

# *Mathematical Journal of Okayama University*

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*Volume 37, Issue 1*

1995

*Article 16*

JANUARY 1995

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## The Existence of $p$ -Harmonic Maps between Spheres

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## THE EXISTENCE OF $p$ -HARMONIC MAPS BETWEEN SPHERES

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**1. Introduction.** Let  $(M, g)$  and  $(N, h)$  be compact Riemannian manifolds. For a smooth map  $\phi: M \rightarrow N$  and  $p \geq 2$ , the  $p$ -energy integral is defined by

$$J_p(\phi) = \int_M |d\phi|^p dV_g$$

where  $|\cdot|$  is the Hilbert-Schmidt norm and  $dV_g$  is the canonical measure associated with the metric  $g$ . A map  $\phi$  is said to be  $p$ -harmonic if it is a critical point of  $J_p$ .

Smith [7] constructs harmonic maps from the join of two harmonic maps between Euclidean spheres. He reduces the harmonic map equation to an ordinary differential equation and studies their properties. The necessary and sufficient condition for the existence of solutions of this equation has been showed by Ding [2]. Pettinati and Ratto [6] obtain similar results by using completely different methods. Recently, Xu and Yang [10] proved the existence of  $p$ -harmonic maps for the case of  $p = \dim M$ .

In this paper, we give a sufficient condition for the existence of  $p$ -harmonic maps. We wish to emphasize that our results fulfill the gap of the range of  $p$  in the previous works although the necessary condition has not been settled yet.

Let  $f: S^q \rightarrow S^m$  and  $g: S^r \rightarrow S^n$  be harmonic homogeneous polynomial maps of degree  $k$  and  $l$ , respectively. Here components of  $f$  (resp.  $g$ ) are eigenfunctions of the Laplacian on  $S^q$  (resp.  $S^r$ ) with eigenvalues  $\lambda_k = k(k+q-1)$  (resp.  $\lambda_l = l(l+r-1)$ ).

**Main Theorem 1.1.** *Suppose one of the following assumptions (1), (2) and (3) is satisfied, then there exists a  $p$ -harmonic map  $\phi: S^{q+r+1} \rightarrow S^{m+n+1}$ .*

- (1)  $p < \max(q, r) + 1$ ,  $(\lambda_k)^{p/2} K(q-p, r) \geq (\lambda_l)^{p/2} K(q, r-p)$  and  $(r-p+1)^2 < 4\lambda_l$ ,
- (2)  $p < \max(q, r) + 1$ ,  $(\lambda_k)^{p/2} K(q-p, r) \geq (\lambda_l)^{p/2} K(q, r-p)$ ,  $(r-p+1)^2 \geq 4\lambda_l$  and  $\sqrt{(r-p+1)^2 - 4\lambda_l} + \sqrt{(q-1)^2 + 4\lambda_k} < q+r-p$ ,
- (3)  $p \geq \max(q, r) + 1$ ,

where  $K(m, n) := \int_0^{\pi/2} \sin^m x \cos^n x \, dx$ .

It is worthwhile remarking that the inequality

$$(\lambda_k)^{\frac{p}{2}} K(q - p, r) \geq (\lambda_l)^{\frac{p}{2}} K(q, r - p)$$

is not a restriction, since it can be always assumed by interchanging the roles of  $f$  and  $g$  whenever necessary.

Smith [7] gets the harmonic representative for all elements in  $\pi_n(S^n)$  ( $1 \leq n \leq 7$ ) by using his equation. We can generalize this result by his method as the following.

**Corollary 1.2.** *For all  $n \geq 1$ , the  $n$ -th homotopy group  $\pi_n(S^n)$  is representable by  $p$ -harmonic map provided*

$$\begin{cases} p \geq 2 & (1 \leq n \leq 7), \\ p > n - 2\sqrt{n-2} - 1 > 2 & (n \geq 8). \end{cases} \quad (1.1)$$

The organization of the remainder of this paper is as follows: In section 2, we reduce the  $p$ -harmonicity equation for  $\phi$  to an ordinary differential equation and show the existence of a weak solution of this equation. It is essential for our proof of Main Theorem 1.1 to study the properties of above solutions. We first exclude constant solutions of this equation in section 3. Then we show the monotonicity and some asymptotic properties in section 4 and regularity in section 5. In final section, we give a proof of Corollary 1.2. Although almost all arguments are a combination of Ding [2] and Eells and Ratto [3], we need some extra arguments in section 4 especially.

