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ON A SUBRING OF AN INTEGRAL DOMAIN OBTAINED BY INTERSECTING A FIELD

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Introduction. Let S be an integral domain and let K be a subfield of the quotient field of S. We are interested in the ring-extension $S/S \cap K$ or the subring $S \cap K$ itself. We call $S \cap K$ a subring with reduced quotient field. It is known that the subring $S \cap K$ inherits some properties from S; for example: if S is integrally closed, so is $S \cap K$; if S is local (not necessarily Noetherian), so is $S \cap K$; if S is a DVR, then $S \cap K$ is either a DVR or a field; if S is a Krull domain, so is $S \cap K$ (see [6],[8]). In these examples, theory of valuations plays an important role.

Our objective of this paper is to show the ring $S \cap K$ maintains several properties of S under certain conditions.

In the section 1, we study the property of Noetherianness. We show mainly the following result:

(1) Let S is a Noetherian normal domain of characteristic zero with quotient field L and let K be a subfield of L such that S is integral over $S \cap K$. Then $S \cap K$ is a Noetherian domain.

In the section 2, we show some basic properties of $S \cap K$ for later use. We consider some conditions for a subring R of S to be of type $S \cap K$ for some subfield K of the quotient field of S. For instance,

(2) The extension $S/S \cap K$ is characterized by behavior of divisorial ideals of $S \cap K$ (Theorem 2.4).

In the section 3, we treat (2,3)-closedness, root-closedness and quasi-normality of a subring $S \cap K$.

In the section 4, we show: Let S be a Noetherian almost factorial domain of characteristic zero. If S is integral over $S \cap K$, then $S \cap K$ is a Noetherian almost factorial domain. (Theorem 4.2).

In the section 5, we have the following:

(3) Let (S, M) be a local factorial domain. If S is LCM-stable over $S \cap K$, then $S \cap K$ is factorial (Theorem 5.3).

When S is not local, the faithful flatness of S over $S \cap K$ does not always ensure the similar result in (3) (Remark 2).

In the section 6, we study the factoriality of $S \cap F$ for a non-local domain S. The obstruction of descent of factoriality is anyway that a certain principal ideal of S is not necessarily generated by elements in $S \cap K$.

In the section 7, we treat Dedekind domains.

In this paper, we mean by a ring a commutative ring with identity and by an *integral domain* (or a *domain*) a ring which has no non-trivial zero-divisors, and for an integral domain S, $K(S)$ denotes the quotient field of S unless otherwise specified. Our unexplained technical terms are standard and are seen in [10] and [13].

1. A subring of a Noetherian domain. An integral domain is called to be *integrally closed* (or *normal*) if it is integrally closed in its quotient field. This section treats the following problem, which means a descent of Noetherianness of ring-extensions:

Problem. *Let S be a Noetherian (normal) domain with quotient field L and let K be a subfield of L. Is the ring $S \cap K$ Noetherian if S is integral over $S \cap K$?*

This problem is a certain converse to the well known result:

If R is a Noetherian normal domain with quotient field K and L a finite separable extension of K, then the integral closure S of R in L is Noetherian (See [10, (31.B)]).

Concerning the descent problem as above, we have known the following results among other things: Let $S \supseteq R$ be a ring-extension with a Noetherian domain S.

(i) (*Faithfully flat descent*) If S is faithfully flat over R, then R is Noetherian.

(ii) (*Eakin-Nagata*) If S is finitely generated as an R-module, then R is Noetherian.

The result (i) is well-known (See [10]) and the result (ii) is seen in [5] and [10], a new proof of which has been given by M. Nagata [14] recently.

Our objective of this section is to settle the problem in the case that S is integral over $S \cap K$ with $\mathrm{char}(K) = 0$ and the case that L is not necessarily algebraic over $S \cap K$ under certain conditions.

Let A be an integral domain with quotient field L. An element α in L is called *almost integral* over A if there exists a non-zero element c in A such that $c\alpha^i \in A$ for all $i \in \boldsymbol{N}$. It is easy to see that the set $A^\natural$ of all almost integral elements over A forms a ring between A and L, which is called the *complete integral closure* of A. We say that A is *completely integrally closed* if $A^\sharp = A$. When A is Noetherian, A being completely integrally closed is equivalent to A being integrally closed. It is known that a Krull domain is completely integrally closed, and if A is a Krull domain $A \cap K$ is also a Krull domain for a field K. Note that a Noetherian normal domain is a Krull domain (See [6] for details).

We require the following lemma.

Lemma 1.1. *Let S be an integral domain and let K be a field. Assume that S is algebraic over $S \cap K$. Let $(\ )^\sharp$ denote the complete integral closure of $(\ )$ in its quotient field. Then $S^\sharp \cap K = (S \cap K)^\sharp$.*

Proof. Since $S \cap K \subseteq S^\sharp \cap K$, we have $(S \cap K)^\sharp \subseteq S^\sharp \cap K$. Take $\beta \in S^\sharp \cap K$. There exists a non-zero element $s \in S$ such that $s\beta^i \in S$ for all $i \in N$ and hence $sS[\beta] \subseteq S$. Since $\beta \in S^\sharp \cap K$, the quotient fields of $S[\beta]$ and S coincide. Since s is algebraic over $S \cap K$, there exists an algebraic dependence:

$$a_0 s^n + a_1 s^{n-1} + \cdots + a_n = 0,$$

where $a_i \in S \cap K$ with $a_n \neq 0$. Then $a_n S[\beta] \subseteq S$. Hence $a_n \beta^i \in S \cap K$ for all $i \in N$. Thus β is almost integral over $S \cap K$, that is, $\beta \in (S \cap K)^\sharp$. Therefore $S^\sharp \cap K = (S \cap K)^\sharp$.

