5 = -32e_2 - \frac{e_7}{4} - \frac{1}{1536}, \ e_6 = -62e_2 - \frac{3}{2}e_7 - \frac{23}{384}, \\
&e_7 = (\text{parameter}),
\end{aligned}$$

where e_2 and e_7 are undetermined. So we express them as parameters. By adding (8) and (9) we obtain

$$(4.3)\ e_2 = -\frac{1}{49152},\ e_3 = -\frac{1}{128},\ e_5 = \frac{1}{512},\ e_6 = -\frac{3}{64},\ e_7 = -\frac{1}{128}.$$

By (3.12) we express $\chi(M)$ in another form

$$\begin{aligned}
(4.4)\ \chi(M^8) = \frac{1}{\pi^4}\int_M \Big(&-\frac{1}{98304}\tau^4 + \frac{23}{49152}\tau^2 R^2 + \frac{21}{4096}\tau R^3 \\
&+ \frac{1}{2048}R^2R^2 + \frac{1}{512}(t) + \frac{13}{1024}R^4 - \frac{1}{128}(c)\Big)\,dv.
\end{aligned}$$

Remark. For a 6-dimensional locally smmetric Einstein space Sakai [12] obtained

$$(4.5)\qquad \chi(M^6) = \frac{1}{\pi^3}\int_M \Big(\frac{1}{3456}\tau^3 - \frac{5}{1152}\tau R^2 - \frac{1}{64}R^3\Big)\,dv.$$

Similarly we can calculate the signature $\sigma(M^8)$. By the following fact

$$(4.6)\qquad \sigma(S^{2n}) = 0,\quad \sigma(CP^{2n}) = 1,\quad \sigma(M\times N) = \sigma(M)\sigma(N),$$

we obtain for the signature

$$(4.7)\ \sigma(M^8) = \frac{1}{\pi^4}\int_M \Big(-\frac{17}{92160}\tau R^3 - \frac{7}{11520}R^4 + \frac{1}{2880}(t) - \frac{7}{2880}(c)\Big)\,dv,$$

up to sign for an oriented 8-dimensional locally symmetric Einstein space.

Remark. For a 4-dimenional Kähler Einstein space we had better refer to Donnelly's paper [4].

$$(4.8)\qquad \chi(M^4) = \frac{1}{32\pi^2}\int_M R^2\,dv,\quad \sigma(M^4) = \frac{1}{96\pi^2}\int_M (\tau^2 - 2R^2)\,dv.$$

Remark. Lovelock [9] also gave the explicit expression of $\chi(M^8)$ by direct tensor calculation.

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5. a_4 for 1-form. We consider the spectrum of the Laplacian for 1-forms. Similarly we can treat the asymptotic expansion for the trace of the heat kernel [5]. The coefficients $a_i{}^1$ contain geometric informations and are spectral invariants. $a_2{}^1$ was calculated by Patodi [11]. $a_3{}^1$ was calculated by Ii [7] (for an Einstein space). Their approach is different from each other. The one is a combinatorial method and the other is a method using Taylor asymptotic expansion. In this section we calculate $a_4{}^1$ on an n-dimensionl locally symmetric Einstein space by a combinatorial method.

Step 1: We can set

$$u_4{}^1 = c_1(n)\tau^4 + c_2(n)\tau^2R^2 + c_3(n)\tau R^3 + c_4(n)R^2R^2 + c_5(n)(t) + c_6(n)R^4 + c_7(n)(c). \tag{5.1}$$

If we express $u_4{}^1$ as a polynomial of the independent contracted values of the product of for curvature tensors in a locally symmetric space, then its coefficients are polynomials of degree 1. However, in our Einstein case, by (1.5) the coefficients $c_i(n)$ corresponding to the term containing τ in (5.1) are polynomials containing powers of the factor $1/n$ (see [7]).

