

Mathematical Journal of Okayama University

Volume 16, Issue 1

1973

Article 6

SEPTEMBER 1973

K0-groups of the stunted real projective spaces

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K_0 -GROUPS OF THE STUNTED REAL PROJECTIVE SPACES

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1. Introduction

The purpose of this note is to calculate the \widetilde{K}_0^t -groups of the stunted real projective spaces. Our results are tabled as follows, where $RP(n)$ is the n -dimensional real projective space and (t) is the cyclic group of order t .

(1) $\widetilde{K}_0^{4m+t}(RP(4m+k)/RP(4m-1)) \quad (k \geq 0)$

| $k \backslash i$ | 0 | -1 | -2 | -3 | -4 | -5 | -6 | -7 |
|--------------------------|-------------------------|------------------------|-------------------------|-------------|-------------------------|------------|-------------|-------------|
| $8r$ | | | | | | | | |
| $r=0$ | (∞) | (2) | (2) | 0 | (∞) | 0 | 0 | 0 |
| $r \neq 0$ | $(\infty) + (2^{4r})$ | $(2) + (2)$ | $(2) + (2) + (2)$ | (2) | $(\infty) + (2^{4r})$ | 0 | 0 | 0 |
| $8r+1$ | | | | | | | | |
| $r=0$ | $(\infty) + (2)$ | $(2) + (2)$ | (2) | (∞) | (∞) | 0 | 0 | (∞) |
| $r \neq 0$ | $(\infty) + (2^{4r+1})$ | $(2) + (2)$ | $(2) + (2)$ | (∞) | $(\infty) + (2^{4r})$ | 0 | 0 | (∞) |
| $8r+2$ | $(\infty) + (2^{4r+2})$ | $(2) + (2)$ | $(2) + (2)$ | 0 | $(\infty) + (2^{4r})$ | 0 | (2) | (2) |
| $8r+3$ | $(\infty) + (2^{4r+2})$ | $(\infty) + (2) + (2)$ | $(2) + (2)$ | 0 | $(\infty) + (2^{4r})$ | (∞) | $(2) + (2)$ | $(2) + (2)$ |
| $8r+4$ | $(\infty) + (2^{4r+3})$ | $(2) + (2)$ | $(2) + (2)$ | 0 | $(\infty) + (2^{4r+1})$ | 0 | (2) | (2) |
| $8r+5$ | $(\infty) + (2^{4r+3})$ | $(2) + (2)$ | $(2) + (2)$ | (∞) | $(\infty) + (2^{4r+2})$ | 0 | 0 | (∞) |
| $8r+6$ | $(\infty) + (2^{4r+3})$ | $(2) + (2)$ | $(2) + (2) + (2)$ | (2) | $(\infty) + (2^{4r+3})$ | 0 | 0 | 0 |
| $8r+7$ | $(\infty) + (2^{4r+3})$ | $(\infty) + (2) + (2)$ | $(2) + (2) + (2) + (2)$ | $(2) + (2)$ | $(\infty) + (2^{4r+3})$ | (∞) | 0 | 0 |

(2) $\widetilde{K}_0^{4m+t}(RP(4m+1+k)/RP(4m)) \quad (k \geq 0)$

| $k \backslash i$ | 0 | -1 | -2 | -3 | -4 | -5 | -6 | -7 |
|------------------------|--------------|------------------|-------------------|-------------|--------------|------------|-------------|-------------|
| $8r$ | | | | | | | | |
| $r=0$ | (2) | (2) | 0 | (∞) | 0 | 0 | 0 | (∞) |
| $r \neq 0$ | (2^{4r+1}) | (2) | (2) | (∞) | (2^{4r}) | 0 | 0 | (∞) |
| $8r+1$ | (2^{4r+2}) | (2) | (2) | 0 | (2^{4r}) | 0 | (2) | (2) |
| $8r+2$ | (2^{4r+2}) | $(\infty) + (2)$ | (2) | 0 | (2^{4r}) | (∞) | $(2) + (2)$ | $(2) + (2)$ |
| $8r+3$ | (2^{4r+3}) | (2) | (2) | 0 | (2^{4r+1}) | 0 | (2) | (2) |
| $8r+4$ | (2^{4r+3}) | (2) | (2) | (∞) | (2^{4r+2}) | 0 | 0 | (∞) |
| $8r+5$ | (2^{4r+3}) | (2) | $(2) + (2)$ | (2) | (2^{4r+3}) | 0 | 0 | 0 |
| $8r+6$ | (2^{4r+3}) | $(\infty) + (2)$ | $(2) + (2) + (2)$ | $(2) + (2)$ | (2^{4r-3}) | (∞) | 0 | 0 |
| $8r+7$ | (2^{4r+4}) | (2) | $(2) + (2)$ | (2) | (2^{4r+3}) | 0 | 0 | 0 |