Corollary 1.1.1. *Let S be a Krull domain and K be a field contained in $K(S)$. Let L be a finite Galois extension of K containing S and let S' be the integral closure of S in L. Then $S' \cap K = S \cap K$.*

Proof. Put $R = S \cap K$. Take $\beta \in S' \cap K$. Then β is integral over R. So $R[\beta]$ is a finite R-module (cf. [13, (10.1)]). Write $R[\beta] = \sum_{i=1}^{s} d_i R$ ($d_i = b_i/c_i$ with $b_i, c_i \in R$), where we note that $R[\beta] \subseteq K$. Put $c = \prod_{i=1}^{s} c_i$. Then $c \in R :_R R[\beta]$, and hence $c\beta^j \in R$ for all $j \in N$. Thus $\beta \in R^\sharp = (S \cap K)^\sharp$ and so $S' \cap K \subseteq (S \cap K)^\sharp \cap K$. Since S' is a Krull domain, $R = S \cap K \subseteq (S \cap K)^\sharp \cap K = S^\sharp \cap K = S \cap K = R$ by Lemma 1.1, that is, $S' \cap K = S \cap K = R$.

We prove the following theorem by using, so-called the Galois-descent.

Theorem 1.2. *Let S be a normal domain of characteristic zero with quotient field L and let K be a subfield of L such that S is integral over $S \cap K$. If S is Noetherian, then so is $S \cap K$.*

Proof. Let $R = S \cap K$. Let I be an ideal of R. Then $IS = (a_1, \ldots, a_t)S$ for some $a_i \in I$. Let J be the ideal of R generated by $a_1, \ldots, a_t$. Take $b \in I$. Then $b = \sum_{i=1}^{t} a_i\alpha_i$ ($\alpha_i \in S$). Put $S' = S \cap K(\alpha_1, \ldots, \alpha_t)$. Then $R \subseteq S' \subseteq S$ and S' is integrally closed in $K(\alpha_1, \ldots, \alpha_t)$. Note that $b \in JS'$. Noting that $\mathrm{char}(K) = 0$, there exists a field L' such that

(a) $L' \supseteq K(\alpha_1, \ldots, \alpha_t) \supseteq K$,

(b) L' is a finite Galois extension of K.

Let G denote the Galois group $G(L'/K)$ with $n = \sharp G$. Let S'' denote the integral closure of R in L'. Then S'' is a Galois extension of R. Note that $S''^g = S''$ for each $g \in G$. Since S is integral over R, we have $S' \subseteq S \cap L' \subseteq S''$ and $S'^g \subseteq S''^g = S''$ for each $g \in G$. Hence $\alpha_i^g \in S''$ for any $g \in G$. By [6, (1.3)], S'' is a Krull domain because L' is a finite extension of K. We see that $nb = \sum_{g\in G} b^g = \sum_{g\in G}\sum_{i=1}^{t}(a_i\alpha_i)^g = \sum_{i=1}^{t}\sum_{g\in G} a_i^g\alpha_i^g = \sum_{i=1}^{t} a_i(\sum_{g\in G}\alpha_i^g)$. Since $\sum_{g\in G}\alpha_i^g$ is invariant under every element in G. Hence $\sum_{g\in G}\alpha_i^g \in K \cap S'' = K \cap S$ by Corollaty 1.1.1. Hence $nb \in \sum_{i=1}^{t} a_iR$. Since $\mathrm{char}(K) = 0$, we have $b \in J$. The implication $I \supseteq J$ is trivial, and hence $I = J = (a_1, \ldots, a_t)R$, a finitely generated ideal of R. Therefore $R = S \cap K$ is Noetherian.

Corollary 1.2.1. *Let R be an integrally closed domain with quotient field K of characteristic zero and let L be a field extension of K. If the integral closure of R in L is a Noetherian ring, then R is Noetherian.*

Proof. This follows from Theorem 1.2.

Let S be an integral domain with quotient field L. We say that S is *N-1* if the integral closure of S in its quotient field L is a finite S-module; and that S is *N-2* if, for any finite extension T of L, the integral closure of S in T is a finite S-module. It is known that N-1 is equivalent to N-2 when S is a Noetherian integral domain of characteristic zero ([10, p.232]). A ring A is called a *Nagata ring* if it is Noetherian and if A/P is N-2 for every $P \in \mathrm{Spec}(A)$.

Corollary 1.2.2. *Let R be an* N-1 *domain with quotient field K of characteristic zero and let L be an algebraic field extension of K. Let S denote the integral closure of R in L. If S is a Noetherian domain, then so is R.*

Proof. Since S is a Noetherian normal domain, $S \cap K$ is Noetherian by Theorem 1.2. Since the quotient field of S is algebraic over K, we have

$S \cap K = S^{\natural} \cap K = (S \cap K)^{\natural}$ by Lemma 1.1. Hence $S \cap K$ is the integral closure of R in K because $S \cap K$ is Noetherian. Since R is a N-1 domain, $S \cap K$ is a finite R-module. So by Eakin-Nagata's Theorem, we conclude that R is Noetherian.

A ring A is called *locally Noetherian* if A_P is a Noetherian ring for each prime ideal P of A.

Remark 1. (1) The following is known in [7, (12.7)]: Let R be an integral closed integral domain with quotient field K and let S be an integral domain containing R such that S is integral over R. Then for each prime ideal M of S, $S_M \cap K = R_{M \cap R}$.

(2) Let S be an integral domain and let K be a subfield of the quotient field $K(S)$ of S such that $K(S)$ is finite algebraic over K. Assume that S is integral over $S \cap K$ and that S is locally Noetherian. Then for each prime ideal p of $S \cap K$, S_p is Noetherian, where S_p denotes $(S \cap K \setminus p)^{-1} S$. Indeed, there are only finitely many prime ideals $P_1, \ldots, P_n$ of S lying over p by [10, p.296]. Let $T = S \setminus \bigcup_{i=1}^{n} P_i$, a multiplicatively closed subset of S. Then $S_p = T^{-1}S$ by [7, (11.10)]. Let I be an ideal of S_p. Then for each $1 \leq i \leq n$, $I_{P_i} = (a_{i1}, \ldots, a_{ir_i}) S_{P_i}$ for some $a_{ij} \in I$. Put $J = \sum a_{ij} S_p$. Then $I_{P_i} = J_{P_i}$ for each $1 \leq i \leq n$. Thus $I = J$, which means that S_p is Noetherian.