**2. The  $p$ -energy integral of the join map and Euler-Lagrange equation.** We regard the sphere  $S^{q+r+1}$  as a subset of  $\mathbf{R}^{q+r+2}$  by

$$\left\{ (\sin t \cdot x, \cos t \cdot y) : x \in S^q, y \in S^r, 0 \leq t \leq \frac{\pi}{2} \right\}.$$

The induced Riemannian metric on  $S^{q+r+1}$  is given by

$$g = \sin^2 t \cdot g_q + \cos^2 t \cdot g_r + dt^2$$

where  $g_\alpha$  is the standard metric of  $S^\alpha$ . When two harmonic homogeneous polynomial maps  $f: S^q \rightarrow S^m$  of degree  $k$  and  $g: S^r \rightarrow S^n$  of degree  $l$

are given, we define the join  $\phi := f * g: S^{q+r+1} \rightarrow S^{m+n+1}$  as follows: For  $x \in S^q$ ,  $y \in S^r$  and  $t \in [0, \pi/2]$ , set

$$\phi(\sin t \cdot x, \cos t \cdot y) = (\sin a(t) \cdot f(x), \cos a(t) \cdot g(y))$$

where  $a(t)$  is a smooth map from  $[0, \pi/2]$  to  $[0, \pi/2]$  satisfying

$$a(0) = 0, \quad a\left(\frac{\pi}{2}\right) = \frac{\pi}{2}, \quad (2.1)$$

$$\dot{a}(t) > 0 \quad \text{for all } t \in (0, \pi/2). \quad (2.2)$$

Then for  $u = (\sin t \cdot x, \cos t \cdot y) \in S^{q+r+1}$ , we have

$$|d\phi|^2(u) = \dot{a}^2(t) + \lambda_k \frac{\sin^2 a(t)}{\sin^2 t} + \lambda_l \frac{\cos^2 a(t)}{\cos^2 t}$$

and

$$J_p(a) = c \int_0^{\pi/2} \left\{ \dot{a}^2(t) + \lambda_k \frac{\sin^2 a(t)}{\sin^2 t} + \lambda_l \frac{\cos^2 a(t)}{\cos^2 t} \right\}^{\frac{p}{2}} f(t) dt,$$

where  $c$  is some positive constant and  $f(t) = \sin^q t \cos^r t$ .

A simple computation leads to the following Euler-Lagrange equation for the functional  $J_p(a)$ :

$$\begin{aligned} \ddot{a}(t) + (q \cot t - r \tan t) \dot{a}(t) + \left( \frac{\lambda_l}{\cos^2 t} - \frac{\lambda_k}{\sin^2 t} \right) \sin a(t) \cos a(t) \\ + \frac{p-2}{2} \dot{a}(t) \frac{d}{dt} \left\{ \log \left( \dot{a}^2(t) + \lambda_k \frac{\sin^2 a(t)}{\sin^2 t} + \lambda_l \frac{\cos^2 a(t)}{\cos^2 t} \right) \right\} = 0. \end{aligned} \quad (2.3)$$

Therefore the equation of  $p$ -harmonic maps is equivalent to (2.1), (2.2) and (2.3). We study this equation in the following. We first introduce a function space

$$X := \left\{ a \in L_1^p([0, \pi/2], \mathbf{R}) : \|a\|^p = \int_0^{\pi/2} (|\dot{a}|^p + |a|^p) f(t) dt < \infty \right\}.$$

Here we allow  $J_p$  to assume the value  $\infty$ . If  $J_p(a) < \infty$ ,  $J_p$  is smooth at  $a$ . It is obvious that  $J_p(a) > 0$  for all  $a \in X$ . We note that  $J_p$  is weak lower semicontinuous on  $X$ , namely, for any sequence  $\{x_n\}$  in  $X$  such that  $x_n \rightarrow x$  weakly, we have  $J_p(x) \leq \liminf J_p(x_n)$ .

In order to investigate the critical points of  $J_p$  subject to the conditions (2.1) and (2.2), it is convenient to introduce the closed convex subset

$$X_0 := \left\{ a \in X : 0 \leq a(t) \leq \frac{\pi}{2} \text{ for all } t \in (0, \pi/2) \right\}.$$

If we restrict  $J_p$  to  $X_0$ , then  $J_p$  satisfies the following inequality:

$$\begin{aligned} J_p(a) &> \int_0^{\pi/2} |\dot{a}|^p \sin^q t \cos^r t \, dt \\ &= \int_0^{\pi/2} (|\dot{a}|^p + |a|^p) \sin^q t \cos^r t \, dt - \int_0^{\pi/2} |a|^p \sin^q t \cos^r t \, dt \\ &\geq \|a\|^p - \left(\frac{\pi}{2}\right)^p \int_0^{\pi/2} \sin^q t \cos^r t \, dt, \end{aligned}$$

namely,

$$J_p(a) > \|a\|^p - \text{constant.} \tag{2.4}$$

So we have

**Lemma 2.1.**  $J_p$  satisfies coercive condition on  $X_0$ .

It follows from the coercive condition that there exists an  $a_0 \in X_0$  such that  $J_p(a_0) = c_0 := \inf\{J_p(a) : a \in X_0\}$ .