Step 2: On the product space $M \times M$, we have ([11])

$$\begin{aligned} u_4{}^1(M\times M) &= 2\sum_{i+j=4} u_i{}^1(M)u_j{}^0(M) \\ &= 2(u_4{}^1u_0 + u_3{}^1u_1 + u_2{}^1u_2 + u_1{}^1u_3 + u_0{}^1u_4). \end{aligned} \tag{5.2}$$

In the above we denote $u_i{}^p = u_i{}^p(M)$ for the sake of simplicity.

$$\begin{aligned} \tau(M\times M) &= 2\tau(M), & R^2(M\times M) &= 2R^2(M), \\ R^3(M\times M) &= 2R^3(M), & R^4(M\times M) &= 2R^4(M), \\ (t)(M\times M) &= 2(t)(M), & (c)(M\times M) &= 2(c)(M). \end{aligned} \tag{5.3}$$

In the following we denote simply $\tau = \tau(M)$, $R^2 = R^2(M)$, etc.

Step 3: On the space M^n, u_i and $u_i{}^1$ take the following forms;

$$u_0 = 1, \quad u_1 = \frac{1}{6}\tau,$$

$$u_2 = \Big(\frac{1}{72} - \frac{1}{180}\frac{1}{n}\Big)\tau^2 + \frac{1}{180}R^2,$$

$$\text{(5.4)} \quad u_3 = \Big(\frac{1}{1296} - \frac{1}{1080}\frac{1}{n} - \frac{1}{2835}\frac{1}{n^2}\Big)\tau^3 + \Big(\frac{1}{1080} - \frac{1}{5670}\frac{1}{n}\Big)\tau R^2 - \frac{1}{1890}R^3,$$

$$u_4 = \Big(\frac{1}{31104} - \frac{1}{12960}\frac{1}{n} - \frac{59}{1360800}\frac{1}{n^2} - \frac{1}{113400}\frac{1}{n^3}\Big)\tau^4 + \Big(\frac{1}{12960} - \frac{41}{680400}\frac{1}{n} - \frac{1}{11340}\frac{1}{n^2}\Big)\tau^2R^2 + \Big(-\frac{1}{11340} + \frac{1}{12600}\frac{1}{n}\Big)\tau R^3 + \frac{1}{64800}R^2R^2 + \frac{11}{113400}(t) + \frac{1}{12600}R^4,$$

$$u_0{}^1 = n, \quad u_1{}^1 = \Big(\frac{n}{6} - 1\Big)\tau,$$

$$u_2{}^1 = \Big(\frac{n}{72} - \frac{31}{180} + \frac{1}{2}\frac{1}{n}\Big)\tau^2 + \Big(\frac{n}{180} - \frac{1}{12}\Big)R^2,$$

$$\text{(5.5)} \quad u_3{}^1 = \Big(\frac{n}{1296} - \frac{2}{135} + \frac{251}{2835}\frac{1}{n} - \frac{1}{6}\frac{1}{n^2}\Big)\tau^3 + \Big(\frac{n}{1080} - \frac{89}{4536} + \frac{7}{90}\frac{1}{n}\Big)\tau R^2 + \Big(-\frac{n}{1890} + \frac{1}{120}\Big)R^3,$$

($u_3{}^1$ was obtained by Ii [7]).

Then we put (5.3), ..., (5.5) into (5.2) and obtain the following

$$\text{(5.6)} \quad 16c_1(2n)\tau^4 + 8c_2(2n)\tau^2R^2 + 4c_3(2n)\tau R^3 + 4c_4(2n)R^2R^2 + 2c_5(2n)(t) + 2c_6(2n)R^4 + 2c_7(2n)(c)$$
$$= \Big(2c_1(n) + \frac{5n}{5184} - \frac{77}{6480} + \frac{10651}{226800}\frac{1}{n} - \frac{571}{9450}\frac{1}{n^2}\Big)\tau^4 + \Big(2c_2(n) + \frac{7n}{6480} - \frac{2917}{226800} + \frac{739}{22680}\frac{1}{n}\Big)\tau^2R^2$$
$$+ \Big(2c_3(n) - \frac{n}{1890} + \frac{151}{37800}\Big)\tau R^3 + \Big(2c_4(n) + \frac{n}{10800} - \frac{1}{1080}\Big)R^2R^2 + \Big(2c_5(n) + \frac{11n}{56700}\Big)(t) + \Big(2c_6(n) + \frac{n}{6300}\Big)R^4 + 2c_7(n)(c).$$

Step 4: Since the curvature data $\tau^4, \ldots, (c)$ are independent, we can compare the coefficients. For example

$$16c_1(2n) = 2c_1(n) + \frac{5n}{5184} - \frac{77}{6480} + \frac{10651}{226800}\frac{1}{n} - \frac{571}{9450}\frac{1}{n^2}. \tag{5.7}$$