Corollary 1.2.3. *Let S be a locally Noetherian, normal domain of characteristic zero and let K be a subfield of the quotient field $K(S)$ of S such that $K(S)$ is finite algebraic over K. Assume that S is integral over $S \cap K$. Then $S \cap K$ is locally Noetherian.*

Proof. Note first that for each prime ideal P of $S \cap K$, there exists a prime ideal M of S such that $M \cap K = P$ because S is integral over $S \cap K$. Hence Remark 1(2) and Theorem 1.2 yield our conclusion.

Example. Let k be a field ($\operatorname{char} k \neq 1$) and let t_i ($i \in N$) and X, Y be indeterminates. Put $S = k(t_1, t_2, \ldots)[X, Y]$, which is a Noetherian domain, and for $i \in N$, put $d_i = t_{2i}X + t_{2i-1}Y$. Let $K = k(d_1, d_2, \ldots)$. Then $S \cap K = k[d_1, d_2, \ldots] := R$, which is not Noetherian. Note that S/R is not integral.

Proposition 1.3 (cf. [8, p.73, Ex.4]). *Let (S, M) be a local domain and K a subfield of the quotient field $K(S)$ of S. Then $S \cap K$ is a local domain with the maximal ideal $M \cap K$.*

Proof. Suppose that there exists a maximal ideal m which properly contains $M \cap K$. Then $mS = S$ and we have $\sum_{i=1}^{n} a_i\beta_i = 1$ in S with $a_i \in m$ and $\beta_i \in S$. Since S is a local domain with maximal ideal M, there exists i, say $i = 1$ such that a_1 is a unit in S. Hence $a_1\alpha = 1$ for some $\alpha \in S$. So we have $\alpha = 1/a_1 \in S \cap K$, which means that a_1 is a unit in $S \cap K$. This is absurd. Therefore $S \cap K$ is a local domain with the maximal ideal $M \cap K$.

2. Basic properties of a subring with reduced quotient field. In this section, we study the conditions for a subring to be a subring with reduced quotient field and show some preliminary results which will be used later. We start with the following lemma.

Lemma 2.1. *Let S be an integral domain, let K be a subfield of the quotient field of S and let R be a subring of S which is contained in K. Then the following statements are equivalent:*

(i) *$aS \cap K = aR$ for any $a \in K$;*

(ii) *$R = S \cap K$.*

If furthermore K is the quotient field of R, (i) *is equivalent to the following:*

(iii) *$aS \cap R = aR$ for any $a \in R$.*

Proof. (ii) $\Longrightarrow$ (i). Take $x \in aS \cap K$. Then $x = as$ for some $s \in S$ and hence $x/a = s \in S \cap K = R$. Thus $x \in aR$.
The implications (i) $\Longrightarrow$ (ii) is trivial.
Assume that K is the quotient field of R. The implications (i) $\Longrightarrow$ (iii) is trivial.

(ii) $\Longrightarrow$ (iii). Take $s \in S \cap K$. Since K is the quotient field of R, $s = b/a$ for some $a, b \in R$. Hence $b = as \in R \cap aS = aR$. Thus $s \in R$.

Corollary 2.1.1. *Let S be an integral domain and let K be a subfield of the quotient field of S. Then for any $a, b \in R := S \cap K$, the following hold:*

(a) *$aR = bR$ if and only if $aS = bS$,*

(b) *$\sqrt{aR} = \sqrt{bR}$ if and only if $\sqrt{aS} = \sqrt{bS}$.*

Moreover for any $\alpha, \beta \in K$,

(a′) *$\alpha R = \beta R$ if and only if $\alpha S = \beta S$.*

Proof. (a) The implication $aR = bR \Longrightarrow aS = bS$ is obvious. Conversely, $aR = aS \cap K = bS \cap K = bR$ by Lemma 2.1 (i) $\Longleftrightarrow$ (ii).

(b) Assume that $\sqrt{aS} = \sqrt{bS}$. Take $x \in \sqrt{aR}$. Then $x^n \in aR \subseteq aS \subseteq \sqrt{bS}$ for some positive integer n. Hence $x^m \in bS \cap K = bR$ for some positive integer m by (a). Thus $x \in \sqrt{bR}$. By symmetry, we have $\sqrt{aR} = \sqrt{bR}$. Conversely, assume that $\sqrt{aR} = \sqrt{bR}$. Then $\sqrt{\sqrt{aR}S} = \sqrt{\sqrt{bR}S}$ and hence $\sqrt{aS} = \sqrt{bS}$.

(a′) There exist $c, d \in R$ such that $c\alpha, d\beta \in R$. By (a), we have $cd\alpha R = cd\beta R \Longleftrightarrow cd\alpha S = cd\beta S$. Hence $\alpha R = \beta R \Longleftrightarrow \alpha S = \beta S$.

Corollary 2.1.2. *Let S, K and R be the same as in the above corollary* 2.1.1. *If S satisfies the ascending chain condition for principal ideals, then so does R.*

Proof. Let $a_1R \subseteq a_2R \subseteq \ldots$ be an ascending chain of principal ideals of R. Then we have the ascending chain $a_1S \subseteq a_2S \subseteq \ldots$ of principal ideals of S. Since S satisfies the ascending chain condition for principal ideals, there exists an integer r such that for any $n > r$, $a_rS = a_nS$. Thus by Corollary 2.1.1, we have $a_rR = a_nR$ for any $n > r$, which means that R has the ascending chain condition for principal ideals.

Proposition 2.2. *Let S be an integral domain, let K be a subfield of the quotient field of S and let R be its subring $S \cap K$. Then $(aS{:}_S bS) \cap K = aR{:}_R bR$ for any $a, b \in R$. In particular, if $a, b \in R$ is an S-sequence, then a, b is an R-sequence.*

Proof. The implication $aR{:}_R bR \subseteq (aS{:}_S bS) \cap K$ is obvious and it is clear that $(aS{:}_S bS) \cap K \subseteq R$. Take $x \in (aS{:}_S bS) \cap K$. Then $xb \in aS \cap K = aR$ by Lemma 2.1 (i) $\Longleftrightarrow$ (ii). Hence $x \in aR{:}_R bR$. Next if $aS{:}_S bS = aS$, then $aR{:}_R bR = aR$ by the above argument, which means that if $a, b \in R$ is an S-sequence, then a, b is an R-sequence.