**Lemma 2.2.** There exists an  $a_0 \in X_0$  such that  $a_0$  satisfies (2.3) weakly.

*Proof.* For any  $a \in X$ , define  $a^* \in X_0$  by

$$a^*(t) := \begin{cases} \frac{\pi}{2} & \text{if } a(t) > \frac{\pi}{2}, \\ a(t) & \text{if } 0 \leq a(t) \leq \frac{\pi}{2}, \\ 0 & \text{if } a(t) < 0. \end{cases}$$

Set  $F(t, a) := \lambda_k \sin^2 a(t) / \sin^2 t + \lambda_l \cos^2 a(t) / \cos^2 t$ , and let  $U : \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$  be given by

$$U(t, a) := \begin{cases} \frac{\lambda_k}{\sin^2 t} & \text{if } a(t) > \frac{\pi}{2}, \\ \lambda_k \frac{\sin^2 a(t)}{\sin^2 t} + \lambda_l \frac{\cos^2 a(t)}{\cos^2 t} & \text{if } 0 \leq a(t) \leq \frac{\pi}{2}, \\ \frac{\lambda_l}{\cos^2 t} & \text{if } a(t) < 0. \end{cases}$$

We define a functional on  $X$  by

$$J_p^*(a) := \int_0^{\pi/2} \{\dot{a}^2 + U(t, a)\}^{\frac{p}{2}} \sin^q t \cos^r t \, dt.$$

We have the identities

$$F(t, a^*(t)) = U(t, a(t)) = U(t, a^*(t)) \text{ for all } a \in X \text{ and } t \in (0, \pi/2).$$

It follows that

$$J_p(a^*) = J_p^*(a^*) \leq J_p^*(a) \text{ for all } a \in X,$$

which implies  $\inf\{J_p^*(a): a \in X\} = c_0$ .

Let  $\{a_i\}$  be a minimizing sequence in  $X$  for  $J_p^*$ . By passing to  $\{a_i^*\}$  if necessary, we may assume  $a_i \in X_0$ . Since  $J_p^*(a_i) \rightarrow c_0$  and  $0 \leq a_i \leq \pi/2$ , the inequality (2.4) shows that  $\{a_i\}$  is bounded in  $X$ . By noting that  $X$  is a reflexive Banach space, we may choose some subsequence of  $\{a_i\}$  which converges weakly to some  $a_0 \in X_0$  in  $X$ . The semicontinuity of  $J_p^*$  yields  $J_p^*(a_0) = c_0$ . The equation satisfied by  $a_0$  is exactly (2.3), because  $a_0 \in X_0$ ,  $J_p^*$  and  $J_p$  coincide on  $X_0$ .

**3. Exclusion of constant solutions.** In order to exclude constant solutions  $a \equiv 0$  and  $a \equiv \pi/2$ , we show that  $c_0 < \min\{J_p(0), J_p(\pi/2)\}$ .

By the mean value theorem, we get for some  $t_0, t_1 \in (0, \pi/2)$ ,

$$J_p(0) = (\lambda_l)^{\frac{p}{2}} \int_0^{\pi/2} \sin^q t \cos^{r-p} t dt \geq (\lambda_l)^{\frac{p}{2}} \sin^q t_0 \int_{t_0}^{\pi/2} \cos^{r-p} t dt,$$

$$J_p(\pi/2) = (\lambda_k)^{\frac{p}{2}} \int_0^{\pi/2} \sin^{q-p} t \cos^r t dt \geq (\lambda_k)^{\frac{p}{2}} \cos^r t_1 \int_0^{t_1} \sin^{q-p} t dt.$$

If  $p \geq \max(q, r) + 1$ , then  $J_p(0) = J_p(\pi/2) = \infty$  and so constant solutions are excluded. The other cases we have to compare  $J_p(0)$  with  $J_p(\pi/2)$ . However, if we change the roles of  $f$  and  $g$  in the definition of the join, we may always assume that  $J_p(0) \leq J_p(\pi/2)$ .