(3) $\tilde{K}_O^{4m+t}(RP(4m+2+k)/RP(4m+1))$ ($k \geq 0$)

| $k \backslash i$ | 0 | -1 | -2 | -3 | -4 | -5 | -6 | -7 |
|------------------|----------------|--------------|----------------------|--------------|----------------|--------------|----------------------|--------------|
| $8r$ | | | | | | | | |
| $r=0$ | (2) | 0 | (∞) | 0 | 0 | 0 | (∞) | (2) |
| $r \neq 0$ | (2^{4r+1}) | 0 | (∞) | 0 | (2^{4r}) | 0 | (∞)+(2) | (2) |
| $8r+1$ | | | | | | | | |
| $r=0$ | (2) | (∞) | (∞) | 0 | 0 | (∞) | (∞)+(2) | (2)+(2) |
| $r \neq 0$ | (2^{4r+1}) | (∞) | (∞) | 0 | (2^{4r}) | (∞) | (∞)+(2)+(2) | (2)+(2) |
| $8r+2$ | (2^{4r+2}) | 0 | (∞) | 0 | (2^{4r+1}) | 0 | (∞)+(2) | (2) |
| $8r+3$ | (2^{4r+2}) | 0 | (∞) | (∞) | (2^{4r+2}) | 0 | (∞) | (∞) |
| $8r+4$ | (2^{4r+2}) | 0 | (∞)+(2) | (2) | (2^{4r+3}) | 0 | (∞) | 0 |
| $8r+5$ | (2^{4r+2}) | (∞) | (∞)+(2)+(2) | (2)+(2) | (2^{4r+3}) | (∞) | (∞) | 0 |
| $8r+6$ | (2^{4r+3}) | 0 | (∞)+(2) | (2) | (2^{4r+4}) | 0 | (∞) | 0 |
| $8r+7$ | (2^{4r+4}) | 0 | (∞) | (∞) | (2^{4r+4}) | 0 | (∞) | (∞) |

(4) $\tilde{K}_O^{4m+t}(RP(4m+3+k)/RP(4m+2))$ ($k \geq 0$)

| $k \backslash i$ | 0 | -1 | -2 | -3 | -4 | -5 | -6 | -7 |
|------------------|----------------|--------------|---------|--------------|----------------|------------------|-------------|--------------|
| $8r$ | | | | | | | | |
| $r=0$ | 0 | (∞) | 0 | 0 | 0 | (∞) | (2) | (2) |
| $r \neq 0$ | (2^{4r}) | (∞) | 0 | 0 | (2^{4r}) | (∞)+(2) | (2)+(2)+(2) | (2)+(2) |
| $8r+1$ | | | | | | | | |
| $r=0$ | (2) | 0 | 0 | 0 | (2) | (2) | (4) | (2) |
| $r \neq 0$ | (2^{4r+1}) | 0 | 0 | 0 | (2^{4r+1}) | (2) | (2)+(2) | (2) |
| $8r+2$ | (2^{4r+1}) | 0 | 0 | (∞) | (2^{4r+2}) | (2) | (2) | (∞) |
| $8r+3$ | (2^{4r+1}) | 0 | (2) | (2) | (2^{4r+3}) | (2) | (2) | 0 |
| $8r+4$ | (2^{4r+1}) | (∞) | (2)+(2) | (2)+(2) | (2^{4r+3}) | (∞)+(2) | (2) | 0 |
| $8r+5$ | (2^{4r+2}) | 0 | (2) | (2) | (2^{4r+4}) | (2) | (2) | 0 |
| $8r+6$ | (2^{4r+3}) | 0 | 0 | (∞) | (2^{4r+4}) | (2) | (2) | (∞) |
| $8r+7$ | (2^{4r+4}) | 0 | 0 | 0 | (2^{4r+4}) | (2) | (2)+(2) | (2) |