Let S be an integral domain with quotient field L. We say that J is a *fractional* ideal of S if J is an S-submodule of L such that $sJ \subseteq S$ for some non-zero element $s \in S$. Let J be a fractional ideal of S. We denote by J^* a fractional ideal $S{:}_L J := \{x \in L \mid xJ \subseteq S\}$. We also write $S{:}J$ for $S{:}_L J$ if no confusion takes place. We say that a fractional ideal J of S is *divisorial* if $J^{**} := S{:}_L(S{:}_L J) = J$.

Lemma 2.3. *Let S be an integral domain with quotient field $K(S)$ and let I be a divisorial integral ideal of S. Then $I = \bigcap_i (b_iS{:}_S a_iS)$ for some $a_i, b_i \in S$.*

Proof. Let $y = z/x$ be an element in $K(S)$ with $x, z \in S$. Then $yS \cap S = zS{:}_S xS$. Indeed, if $\alpha \in zS{:}_S xS$, then $\alpha x \in zS$ and hence $\alpha \in (z/x)S \cap S = yS \cap S$. Conversely, if $\alpha \in yS \cap S$, then $\alpha = ys = (z/x)s$ for some $s \in S$. So $x\alpha = zs \in zS$. Hence $\alpha \in zS{:}_S xS$. Since I is a divisorial integral ideal of S, I is an intersection of principal fractional ideals, that is, $I = \bigcap yS \cap S$, where $I \subseteq yS$, $y \in K(S)$ (See [6, p.12] for details). By the above argument, I is written as $\bigcap_i (a_i S{:}_S b_i S)$ for some $a_i, b_i \in S$.

Theorem 2.4. *Let S be an integral domain and let R be its subring with quotient field K. Then the following statements are equivalent:*

(i) $R = S \cap K$;

(ii) $aS \cap R = aR$ *for each* $a \in R$;

(ii′) $aS \cap K = aR$ *for each* $a \in K$;

(iii) $IS \cap R = I$ *for each divisorial integral ideal* I *of* R;

(iii′) $IS \cap K = I$ *for each divisorial fractional ideal* I *of* R;

(iv) $(IS)^{**} \cap R = I$ *for each divisorial integral ideal* I *of* R;

(iv′) $(IS)^{**} \cap K = I$ *for each divisorial fractional ideal* I *of* R.

Proof. (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (ii′) have been shown in Lemma 2.1.
Let J be a fractional ideal of R. Then there exists a non-zero element d in R such that $dJ \subseteq R$. It is easy to see that if $(dJS) \cap K = dJ$ holds, then $JS \cap K = J$ holds. Hence in (iii′) and (iv′), we can assume that I is an integral ideal, i.e., $I \subseteq R$.

(iv) $\Longrightarrow$ (iii) (resp. (iv′) $\Longrightarrow$ (iii′)) follows from the implications: $I \subseteq IS \cap R \subseteq (IS)^{**} \cap R = I$ (resp. $I \subseteq IS \cap K \subseteq (IS)^{**} \cap K = I$).

(iv) $\Longrightarrow$ (ii) and (iv′) $\Longrightarrow$ (ii′) are trivial because a principal ideal is divisorial.
We must show the implication (i) $\Longrightarrow$ (iv) (resp. (i) $\Longrightarrow$ (iv′)). The ideal I is written as $\bigcap_i (a_i R{:}_R b_i R)$ for some $a_i, b_i \in R$ by Lemma 2.3. Hence we have $IS \subseteq \bigcap_i ((a_i R{:}_R b_i R)S) \subseteq \bigcap_i (a_i S{:}_S b_i S)$. Thus $IS \subseteq (IS)^{**} \subseteq \bigcap_i (a_i S{:}_S b_i S)$. So we have $I \subseteq IS \cap R \subseteq (IS)^{**} \cap R \subseteq \bigcap_i (a_i S{:}_S b_i S) \cap R = \bigcap_i (a_i R{:}_R b_i R) = I$ (resp. $I \subseteq IS \cap K \subseteq (IS)^{**} \cap K \subseteq \bigcap_i (a_i S{:}_S b_i S) \cap K = \bigcap_i (a_i R{:}_R b_i R) = I$) by Proposition 2.2, which means that $(IS)^{**} \cap R = I$. (resp. $(IS)^{**} \cap K = I$).

Corollary 2.4.1. *Let S, K and R be the same as in Theorem 2.4 and assume that $R = S \cap K$. Let I and J be divisorial fractional ideal of R. Then $I = J$ if and only if $(IS)^{**} = (JS)^{**}$.*

Proof. The implication $I = J \Longrightarrow (IS)^{**} = (JS)^{**}$ is obvious. Let I, J be divisorial fractional ideals with $(IS)^{**} = (JS)^{**}$. Then there exist non-zero elements $a, b \in R$ such that both aI and bJ are integral ideals of R, which are divisorial. Then $(abIS)^{**} = ab(IS)^{**} = ab(JS)^{**} = (abJS)^{**}$. By Theorem 2.4, we have $abI = (abIS)^{**} \cap R = (abJS)^{**} \cap R = abJ$. Thus we have $I = J$.

For a domain D, $\mathrm{Inv}(D)$ denotes the set of the invertible ideals of D. Define $\mathrm{Prin}(D)$ to be the set $\{aD | a \in K(D), a \neq 0\}$. It is easy to see that $\mathrm{Prin}(D)$ is a subgroup of $\mathrm{Inv}(D)$. Define $\mathrm{Pic}(D) = \mathrm{Inv}(D)/\mathrm{Prin}(D)$, which is equipped with the commutative group structure induced from that of $\mathrm{Inv}(D)$. We call $\mathrm{Pic}(D)$ the *Picard group* of D, which can be regarded as the group of isomorphic classes of invertible D-modules. We denote the composition in $\mathrm{Pic}(D)$ additively.

Let S and K be the same as in Theorem 2.4. The inclusion $S \cap K \hookrightarrow S$ induces the canonical map φ: $\mathrm{Inv}(S \cap K) \to \mathrm{Inv}(S)$ defined by sending $I \in \mathrm{Inv}(S \cap K)$ to $IS \in \mathrm{Inv}(S)$.