Thus, it suffices to show that  $c_0 < J_p(0)$ . We assume  $r + 1 > p$  because  $J_p(0) < \infty$ . Following an argument of Ding [2], we verify this condition by considering the second variation  $d^2(J_p)_0$  of  $J_p$  at the constant critical point 0, which is given by

$$d^2(J_p)_0(a, a) = p(\lambda_l)^{\frac{p}{2}-1} \int_0^{\pi/2} \{\dot{a}^2 + W(t)a^2\}g(t) dt,$$

where  $a \in X$ ,  $W(t) = \lambda_k/\sin^2 t - \lambda_l/\cos^2 t$  and  $g(t) = \sin^q t \cos^{r-p+2} t$ .

Set

$$\tilde{X} := \left\{ a \in L_1^2([0, \pi/2], \mathbf{R}) : \|a\|^2 = \int_0^{\pi/2} (\dot{a}^2 + a^2)f(t) dt < \infty \right\}.$$

**Lemma 3.1.** *If there exists  $a \in \tilde{X}$  satisfying  $a \geq 0$  and*

$$I(a) := \int_0^{\pi/2} \{\dot{a}^2 + W(t)a^2\}g(t) dt < 0,$$

then  $c_0 < J_p(0)$ .

*Proof.* For any  $a \in \tilde{X}$ , define  $a^M$  by

$$a^M(t) := \begin{cases} a(t) & \text{if } a(t) < M, \\ M & \text{if } a(t) \geq M, \end{cases}$$

where  $M$  is positive constant. Then  $I(a^M) \rightarrow I(a)$  as  $M \rightarrow \infty$ . It follows that  $I(a^M) < 0$  for sufficiently large  $M$ . Furthermore, approximating  $a^M$  by smooth function  $a_M$ , we can take  $a_M \in X$  and  $I(a_M) < 0$ . Fixing such a large  $M$ , we have

$$\frac{\partial}{\partial s} J_p(sa_M) \Big|_{s=0} = 0$$

and

$$\frac{\partial^2}{\partial s^2} J_p(sa_M) \Big|_{s=0} = p(\lambda_l)^{\frac{p}{2}-1} I(a_M) < 0.$$

Therefore,  $J_p(sa_M) < J_p(0)$  holds for small  $s > 0$ . But  $a_M$  is non-negative and bounded, so  $sa_M \in X_0$  for small  $s > 0$ . It follows that  $c_0 < J_p(0)$ .

In order to find a function satisfying the assumption of Lemma 3.1, we put  $a(t) := \sin^\sigma t \cos^{-\tau} t$  with  $0 < \sigma$  and  $0 < \tau < (r - p + 1)/2$ . Then  $a \in \tilde{X}$  and  $a \geq 0$ .

**Lemma 3.2.**

$$I(a(t)) = \frac{K\{(r^2 + p^2 - 2pr - 4\tau + 4r - 4p - 4\lambda_l + 3)\sigma + c_2 + \frac{c_3}{\sigma}\}}{\left(2 + \frac{q-1}{\sigma}\right)(-2\tau + r - p + 1)},$$

where  $K = K(2\sigma + q, -2\tau + r - p + 2)$  in Main Theorem 1.1,  $c_2$  and  $c_3$  are constants which do not depend on  $\sigma$ .

*Proof.* By substitution, we have

$$\begin{aligned} I(a) = \int_0^{\pi/2} & (\sigma^2 \sin^{2\sigma+q-2} t \cos^{-2\tau+r-p+4} t \\ & + 2\sigma\tau \sin^{2\sigma+q} t \cos^{-2\tau+r-p+2} t \\ & + \tau^2 \sin^{2\sigma+q+2} t \cos^{-2\tau+r-p} t \\ & + \lambda_k \sin^{2\sigma+q-2} t \cos^{-2\tau+r-p+2} t \\ & - \lambda_l \sin^{2\sigma+q} t \cos^{-2\tau+r-p} t) dt. \end{aligned}$$

Put  $q' := 2\sigma + q$ ,  $r' := -2\tau + r - p + 2$ . By noting  $q' \geq 1$  and  $r' > 1$ , integration by parts leads to

$$\begin{aligned} K(q', r') &= \frac{q' - 1}{q' + r'} K(q' - 2, r') = \frac{q' - 1}{r' + 1} K(q' - 2, r' + 2) \\ &= \frac{r' - 1}{q' + 1} K(q' + 2, r' - 2) = \frac{r' - 1}{q' + r'} K(q', r' - 2). \end{aligned}$$

Then we have

$$I(a) = K \left( \sigma^2 \frac{r' + 1}{q' - 1} + 2\sigma\tau + \tau^2 \frac{q' + 1}{r' - 1} + \lambda_k \frac{q' + r'}{q' - 1} - \lambda_l \frac{q' + r'}{r' - 1} \right).$$

**Theorem 3.3.** *If the condition (1) or (2) of Main Theorem 1.1 is satisfied, then we have  $c_0 < J_p(0)$ .*

*Proof.* Case 1.  $(r - p + 1)^2 < 4\lambda_l$ ;

Set  $H := (r - p + 1)^2 - 4\lambda_l$ .