Then we can set $c_1(n) = p_1 n + p_2 + p_3/n + p_4/n^2 + c_1/n^3$, and we obtain $p_1 = 1/31104$, $p_2 = -11/12960$, $p_3 = 10651/1360800$, $p_4 = -571/18900$, $c_1 =$ (parameter), that is,

$$c_1(n) = \frac{n}{31104} - \frac{11}{12960} + \frac{10651}{1360800}\frac{1}{n} - \frac{571}{18900}\frac{1}{n^2} + \frac{c_1}{n^3}. \tag{5.8}$$

Similarly

$$\begin{aligned} c_2(n) &= \frac{n}{12960} - \frac{2917}{1360800} + \frac{739}{45360}\frac{1}{n} + \frac{c_2}{n^2}, \\ c_3(n) &= -\frac{n}{11340} + \frac{151}{75600} + \frac{c_3}{n}, \\ c_4(n) &= \frac{n}{64800} - \frac{1}{2160} + \frac{c_4}{n}, \end{aligned} \tag{5.9}$$

(since R^2R^2 does not contain τ, the factor of $1/n$ does not appear, i.e. $c_4 = 0$.)

$$c_5(n) = \frac{11n}{113400} + c_5, \quad c_6(n) = \frac{n}{12600} + c_6, \quad c_7(n) = c_7.$$

Step 5: To determine $c_1, c_2, \ldots, c_7$ we need the explicit data of ${u_4}^1$ for some model spaces. We calculate ${u_4}^1$ for model spaces by the following formula (5.10)

$$Z^1(t) = \sum_{i=0}^{\infty} \mathrm{e}^{-\lambda_i t} \sim \frac{\mathrm{Vol}(M^n, g)}{(4\pi t)^{\frac{n}{2}}} \sum_{i=0}^{\infty} {u_i}^1 t^i, \quad t \to +0, \tag{5.10}$$

where λ_i's are the eigenvalues counted with multiplicities of the Laplacian for 1-forms. For the perpose we need the explicit data of the spectrum for $S^n(1)$ and $CP^n(4)$ which are determined by Ikeda [8] (see Table 3).

We use the following formulae (see [3],[10])

$$\begin{aligned} &\sum_{n=0}^{\infty} (2n+1)\mathrm{e}^{-\left(n+\frac{1}{2}\right)^2 t} \sim \frac{1}{t} + \sum_{n=0}^{\infty} p_n t^n, \\ &\sum_{n=0}^{\infty} 2n\mathrm{e}^{-n^2 t} \sim \frac{1}{t} + \sum_{n=0}^{\infty} q_n t^n, \qquad \sum_{n=-\infty}^{\infty} \mathrm{e}^{-n^2 t} \sim \sqrt{\pi} t^{-\frac{1}{2}}, \end{aligned} \tag{5.11}$$

Table 3

	Eigenvalue	Multiplicity	Range
$S^n(1)$	$(k+1)(n+k)$	$\frac{(n+2k+1)(n+k-1)!}{(n-1)!(k+1)!}$	$k \geq 0$
	$(k+1)(n+k-2)$	$\frac{(n+2k-1)(n+k-1)!}{(n-2)!(k-1)!(k+1)(n+k-2)}$	$k \geq 1$
$CP^n(4)$	$4(k+1)(k+n+1)$	$\frac{n(2k+n+2)(k+n)!^2}{(k+1)!^2 n!^2}$	$k \geq 0$
	$4(k+2)(k+n+1)$	$\frac{(2k+n+3)(k+n+1)!(k+n+2)!}{(k+2)^2(k+n+1)^2(k+1)!k!(n-2)!n!}$	$k \geq 0$
	$4k(k+n)$	$\frac{n(2k+n)(k+n-1)!^2}{k!^2 n!^2}$	$k \geq 1$
	$4k(k+n-1)$	$\frac{(2k+n-1)(k+n)!(k+n-1)!}{k^2(k+n-1)^2(k-1)!(k-2)!(n-2)!n!}$	$k \geq 2$

where

$$p_n = \frac{(-1)^n}{(n+1)!}B_{2n+2}\left(1 - \frac{1}{2^{2n+1}}\right), \quad q_n = -\frac{(-1)^n}{(n+1)!}B_{2n+2}.$$

B_i is the Bernolli number, so that $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$, $B_8 = -1/30$, $B_{10} = 5/66$. Then we obtain the following table.