Corollary 2.4.2. *Let S and K be the same as above. Then φ: $\mathrm{Inv}(S \cap K) \to \mathrm{Inv}(S)$ is injective.*

Proof. Take two invertible ideals I and J of $S \cap K$ such that $IS = JS$. Then $I = IS \cap K = JS \cap K = J$ by Theorem 2.4, which means φ is injective.

Question. Let S and K be the same as above. When is the canonical group homomorphism $\mathrm{Pic}(S \cap K) \to \mathrm{Pic}(S)$ injective i.e., $\mathrm{Inv}(S \cap K) \cap \mathrm{Prin}(S) = \mathrm{Prin}(S \cap K)$?

Let S be an integral domain and let $D(S)$ denote the collection of divisorial fractional S-ideals. Define $D(S) \times D(S) \to D(S)$ by $(a, b) \mapsto S{:}(S{:}ab)$. Then $D(S)$ is a commutative monoid. It is known that $D(S)$ is a group if and only if S is completely integral closed [6, (3.4)]. Note here that a Krull domain is completely integral closed [6, (3.6)].

Let $R \subseteq S$ be Krull domains. We say that S/R satisfies the condition **(PDE)** if $\mathrm{ht}(P \cap R) \leq 1$ for each $P \in Ht_1(S)$.

It is known that if S is a Krull domain, then $S \cap K$ is also a Krull domain for any field [6, (1.2)].

Proposition 2.5. *Let S be a Krull domain and let K be a subfield of the quotient field of S. Then the extension $S \cap K \subseteq S$ satisfies (PDE)*

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and the canonical group homomorphism $D(S \cap K) \to D(S)$ *defined by* $I \mapsto (IS)^{**}$ *is injective.*

Proof. The second statement follows from Corollary 2.4.1. Since S is a Krull domain, $S = \bigcap_i V_i$, where V_i is a DVR on the quotient field of S which contains S. Let m_i denote the maximal ideal of V_i. Then $S \cap K = \bigcap_i (V_i \cap K)$, where $V_i \cap K$ is either a DVR with maximal ideal $m_i \cap K$ or a field. Take $P \in Ht_1(S)$. Then there exists a DVR V_i such that $m_i \cap S = P$. Hence $P \cap K = m_i \cap S \cap K = m_i \cap K$ is (0) or in $Ht_1(S \cap K)$.

3. (2,3)-closed, root-closed and quasinormal. Let D be an integral domain with quotient field $K(D)$ and let L be a field containing $K(D)$. We say that D is (2,3)-*closed in* L if every element $\alpha \in L$ such that $\alpha^2, \alpha^3 \in D$ is an element of D, and we say "(2,3)-closed" when $L = K(D)$. We say that D is *root-closed in* L if every element $\alpha \in L$ such that $\alpha^n \in D$ for some $n \in N$ is an element of D. We say that D is *quasinormal* if the canonical homomorphism: $\text{Pic}(D) \to \text{Pic}(D[X, X^{-1}])$ is an isomorphism, where X denotes an indeterminate over D.

Theorem 3.1. *Let* S *be an integral domain and let* L *be a field containing the quotient field* $K(S)$ *of* S. *Let* K *be a field. If* S *is* (2,3)-*closed in* L, *then* $S \cap K$ *is* (2,3)-*closed in* $L \cap K$.

Proof. Take $\alpha \in L \cap K$ with $\alpha^2, \alpha^3 \in S \cap K$. Then $\alpha^2, \alpha^3 \in S$ implies $\alpha \in S$ because S is (2,3)-closed in L. Hence $\alpha \in S \cap K$, which means that $S \cap K$ is (2,3)-closed in $L \cap K$.

In [4], the following is proved:

Lemma 3.2. *Let* D *be an integral domain and let* X *be an indeterminate over* D. *Then the following conditions are equivalent*:

(i) D *is* (2,3)-*closed,*

(ii) *the canonical homomorphism* $\text{Pic}(D) \to \text{Pic}(D[X])$ *is an isomorphism.*

Corollary 3.2.1. *Let* S, K *be the same as in Theorem* 3.1 *and let* $S[X]$ *be a polynomial ring. If* $\text{Pic}(S) \to \text{Pic}(S[X])$ *is an isomorphism, then* $\text{Pic}(S \cap K) \to \text{Pic}((S \cap K)[X])$ *is an isomorphism.*

Proof. This follows from Theorem 3.1 and Lemma 3.2.

Theorem 3.3. *Let S, L and K be the same as in Theorem 3.1. If S is root-closed in L, then $S \cap K$ is root-closed in $L \cap K$.*

Proof. Take $\alpha \in L \cap K$ with $\alpha^n \in S \cap K$ for some $n \in N$. Then $\alpha^n \in S$ implies $\alpha \in S$ because S is root-closed in L. Hence $\alpha \in S \cap K$, which means that $S \cap K$ is root-closed in L.

Let D be integral domain and let I be an invertible ideal of D. We denote by $[I]$ the equivalence class containing I in $\mathrm{Pic}(D)$.

Theorem 3.4. *Let S be an integral domain, let X be indeterminate and let K be a field. Assume that the canonical homomorphism $\mathrm{Pic}((S \cap K)[X, X^{-1}]) \to \mathrm{Pic}(S[X, X^{-1}])$ is injective. If S is quasinormal, then so is $S \cap K$.*

Proof. Put $R := S \cap K$. Take $I \in \mathrm{Inv}(R[X, X^{-1}])$. Consider the commutative diagram:

$$\begin{array}{ccc} \mathrm{Pic}(R) & \xrightarrow{i_1} & \mathrm{Pic}(S) \\ {\scriptstyle \varphi_{/K}}\downarrow\uparrow{\scriptstyle \psi_{/K}} & & {\scriptstyle \varphi}\downarrow\uparrow{\scriptstyle \psi} \\ \mathrm{Pic}(R[X, X^{-1}]) & \xrightarrow{i_2} & \mathrm{Pic}(S[X, X^{-1}]) \end{array}$$

where φ and $\varphi_{/K}$ are the canonical maps and ψ and $\psi_{/K}$ are the ones induced from the maps sending X to 1. It is clear that $\psi_{/K} \cdot \varphi_{/K} = 1$ and $\psi \cdot \varphi = 1$. So φ and $\varphi_{/K}$ are injective. By definition, $\psi_{/K}([I]) = [I']$ for some $I' \in \mathrm{Inv}(R)$. Since $\varphi \cdot i_1([I']) = \varphi([I'S]) = [I'S[X, X^{-1}]]$, we have $[I'S[X, X^{-1}]] \in \mathrm{Im}\, i_2$. By the diagram above, we have $i_2([I]) = \varphi \cdot \psi \cdot i_2([I]) = \varphi \cdot i_1([I']) = i_2 \cdot \varphi_{/K}([I'])$. Since i_2 is injective, we have that $[I] = \varphi_{/K}([I'])$. Thus $\varphi_{/K}$ is bijective.