Then we have

$$r^2 + p^2 - 2pr - 4\tau + 4r - 4p - 4\lambda_l + 3 = H - 4 \left( \tau - \frac{r - p + 1}{2} \right).$$

So if we take  $\sigma$  sufficiently large and  $\tau$  as follows:

$$\max \left( 0, \frac{H}{4} + \frac{r - p + 1}{2} \right) < \tau < \frac{r - p + 1}{2},$$

then  $I(a) < 0$  because of

$$\frac{K}{\left( 2 + \frac{q - 1}{\sigma} \right) (-2\tau + r - p + 1)} > 0.$$

Case 2.  $(r - p + 1)^2 \geq 4\lambda_l$ ;

We put  $a_0(t) := \sin^\sigma t \cos^{-\tau} t$ , where

$$\begin{aligned} \sigma &= \frac{-(q - 1) + \sqrt{(q - 1)^2 + 4\lambda_k}}{2}, \\ \tau &= \frac{(r - p + 1) - \sqrt{(r - p + 1)^2 - 4\lambda_l}}{2}. \end{aligned}$$

Then  $a_0$  is a solution of the following equation:

$$-\frac{d}{dt} (g(t)\dot{a}) + aW(t)g(t) = \mu a g(t), \quad (3.1)$$



where  $\mu = (\tau - \sigma)^2 - (\tau - \sigma)(r + q - p + 2)$ .

Take  $a = a_0$  in (3.1), multiply the both sides of (3.1) by  $a_0$ , and integrate over  $[0, \pi/2]$ , we have

$$-\int_0^{\pi/2} \left\{ \frac{d}{dt}(g(t)a_0) \right\} a_0 dt + \int_0^{\pi/2} a_0^2 W(t)g(t) dt = \mu \int_0^{\pi/2} a_0^2 g(t) dt.$$

By applying the integral by parts to the first term, we obtain

$$I(a_0) = \mu \int_0^{\pi/2} a_0^2 g(t) dt.$$

Thus,  $I(a_0) < 0$  if and only if  $\mu < 0$ . But an easy computation leads to that  $\mu < 0$  if and only if

$$\sqrt{(r - p + 1)^2 - 4\lambda_l} + \sqrt{(q - 1)^2 + 4\lambda_k} < q + r - p.$$

**4. Monotonicity and asymptotic properties of solutions.** By changing variables,  $t = \arctan e^s$ , we rewrite the equation (2.3) as follows:

$$\begin{aligned} A''(s) + \left\{ \frac{(q - 1)e^{-s} - (r - 1)e^s}{e^s + e^{-s}} + \frac{p - 2}{2} \left( \frac{k'(s)}{k(s)} + \tanh s \right) \right\} A'(s) \\ = \frac{\lambda_k e^{-s} - \lambda_l e^s}{e^s + e^{-s}} \sin A(s) \cos A(s), \end{aligned} \tag{4.1}$$

where  $A(s) = a(\arctan e^s)$  and  $k(s) = A'^2(s)(e^s + e^{-s}) + \lambda_k e^{-s} \sin^2 A(s) + \lambda_l e^s \cos^2 A(s)$ .

We note that if  $a \in X_0$  then  $0 \leq A \leq \pi/2$  on  $\mathbf{R}$ . We analyse properties of this equation's solutions by following an argument of Eells and Ratto [3].

**Theorem 4.1.** *Let  $A$  be a solution of (4.1). Suppose one of the following assumptions is satisfied:*

- (1)  $p < \max(q, r) + 1$  and  $A$  is non-constant,
- (2)  $p \geq \max(q, r) + 1$ ,

then  $A' > 0$  on  $\mathbf{R}$ .

**Lemma 4.2.** *If  $A$  is a non-constant solution of (4.1), then  $0 < A < \pi/2$  on  $\mathbf{R}$ .*

*Proof.* If  $A(s) = 0$  for some  $s \in \mathbf{R}$ , then  $A'(s) \neq 0$ ; for otherwise  $A \equiv 0$  by (4.1). Thus  $A$  assumes negative values, and consequently  $a \notin X_0$ . Similarly,  $A$  does not assume the value  $\pi/2$ .