Table 4

	$u_0{}^1$	$u_1{}^1$	$u_2{}^1$	$u_3{}^1$	$u_4{}^1$
$S^2(1)$	2	$-\frac{4}{3}$	$\frac{2}{15}$	$\frac{8}{315}$	$\frac{2}{315}$
$S^3(1)$	3	-3	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{24}$
$S^4(1)$	4	-4	$-\frac{4}{15}$	$\frac{44}{63}$	$\frac{116}{315}$
$S^5(1)$	5	$-\frac{10}{3}$	$-\frac{10}{3}$	$\frac{2}{3}$	$\frac{35}{18}$
$S^6(1)$	6	0	-8	$-\frac{88}{21}$	$\frac{22}{5}$
$CP^2(4)$	4	-8	$-\frac{104}{15}$	$\frac{1312}{315}$	$\frac{4384}{315}$
$CP^3(4)$	6	0	-32	$-\frac{5312}{105}$	$\frac{64}{35}$

By putting (3.1), (3.2) and the data in table 4 into (5.8) and (5.9), we obtain the following system of equations

$$
\begin{aligned}
&2c_1+4c_2-8c_3+8c_5+16c_6+2c_7-\frac{4}{945}=\frac{2}{315},\\
&48c_1+48c_2-48c_3+48c_5+48c_6+24c_7-\frac{17}{24}=\frac{1}{24},\\
&324c_1+216c_2-144c_3+144c_5+96c_6+84c_7-\frac{2164}{315}=\frac{116}{315},\\
(5.12)\ &1280c_1+640c_2-320c_3+320c_5+160c_6+200c_7-\frac{860}{27}=\frac{35}{18},\\
&3750c_1+1500c_2-600c_3+600c_5+240c_6+390c_7-\frac{10988}{105}=\frac{22}{5},\\
&5184c_1+6912c_2-11520c_3+9216c_5+21504c_6+2112c_7-\frac{15392}{315}=\frac{4384}{315},\\
&24576c_1+24576c_2-36864c_3+24576c_5+67584c_6+5376c_7-\frac{45056}{105}=\frac{64}{35}.
\end{aligned}
$$

Then we have

$$
(5.13)\qquad \begin{aligned}
&c_1=\frac{1}{24}, \quad c_2=-\frac{107}{3024}, \quad c_3=-\frac{1}{180}, \quad c_4=0,\\
&c_5=\frac{37}{15120}, \quad c_6=-\frac{1}{20160}, \quad c_7=\frac{1}{360},
\end{aligned}
$$

$$
\begin{aligned}
(5.14)\ u_4{}^1 = &\Big(\frac{n}{31104}-\frac{11}{12960}+\frac{10651}{1360800}\frac{1}{n}-\frac{571}{18900}\frac{1}{n^2}+\frac{1}{24}\frac{1}{n^3}\Big)\tau^4\\
&+\Big(\frac{n}{12960}-\frac{2917}{1360800}+\frac{739}{45360}\frac{1}{n}-\frac{107}{3024}\frac{1}{n^2}\Big)\tau^2R^2\\
&+\Big(-\frac{n}{11340}+\frac{151}{75600}-\frac{1}{180}\frac{1}{n}\Big)\tau R^3+\Big(\frac{n}{64800}-\frac{1}{2160}\Big)R^2R^2\\
&+\Big(\frac{11n}{113400}+\frac{37}{15120}\Big)(t)+\Big(\frac{n}{12600}-\frac{1}{20160}\Big)R^4+\frac{1}{360}(c).
\end{aligned}
$$

Remark. $c_1, c_2, \dots, c_7$ are determined except $S^6(1)$.