4. A subring of an almost factorial domain. Let S be an integral domain and let K be a subfield of the quotient field of S. An ideal I of S is called *radically principal* if $I = \sqrt{fS}$ for some $f \in S$. A Krull domain is called *almost factorial* if its divisor class group is a torsion group.

Lemma 4.1 ([16, Proposition 7]). *Let R be a Krull domain. Then R is almost factorial if and only if any $P \in Ht_1(R)$ is radically principal.*

Theorem 4.2. *Let S be a Noetherian almost factorial domain of characteristic zero. Assume that S is integral over $S \cap K$. Then $S \cap K$ is a Noetherian almost factorial domain.*

Proof. By Theorem 1.2, $S \cap K$ is Noetherian. Since S is normal, so is $S \cap K$. Since S is almost factorial, any prime ideal of height one is radically principal by Lemma 4.1. Take $P \in Ht_1(S \cap K)$. Then any prime divisor of $\sqrt{PS}$ is of height one by Going-Down Theorem. So $\sqrt{PS} = \sqrt{fS}$ for some $f \in PS$. Let $P = (a_1, \dots, a_n)(S \cap K)$. Then taking a non-negative integer s, we have $a_i^s = fb_i$ for some $b_i \in S$. Put $S' = S \cap K(f, b_1, \dots, b_n)$. Then $S \cap K \subseteq S' \subseteq S$ and S' is integrally closed in $K(f, b_1, \dots, b_n)$. Note here that $\mathrm{char}(K) = 0$. There exists a field L' such that

(a) $L' \supseteq K(f, b_1, \dots, b_n) \supseteq K$,

(b) L' is a finite Galois extension of K.

Let G denote the Galois group $G(L'/K)$ with $m = \sharp G$. Let S'' denote the integral closure of $S \cap K$ in L'. Then S'' is a Galois extension of $S \cap K$. Note that $S''^{\sigma} = S''$ for each $\sigma \in G$. Since S is integral over R, we have $S' \subseteq S \cap L' \subseteq S''$ and $S'^{\sigma} \subseteq S''^{\sigma} = S''$ for each $\sigma \in G$. Hence $f^{\sigma}, b_1^{\sigma}, \dots, b_n^{\sigma} \in S''$ for any $\sigma \in G$. By [6, (1.3)], S'' is a Krull domain. The elements $\prod_{\sigma \in G} f^{\sigma}, \prod_{\sigma \in G} b_i^{\sigma}$ $(i = 1, \dots, n)$ are invariant under every element in G. Hence $\prod_{\sigma \in G} f^{\sigma}, \prod_{\sigma \in G} b_i^{\sigma} \in K \cap S''$ for $(i = 1, \dots, n)$. By Corollaty 1.1.1, we have $S'' \cap K = S \cap K$. Thus $\prod_{\sigma \in G} f^{\sigma}, \prod_{\sigma \in G} b_i^{\sigma} \in K \cap S$ for $(i = 1, \dots, n)$. So $f = a_i/b_i$ and $\prod_{\sigma \in G} f^{\sigma} = \prod_{\sigma \in G} a_i^{\sigma} / \prod_{\sigma \in G} b_i^{\sigma} \in S \cap K$. Put $g = \prod_{\sigma \in G} f^{\sigma}$. Then $a_i^{sm} = \prod_{\sigma \in G} f^{\sigma} \cdot \prod_{\sigma \in G} b_i^{\sigma}$, where $\sharp G = m$. Hence for a sufficiently large integer ℓ, $P^{\ell} \subseteq g(S \cap K)$. Thus we have $P = \sqrt{g(S \cap K)}$, and hence $S \cap K$ is almost factorial by Lemma 4.1.

Theorem 4.3. *Let S be an almost factorial domain. Assume that S is integral over $S \cap K$. Then $S \cap K$ is an almost factorial domain.*

Proof. The proof is similar to that of Theorem 4.2.

Corollary 4.3.1. *Let R be a Krull domain and let L be a field extension of $K(R)$. If the integral closure S of R in L is almost factorial, then so is R.*

Proof. Note that S is a Krull doamin. Since $S \cap K(R) = R$, our conclusion follows from Theorem 4.3.

5. A subring of a locally factorial domain and LCM-stableness. We mean by a local ring a ring with unique maximal ideal. It is known that an integral domain S is factorial domain if and only if S is a Krull domain in which each $P \in Ht_1(S)$ is principal [6, (6.1)].

Lemma 5.1. *Let (S, M) be a local domain and let K be a subfield of the quotient field of S. Let I be an ideal of $S \cap K$. If IS is principal, then so is I.*

Proof. Let I be generated by a set $\{a_i\}_{i \in \Delta}$. Since IS is a principal ideal of S, there exists $\alpha S = IS$. So for each $i \in \Delta$, $a_i = \alpha s_i$ for some $s_i \in S$. Suppose that the set $\{s_i | i \in \Delta\}$ generates a proper ideal of S. Then $\alpha S = IS \subseteq \alpha MS \subseteq \alpha S$, that is, $\alpha S = \alpha MS$. Hence $S = M$, a contradiction. So there exists a unit s_i so that $a_i S = \alpha s_i S = \alpha S = IS$. We have $I \subseteq IS \cap K = a_i S \cap K = a_i(S \cap K) \subseteq I$ by Lemma 2.1 (i) $\Longleftrightarrow$ (ii). Therefore $I = a_i(S \cap K)$.