Next we set

$$G(s) := \frac{\lambda_k e^{-s} - \lambda_l e^s}{e^s + e^{-s}}$$

and let  $\bar{s}$  be the unique solution of  $G(s) = 0$ .

**Lemma 4.3.** *The zeros of  $A'$  are isolated.*

*Proof.* If  $A'(s_0) = 0$  and  $s_0 \neq \bar{s}$ , then (4.1) shows that  $A''(s_0) \neq 0$ , which implies that  $s_0$  is an isolated zero of  $A'$ . If  $s_0 = \bar{s}$ , we have  $A''(\bar{s}) = 0$ . Then differentiate both sides of (4.1) leads to

$$A'''(\bar{s}) = -2 \frac{\lambda_k + \lambda_l}{(e^{\bar{s}} + e^{-\bar{s}})^2} \sin A(\bar{s}) \cos A(\bar{s}) < 0,$$

which implies that  $s_0$  is again an isolated zero of  $A'$ .

**Lemma 4.4.** *Let  $A$  be a non-constant solution of (4.1).*

- (1) *If  $p \leq q + 1$ , then  $A' > 0$  on  $(-\infty, \bar{s}]$ .*
- (2) *If  $p \leq r + 1$ , then  $A' > 0$  on  $[\bar{s}, +\infty)$ .*

*Proof.* We only prove (1), since (2) can be proved similarly. First, we consider the following linear equation on the interval  $I$ , where  $A'$  does not vanish:

$$y'(s) + P_A(s)y(s) = Q_A(s) \tag{4.2}$$

where

$$P_A(s) = 2 \left\{ \frac{A''}{A'} + \frac{(q-1)e^{-s} - (r-1)e^s}{e^s + e^{-s}} + \frac{p-2}{2} \left( \frac{k'(s)}{k(s)} + \tanh s \right) \right\},$$

$$Q_A(s) = 2 \frac{\lambda_k e^{-s} - \lambda_l e^s}{A'(e^s + e^{-s})} \sin A \cos A.$$

The solution of (4.2) with  $y(\bar{s}) = 1$  for  $\bar{s} \in I$  can be written as

$$y(s) = \frac{N(s)}{D(s)}, \tag{4.3}$$

where

$$D(s) = \exp\left(\int_{\bar{s}}^s P_A(u) du\right)$$

and

$$N(s) = \int_{\bar{s}}^s Q_A(\theta) \exp\left(\int_{\bar{s}}^{\theta} P_A(\mu) d\mu\right) d\theta + 1.$$

By direct computation, we have

$$D(s) = c \cdot A'^2(s) k^{p-2}(s) e^{(2q-p)s} (\epsilon^{2s} + 1)^{p-q-r},$$

where  $c$  is a positive constant. Since  $k(s) > 0$ , we have  $D(s) \geq 0$  and  $D(s) = 0$  if and only if  $A'(s) = 0$ .

On the other hand, since  $A$  satisfies (4.1),  $y(s) = 1$  is a solution of (4.2). By uniqueness, (4.3) implies

$$N(s) \equiv D(s) \quad \text{on } I. \tag{4.4}$$

Suppose that  $A'(s_0) = 0$  and  $s_0 \leq \bar{s}$ . Put

$$\Lambda := \{s \in [-\infty, s_0) : A'(s) = 0\}.$$

Here we note that  $p \leq q + 1$  implies  $\lim_{s \rightarrow -\infty} D(s) := D(-\infty) = 0$ . Thus  $A'(-\infty) = 0$ , so  $\Lambda \neq \emptyset$ .

Next let

$$s_1 := \sup \Lambda. \tag{4.5}$$

Since  $s_1 < s_0$  by Lemma 4.3, we get  $N'(\hat{s}) = 0$  for some  $\hat{s} \in (s_1, s_0)$  because  $N(s_1) = N(s_0) = 0$  and  $N(s) = D(s) > 0$  for all  $s \in (s_1, s_0)$ . By noting that  $N'(s) = 0$  holds for some  $s \in (s_1, s_0)$  if and only if  $A'(s) = 0$ , we have  $A'(\hat{s}) = 0$  and  $\hat{s} \in (s_1, s_0)$ , which contradicts (4.5).

Finally we show that  $A' > 0$  on  $(-\infty, \bar{s}]$ . Suppose  $A' < 0$  on  $(-\infty, \bar{s}]$ , then

$$N'(\bar{s}) = Q_A(\bar{s})D(\bar{s}) = 0$$

and

$$\begin{aligned} N''(\bar{s}) &= -4\lambda_l e^{\bar{s}} k^{p-2}(\bar{s}) \frac{e^{(2q-p)\bar{s}} (\epsilon^{2\bar{s}} + 1)^{p-q-r}}{e^{\bar{s}} + e^{-\bar{s}}} A'(\bar{s}) \sin A(\bar{s}) \cos A(\bar{s}) \\ &> 0 \end{aligned}$$

implies that  $\bar{s}$  is a minimal of  $N$ . By combining this with  $N(-\infty) = 0$ , we have  $N'(s_2) = 0$  and  $s_2 \in (-\infty, \bar{s}]$ , which is a contradiction.