2. Proof of the table (1)

Let ξ_k be the canonical line bundle over $RP(k)$ and θ^n be the trivial n -dimensional vector bundle over $RP(k)$. Then the Thom space $T(m\xi_k + \theta^n)$ and the n -fold suspension $S^n(RP(m+k)/RP(m-1))$ of the stunted projective space are homeomorphic (cf. [4, Chap. 15]).

According to [2, § 12], there is the K_O -theory Thom isomorphism $\psi: K_O^{-t}(X) \cong \tilde{K}_O^{-t}(T(\zeta))$ for $8n$ -dimensional vector bundle ζ over X which admits a reduction to $Spin(8n)$. Moreover, it is well known that ζ has a

spin-reduction if and only if its first and second Stiefel-Whitney classes vanish : $w_1(\xi) = w_2(\xi) = 0$.

Since $w_1(l\xi_k) = w_2(l\xi_k) = 0$ iff $l \equiv 0 \pmod{4}$, we have an isomorphism $K_0^i(RP(k)) \cong \widetilde{K}_0^i(T(4m\xi_k + \theta^{4m}))$. Hence, we have the following

Proposition (2. 1).

$$\widetilde{K}_0^{i-4m}(RP(4m+k)/RP(4m-1)) \cong K_0^i(RP(k)).$$

By Proposition (2. 1) and [3, Theorem 1], we obtain the table (1).

3. $\widetilde{K}_0^0(RP(n+k)/RP(n)), \widetilde{K}_0^{-4}(RP(n+k)/RP(n)) \quad n \not\equiv 3 \pmod{4}$

From [1, Theorem 7. 4], we have

$$(3. 1) \quad \widetilde{K}_0^0(RP(n+k)/RP(n)) = Z_2^{\varphi(n+k, n)},$$

where $\varphi(n+k, n)$ is the number of the integers s such that $n < s \leq n+k$ and $s \equiv 0, 1, 2$ or $4 \pmod{8}$.

Since $n \not\equiv 3 \pmod{4}$, by [3, Theorem 1], we obtain the following short exact sequence

$$0 \longrightarrow \widetilde{K}_0^{-4}(RP(n+k)/RP(n)) \longrightarrow \widetilde{K}_0^{-4}(RP(n+k)) \longrightarrow \widetilde{K}_0^{-4}(RP(n)) \longrightarrow 0.$$

Hence, we have

$$(3. 2) \quad \widetilde{K}_0^{-4}(RP(n+k)/RP(n)) \cong Z_2^{\psi(n+k, n)},$$

where $\psi(n+k, n)$ is the number of the integers s such that $n < s \leq n+k$ and $s \equiv 0, 4, 5$ or $6 \pmod{8}$.

From (3. 1) and (3. 2), we obtain the parts of $i=0$ and -4 in the tables (2), (3) and (4).

4. $\widetilde{K}_0^i(RP(4m+1+k)/RP(4m)), \widetilde{K}_0^i(RP(4m+3+k)/RP(4m+2))$
(k : odd)

By [5, Corollary to Theorem 3. 8], we have the parts of k : odd in the tables (2) and (4) except for the groups

$$(4. 1) \quad \widetilde{K}_0^{4m-2}(RP(4m+8r+6)/RP(4m)),$$

$$(4. 2) \quad \widetilde{K}_0^{4m-2}(RP(4m+8r+8)/RP(4m)),$$

$$(4. 3) \quad \widetilde{K}_0^{4m+2}(RP(4m+8r+4)/RP(4m+2)),$$

$$(4. 4) \quad \widetilde{K}_0^{4m+2}(RP(4m+8r+10)/RP(4m+2)),$$

and it is also known that these are groups of order 4.