Corollary 5.1.1. *Let (S, M) and K be the same as in Lemma 5.1. Assume that for each $P \in Ht_1(S \cap K)$, $Ass_S(S/PS) \subseteq Ht_1(S)$. If S is a factorial domain, then so is $S \cap K$.*

Proof. Take $P \in Ht_1(S \cap K)$. Since $Ass_S(S/PS) \subseteq Ht_1(S), PS$ is a divisorial ideal of S because S is a Krull domain. Since S is factorial, PS is a principal ideal and hence P is principal by Lemma 5.1.

A ring A is called *locally factorial* if A_P is factorial for each prime ideal P.

Theorem 5.2. *Let S be a locally factorial domain and K a field. Assume that S is integral over $S \cap K$. Then $S \cap K$ is locally factorial.*

Proof. Note first that for each prime ideal P of $S \cap K$, there exists a prime ideal M of S such that $M \cap K = P$ because S is integral over $S \cap K$ and that K can be assumed to be the quotient field of $S \cap K$. Hence our assertion follows from Lemma 5.1 and Remark 1(1) in the section one.

Remark 2. In [6, (6.11)], it is seen that when a local R-algebra S is faithfully flat over R, R is a factorial domain if S is factorial. But in general, not even factoriality descends through faithfully flat extensions. That is, if S is not local, then the above conclusion does not always hold. Indeed, we have the following example (cf. [6, p.39],[8, p.74],[18, p.105]): Consider a Dedekind domain R which is not a principal ideal domain. Let T be the multiplicative subset of the polynomial ring $R[X]$ generated by the polynomials whose coefficients generate R. Then the ring $S :=$ $T^{-1}R[X]$ is factorial (more precisely, a principal ideal domain) and it is a faithfully fat extension of R. But R is not factorial. Let K denote the quotient field of R. Then $S \cap K = R$. This example shows that even if S is a factorial domain, $S \cap K$ is not necessarily factorial for a field K.

Moreover even if a Noetherian normal domain S is a finite Galois extension of $S \cap K$, the factoriality of S does not necessarily yield that of $S \cap K$ [6, (16.5)].

Let S be a ring and let M be a S-module. We say that M is *LCM-stable* over S if $aM \cap bM = (aS \cap bS)M$ for any $a, b \in S$ and that M is *Q-stable* over S if $aM:_M b = (aS:_S b)M$ for any $a, b \in S$. It is easy to see that if a S-module M is flat, then M is LCM-stable over S, but the converse does not always hold.

Let $R \subseteq S$ be integral domains. It is known that S is LCM-stable over R if and only if S is Q-stable over R [1, Lemma 1].

We know that a maximal proper divisorial integral ideal of a Krull domain S is a prime ideal of height one with the form $S:(xS + S)$ for some $x \in K(S)$, the quotient field of S, which is equal to $yS:_S xS$ for some $y, x \in S$ [6, (3.5)]. Moreover in a Krull domain S, $P \in Ht_1(S)$ if and only if P is a maximal divisorial prime ideal [6, (3.11)].

Theorem 5.3. *Let (S, M) be a local domain and let K be a subfield of the quotient field of S. Assume that S is LCM-stable over $S \cap K$. If S is a factorial domain, then so is $S \cap K$.*

Proof. Put $R = S \cap K$. Let P be a prime ideal of R of height one. Then $P = aR:_R bR$ for some $a, b \in R$. Since S is LCM-stable over R, equivalently Q-stable over R, $PS = (aR:_R bR)S = aS:_S bS$, which is a divisorial integral ideal of S. Hence $Ass_S(S/PS) \subseteq Ht_1(S)$, which yields that R is a factorial domain by Corollary 5.1.1.

The following result is known: let (R, m) be a local domain with quotient field K and let S be an integral domain containing R with $mS \neq S$. If S is LCM-stable over R, then $S \cap K = R$ (cf. [17, (1.11)]).

Corollary 5.3.1. *Let (R, m) be a local domain and let S be an integral domain containing R with $mS \neq S$. Assume that S is LCM-stable over R. If S is a factorial domain, then so is R.*

Proof. This follows from Theorem 5.3 and the preceding known result.

6. A subring of a factorial domain. Let S be an integral domain and let K be a subfield of the quotient field of S. In this section, we treat mainly factoriality. Recall that an integral domain S is a *factorial* domain (or a unique factorization domain or a UFD) provided every element in S is uniquely (up to multiplication by a unit) a finite product of irreducible (or prime) elements. Even if S is a factorial domain, $S \cap K$ is not always factorial (see [6, (16.5)] or [3, VII,§3,Ex.11] for instance). In fact, we can see the following example in [3, VII,§3,Ex.11]:

Example. Let K be a field, $S = K[X,Y]$ be a polynomial ring and $L = K(X^2, Y/X) \subsetneq K(X,Y)$. Then S is factorial but $S \cap L$ is not.

So our aim is to study when $S \cap K$ is factorial if S is factorial.

In [6, (6.1)], we see that an integral domain S is factorial if and only if S has the ascending chain condition for principal ideals and a maximal proper principal ideal is a prime ideal.

Theorem 6.1. *Let S be an integral domain and let K be a subfield of the quotient field of S. Assume that S satisfies the ascending chain condition for principal ideals. Then $S \cap K$ is factorial if for each $P \in Ht_1(S)$ there exists a non-unit $a \in S \cap K$ such that $P \cap K \subseteq a(S \cap K)$.*

Proof. Put $R = S \cap K$. By Corollary 2.1.2, R has the ascending chain condition for principal ideals. Let dR be a maximal proper principal ideal of R. Then dS is contained in a prime ideal in $Ht_1(S)$. Indeed, if $dS = S$ then $dR = dS \cap K = S \cap K$ by Lemma 2.1, a contradiction. So by assumption, $dR \subseteq P \cap K$ and $P \cap K \subseteq aS$ for some non-unit a in R. By the maximality, we have $dR = P \cap K = aR$ and hence dR is a prime ideal. Thus R is a factorial domain.