**Lemma 4.5.** *Let  $A$  be a non-constant solution of (4.1).*

- (1) *If  $p \geq q + 1$ , then  $A' > 0$  on  $(-\infty, \bar{s})$ .*
- (2) *If  $p \geq r + 1$ , then  $A' > 0$  on  $(\bar{s}, +\infty)$ .*

*Proof.* We also give a proof of (1) because (2) may be proved in the same manner. To begin with, we write (4.1) in the form

$$A'' + E(s)A' = G(s) \sin A \cos A,$$

where

$$E(s) = \frac{(q-1)e^{-s} - (r-1)e^s}{e^s + e^{-s}} + \frac{p-2}{2} \left( \frac{k'(s)}{k(s)} + \tanh s \right).$$

This equation may be also written as

$$\frac{d}{ds} \left\{ A' \exp \left( \int_{\bar{s}}^s E(\mu) d\mu \right) \right\} = \exp \left( \int_{\bar{s}}^s E(\mu) d\mu \right) G(s) \sin A \cos A \quad (4.6)$$

for some constant  $\bar{s}$ .

Suppose  $A'(s_0) \leq 0$  for  $s_0 < \bar{s}$ , integrating (4.6) over  $[s_1, s_0]$ . Then we have  $A'(s_1) < 0$  for all  $s_1 < s_0$  because the right-hand side is positive due to Lemma 4.2. Thus  $A' < 0$  on  $(-\infty, s_0)$ , that is,  $A(s) > A(s_0) > 0$  for all  $s \in (-\infty, s_0)$ . Now  $a(t) = A(\log \tan t)$  is a solution of (2.3) and we have  $a(t) > a(t_0) =: c > 0$  for all  $t \in (0, t_0)$ . Then we have

$$J_p(a) \geq (\lambda_k)^{\frac{p}{2}} \sin^p c \cos^r t_0 \int_0^{t_0} \sin^{q-p} t dt.$$

From the assumption  $p \geq q + 1$ , it follows that  $J_p(a) = \infty$ , which is a contradiction. Therefore  $A' > 0$  on  $(-\infty, \bar{s})$ .

**Lemma 4.6.** *Let  $A$  be a non-constant solution of (4.1). If  $p < \max(q, r) + 1$ , then  $A' > 0$  on  $\mathbf{R}$ , which implies the case (1) of Theorem 4.1.*

*Proof.* Lemma 4.4 tells us the case  $p \leq \min(q, r) + 1$ . If the case  $r + 1 \leq p < q + 1$  happens, we have to use (1) of Lemma 4.4 and (2) of Lemma 4.5. Similarly, the case  $q + 1 \leq p < r + 1$  is in need of (2) of Lemma 4.4 and (1) of Lemma 4.5.

**Lemma 4.7.** *If  $p \geq \max(q, r) + 1$ , then  $A' > 0$  on  $\mathbf{R}$ , which implies the case (2) of Theorem 4.1.*

*Proof.* Note that the assumption implies  $A$  is not constant, because  $J_p(0) = J_p(\pi/2) = \infty$ . We have  $0 < A < \pi/2$  on  $\mathbf{R}$  by Lemma 4.2 and Lemma 4.5 tells us that  $A' > 0$  on  $\mathbf{R} \setminus \{\bar{s}\}$ . Here,  $A'(\bar{s}) = 0$  is not possible.

For otherwise  $A''(\bar{s}) = 0$  by (4.1) and  $A'$  assumes minimum at  $\bar{s}$ . However this contradicts  $A'''(\bar{s}) < 0$ .

The proof of Theorem 4.1 is completed and we obtain

**Proposition 4.8.** *Suppose the same assumption as Theorem 4.1. A solution  $A$  of (4.1) has the following properties:  $0 < A < \pi/2$  on  $\mathbf{R}$  and*

$$\lim_{s \rightarrow -\infty} A(s) = 0, \quad \lim_{s \rightarrow +\infty} A(s) = \frac{\pi}{2}.$$

*Proof.* Suppose  $\lim_{s \rightarrow +\infty} A(s) < \pi/2$ . From the equation (4.1), we observe that there exist large  $s_0 \in \mathbf{R}$  such that  $A''(s) < \text{constant} < 0$  for all  $s > s_0$ , which contradicts  $A' > 0$  on  $\mathbf{R}$ . Similarly,  $\lim_{s \rightarrow -\infty} A(s) = 0$ .