6. Applications. Summing up we obtain the following formulae for an oriented 8-dimensional locally symmetric Einstein space (M, g);

$$
\begin{aligned}
(6.1)\qquad a_4 = \int_M\Big(&\frac{3799}{174182400}\tau^4+\frac{743}{10886400}\tau^2R^2-\frac{71}{907200}\tau R^3\\
&+\frac{1}{64800}R^2R^2+\frac{11}{113400}(t)+\frac{1}{12600}R^4+0(c)\Big)dv,
\end{aligned}
$$

$$(6.2)\qquad a_4{}^1 = \int_M\Big(-\frac{673}{174182400}\tau^4 - \frac{1859}{43545600}\tau^2R^2 + \frac{271}{453600}\tau R^3 - \frac{11}{32400}R^2R^2 + \frac{731}{226800}(t) + \frac{59}{100800}R^4 + \frac{1}{360}(c)\Big)dv,$$

$$(6.3)\quad \chi(M^8) = \frac{1}{\pi^4}\int_M\Big(-\frac{1}{98304}\tau^4 + \frac{23}{49152}\tau^2R^2 + \frac{21}{4096}\tau R^3 + \frac{1}{2048}R^2R^2 + \frac{1}{512}(t) + \frac{13}{1024}R^4 - \frac{1}{128}(c)\Big)dv,$$

$$(6.4)\quad \sigma(M^8) = \frac{1}{\pi^4}\int_M\Big(-\frac{17}{92160}\tau R^3 - \frac{7}{11520}R^4 + \frac{1}{2880}(t) - \frac{7}{2880}(c)\Big)dv.$$

Remark. As for the signature the ambiguity of the sign occures by the orientation of M.

Proposition 3. *Let (M,g) and (M',g') be oriented 8-dimensional locally symmetric Einstein spaces. Assume that* $\mathrm{Spec}(M,g) = \mathrm{Spec}(M',g')$ *holds. Then if* (1) $R^4 = R'^4$ *and* (2) $(c) = (c)'$ *hold, we have* $\chi(M) = \chi(M')$ *and* $|\sigma(M)| = |\sigma(M')|$.

Proof. By $\mathrm{Spec}(M,g) = \mathrm{Spec}(M',g')$, $a_i = a_i'$ hold for each i. From $a_0 = a_0'$, $a_1 = a_1'$, $a_2 = a_2'$, $a_3 = a_3'$ and the local symmetricity we have

$$(6.5)\qquad \begin{aligned} &\mathrm{Vol}(M) = \mathrm{Vol}(M'), \qquad \tau^4 = \tau'^4, \qquad \tau^2R^2 = \tau'^2R'^2, \\ &\tau R^3 = \tau' R'^3, \qquad R^2R^2 = R'^2R'^2, \end{aligned}$$

and they are constant on M, M'. If (1) and (2) hold, by putting (6.5), (1) and (2) into $a_4 = a_4'$ of (6.1) we obtain $(t) = (t)'$. Then we can conclude $\chi(M) = \chi(M')$ and $|\sigma(M)| = |\sigma(M')|$.

Remark. For 6-dimensional locally symmetric Einstein spaces this proposition holds without conditions (1) and (2) (see [12]).

Proposition 4 (Patodi [11]). *Let (M,g), (M',g') be closed Riemannian manifolds. Assume that (M,g) is a locally symmetric Einstein space. If* $\mathrm{Spec}(M,g) = \mathrm{Spec}(M',g')$ *and* $\mathrm{Spec}^1(M,g) = \mathrm{Spec}^1(M',g')$ *hold, then the other (M',g') is also a locally symmetric Einstein space with the same dimension.*

Proposition 5. *For two oriented closed Riemannian manifolds (M,g) and (M',g') assume that one of them is an 8-dimensional locally symmetric Einstein space. If (M,g) and (M',g') have the same* *spectra for functions and for 1-forms, respectively, i.e.,* $\mathrm{Spec}(M,g) = \mathrm{Spec}(M',g')$ *and* $\mathrm{Spec}^1(M,g) = \mathrm{Spec}^1(M',g')$, *then* (1) $\chi(M) = \chi(M')$ *and* (2) $|\sigma(M)| = |\sigma(M')|$ *are equivalent.*

Proof. By Proposition 4 M' is also an 8-dimensional locally symmetric Einstein space. Then we can apply (6.1), ..., (6.4) for M and M'. From the assumptions $a_0 = a_0'$, $a_1 = a_1'$, $a_2 = a_2'$, $a_3 = a_3'$ and the local symmetricity, we obtain (6.5) and they are constant on M, M'. If (1) (resp. (2)) holds, we obtain a system of equations (6.1), (6.2) and (6.3) (resp. (6.4)) of (t), R^4, and (c). Then it suffices to solve them.

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