Corollary 6.1.1. *Let S be a factorial domain and let K be a subfield of the quotient field of S. Then $S \cap K$ is factorial if for each non-unit $x \in S$ there exists a non-unit $a \in S \cap K$ such that $xS \cap K \subseteq a(S \cap K)$.*

Proof. Since S is factorial, any $P \in Ht_1(S)$ is a principal ideal. So apply Theorem 6.1 and we get our conclusion.

Recall that a ring extension $S \supseteq R$ is called to be *innert* if $x, y \in S$ with $xy \in R$ yields $xs, ys^{-1} \in R$ for some unit s in S (cf. [2]). For example, let S be a polynomial ring $R[X]$. Then the extension $S \supseteq R$ is innert.

Let S be an integral domain and let K be a subfield of the quotient field of S. We say that K is *innert* with respect to S if $x, y \in S$ with

$xy \in K$ yields that $xs, ys^{-1} \in K$ for some unit s in S. This is equivalent to the extension $S/S \cap K$ being innert in the above sense.

Theorem 6.2. *Let S be an integral domain and let K be a subfield of the quotient field of S. Assume that K is innert with respect to S. If S is factorial, then so is $S \cap K$.*

Proof. Put $R = S \cap K$. Then R is a Krull domain because S is a Krull domain. By Corollary 2.1.2, R has the ascending chain condition for principal ideals. Let dR be a maximal proper principal ideal. We have only to show that dR is prime. Suppose that dR is not prime. Then dS is not prime because $dS \cap K = dR$ by Lemma 2.1. So there exists a prime ideal bS in S containing dS properly. Thus we can write $d = bs$ for some non-unit $s \in S$. Since $bs = d \in S \cap K = R$, there exists a unit t in S such that $bt, st^{-1} \in R$ by assumption. Hence $dS \subseteq btS \cap st^{-1}S$ and hence $dR = dS \cap K \subseteq (btS \cap K) \cap (st^{-1}S \cap K) = btR \cap st^{-1}R$ by Lemma 2.1. Since dR is not prime, $dR \neq bS \cap K = btS \cap K = btR$ by Lemma 2.1 but by the maximality, $dR = btR$, which is a prime ideal of R, a contradiction.

We close this section by showing the following result.

Proposition 6.3. *Let S be an integral domain and let K be a subfield of the quotient field of S. Assume that K is innert with respect to S and that $U(S) = U(S \cap K)$, where $U(\ )$ denotes the group of the units. Then $S \cap K$ is algebraically closed in S.*

Proof. Take $\alpha \in S$. Then there is an algebraic dependence:

$$a_0\alpha^n + a_1\alpha^{n-1} + \cdots + a_n = 0,$$

where $a_i \in S \cap K$. Thus $\alpha(a_o\alpha^{n-1} + a_1\alpha^{n-2} + \cdots + a_{n-1}) \in S \cap K$. Hence there exists a unit t in S such that $\alpha t \in S \cap K$. By assumption, t is also a unit in $S \cap K$, we have $\alpha \in S \cap K$. This shows that $S \cap K$ is algebraically closed in S.

7. Remarks on Dedekind domains. In this section, we investigate Dedekind domains.

Proposition 7.1. *Let S be a Noetherian domain, let $K(S)$ be its quotient field and let K be a subfield of $K(S)$. Let m be a maximal ideal of a subring $K \cap S$ of S such that $mS \neq S$. Then* $\mathrm{ht}(m) \leq \dim S$.

Proof. Put $B = K \cap S$. Since $mS \neq S$ and m is a maximal ideal of B, there exists a prime ideal M of S with $M \cap B = m$. There exists a valuation ring (W, N) in $K(S)$ such that $S \subseteq W$, $N \cap S = M$ and $\dim W = \mathrm{ht}(M)$ by [13, (11.9) and its proof]. Similarly there exists a valuation ring (V, n) in $K(B)$ such that $B \subseteq V$, $n \cap B = m$ and $\dim V = \mathrm{ht}(m)$. Let W' be a subring generated by V and W in $K(S)$. Since $W \subseteq W' \subseteq K(S)$ and W is a valuation ring, W' is also a valuation ring by [13, (11.3)]. Let N' be the maximal ideal of W'. Then $N' \cap W \subseteq N$ and $W' = W_{N' \cap W}$ by [13, (11.3)]. Note that $mW' \neq W'$. Hence $m \subseteq N' \cap B \subseteq N \cap B = m$, that is, $N' \cap B = m$. Since $V \subseteq W' \cap K$, we have $N' \cap V \subseteq n$. Since $\mathrm{ht}(m) = \mathrm{ht}(n)$, we have $n = N' \cap V$, which yields that $W' \cap K = V$ by [13, (11.3)]. Hence $\mathrm{ht}(m) = \dim V = \dim W' \cap K \leq \dim W' \leq \dim W = \mathrm{ht}(M) \leq \dim S$.

We require the following Lemma:

Lemma 7.2 ([11, (12.5)]). *An integral domain A is a Dedekind domain if and only if A is a one-dimensional Krull domain.*

We have known the following example:

Example (cf. [3, VII,§2,Ex.5(a)]). Let k be a field and $L = k(X, Y)$, where X, Y are indeterminates. Let $S = L[Z]$ be a polynomial ring, which is actually a PID, and let $K = k(Z, X+YZ)$. Then $S \cap K$ is not a Dedekind domain. In fact, $\dim S \cap K = 2$ and $(Z, X+YZ)S = S$ for a maximal ideal $(Z, X+YZ)$ of $S \cap K$.

Proposition 7.3. *Let S be a Dedekind domain and K a subfield of $K(S)$. Assume that $mS \neq S$ for each $m \in \mathrm{Spec}(S \cap K)$. Then $S \cap K$ is a Dedekind domain.*

Proof. Note that a Dedekind domain is a Noetherian normal domain of dimension one. Since $mS \neq S$ for any maximal ideal m of $S \cap K$, $\dim S \cap K \leq 1$ by Proposition 7.1. Hence $S \cap K$ is a Krull domain of dimension one. So by Lemma 7.2, $S \cap K$ is a Dedekind domain.

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