**5. Regularity of solutions.** Let  $a(t)$  be a solution of (2.3). For any  $t_0, t_1 \in (0, \pi/2)$ , we have

$$\begin{aligned} & |a(t_1) - a(t_0)| \\ & \leq \int_{t_0}^{t_1} |\dot{a}(t)| dt \\ & \leq \left( \int_{t_0}^{t_1} |\dot{a}(t)|^p \sin^q t \cos^r t dt \right)^{\frac{1}{p}} \left( \int_{t_0}^{t_1} \sin^{-\frac{q}{p-1}} t \cos^{-\frac{r}{p-1}} t dt \right)^{\frac{p-1}{p}} \\ & \leq \|a\| \left( \int_{t_0}^{t_1} \sin^{-\frac{q}{p-1}} t \cos^{-\frac{r}{p-1}} t dt \right)^{\frac{p-1}{p}}. \end{aligned}$$

This shows  $a(t)$  is continuous in  $(0, \pi/2)$ . Similarly, we observe that  $\dot{a}(t)$  is continuous in  $(0, \pi/2)$  due to (2.3). Therefore all solutions of (2.3) are smooth in  $(0, \pi/2)$ .

**Lemma 5.1.** *The solution  $a(t)$  satisfies a Hölder condition at  $t = 0$  (resp.  $t = \pi/2$ ) with exponent  $q/(p-1)$  (resp.  $r/(p-1)$ ).*

*Proof.* We consider again  $A(s)$  instead of  $a(t)$  by change of variables as before. The equality (4.4) can be rewritten with  $\bar{s} = 0$  as

$$\begin{aligned} & 2 \int_0^s \left\{ (\lambda_k e^{-\theta} - \lambda_l e^{\theta}) k^{p-2}(\theta) \frac{e^{(2q-p)\theta} (e^{2\theta} + 1)^{p-q-r}}{e^{\theta} + e^{-\theta}} \right. \\ & \quad \left. \cdot A'(\theta) \sin A(\theta) \cos A(\theta) \right\} d\theta + 1 \\ & = \frac{A'^2(s) k^{p-2}(s) e^{(2q-p)s} (e^{2s} + 1)^{p-q-r}}{2^{p-q-r} A'^2(0) k^{p-2}(0)}. \end{aligned}$$

The left-hand side of this equation is a decreasing function of  $s$ , hence remain bounded for large  $s$ , we have for large  $s$

$$A'^2(s)k^{p-2}(s)e^{(2q-p)s}(e^{2s} + 1)^{p-q-r} \leq C.$$

By recalling the definition of  $k(s)$ , this implies  $A'(s) \leq Ce^{(r/(p-1)-1)s}$ . Namely we get for  $t$  close to  $\pi/2$ ,  $a'(t) \leq C(\pi/2 - t)^{-r/(p-1)}$ . Similarly, we have for  $t$  close to 0,  $a'(t) \leq Ct^{-q/(p-1)}$ . Here,  $C$  are positive constants.

By using the smoothness of the solution  $a(t)$ , the map  $\phi$  is Hölder continuous with the smaller exponent of  $q/(p-1)$  and  $r/(p-1)$ . Thus the regularity theory of  $p$ -harmonic maps (cf. [4]) tells us that  $\phi$  has Hölder continuous gradient. Eventually, the map  $\phi$  actually satisfies an elliptic system, so  $\phi$  is a smooth  $p$ -harmonic map (cf. [10]).

**6. Proof of Corollary 1.2.** Let  $d_k: S^1 \rightarrow S^1$  be the complex polynomial  $z \mapsto z^k$  of degree  $|k|$ , and let  $id: S^r \rightarrow S^r$  be the identity map. Then the join map  $d_k * id$  is  $r$ -th suspension of  $d_k$ . If  $p$  satisfies (1.1), then the assumption (1) or (3) of Main Theorem 1.1 can be fulfilled for all  $k \in \mathbf{Z}$  with  $q = 1$ ,  $r = \lambda_l = n - 2$  and  $\lambda_k = k^2$ .

**Acknowledgement.** The results of this paper appear in the author's master thesis [5]. He would like to express sincere gratitude to Professors T. Sakai, A. Katsuda and H. Takeuchi for their helpful advice and continuous encouragements. He also thanks Professor H. Urakawa to be interested in his works.

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*(Received September 19, 1994)*

*(Revised May 19, 1995)*

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