4.1. Let $X = RP(4m + 8r + 6)/RP(4m)$ and $A = RP(4m + 8r + 6)/RP(4m - 1)$, and consider the exact sequence of the triple

$$0 = \widetilde{K}_0^{4m-3}(S^{4m}) \longrightarrow \widetilde{K}_0^{4m-2}(X) \longrightarrow \widetilde{K}_0^{4m-2}(A).$$

By Proposition (2. 1) we have

$$\widetilde{K}_0^{4m-2}(A) \cong K_0^{-2}(RP(8r + 6)) = Z_2 + Z_2 + Z_2.$$

Therefore, we have $\widetilde{K}_0^{4m-2}(X) = Z_2 + Z_2$.

4.2. Similarly to the proof of 4. 1, we have $\widetilde{K}_0^{4m-2}(RP(4m + 8r + 8)/RP(4m)) = Z_2 + Z_2$.

4.3. Let $X = RP(4m + 8r + 4)/RP(4m + 2)$ and $A = RP(4m + 8r + 4)/RP(4m + 3)$, and consider the exact sequence of the triple

$$Z_2 = \widetilde{K}_0^{4m+1}(S^{4m+3}) \longrightarrow \widetilde{K}_0^{4m+2}(A) \longrightarrow \widetilde{K}_0^{4m+2}(X).$$

If $r \neq 0$, by Proposition (2. 1) we have

$$\widetilde{K}_0^{4m+2}(A) \cong K_0^{-2}(RP(8r)) = Z_2 + Z_2 + Z_2.$$

Therefore, $\widetilde{K}_0^{4m+2}(X) = Z_2 + Z_2$.

If $r = 0$, we have $\widetilde{K}_0^{4m+2}(X) = Z_4$ by [5, Theorem 3. 2].

4.4 Similarly to the proof of 4. 3 in the case of $r \neq 0$, we have $\widetilde{K}_0^{4m+2}(RP(4m + 8r + 10)/RP(4m + 2)) = Z_2 + Z_2$.

5. Some lemmas

Lemma (5. 1). *The following homomorphisms which are induced from the inclusion $i: RP(n) \subset RP(n + 1 + k)$*

$$i_1: \widetilde{K}_0^{-1}(RP(n + 1 + k)) \longrightarrow \widetilde{K}_0^{-1}(RP(n)) \quad (n \geq 2),$$

$$i_2: \widetilde{K}_0^{-2}(RP(n + 1 + k)) \longrightarrow \widetilde{K}_0^{-2}(RP(n)) \quad (n \geq 2)$$

are non-zero.

Proof. Consider the exact sequence of the pair

$$\widetilde{K}_0^{-1}(RP(8r + 6)/RP(2)) \longrightarrow \widetilde{K}_0^{-1}(RP(8r + 6)) \xrightarrow{i_1} \widetilde{K}_0^{-1}(RP(2)).$$

By §4, $\widetilde{K}_o^{-1}(RP(8r+6)/RP(2)) = 0$ and by [3, Theorem 1], two groups $\widetilde{K}_o^{-1}(RP(8r+6))$ and $\widetilde{K}_o^{-1}(RP(2))$ are Z_2 . Therefore, we obtain the isomorphism i'_1 .

In the same way as the above, we have the isomorphism $i'_2: \widetilde{K}_o^{-2}(RP(8r+4)) \rightarrow \widetilde{K}_o^{-2}(RP(2))$. The rest of the proof is immediate from the isomorphisms i'_1 and i'_2 .

Moreover, in virtue of the proof of [3, Theorem 1, i)], we have the following

Lemma (5.2). *The following exact sequence*

$$0 \longrightarrow \widetilde{K}_o^{-1}(S^{4s+3}) \longrightarrow \widetilde{K}_o^{-1}(RP(4s+3)) \xrightarrow{i_1} \widetilde{K}_o^{-1}(RP(4s+2)) \longrightarrow 0$$

splits.

6. $\widetilde{K}_o^k(RP(4m+1+k)/RP(4m))$ (k : even)

6.1. $k \equiv 0 \pmod{8}$. If $k = 0$, the results are obvious, because $RP(4m+1)/RP(4m) \approx S^{4m+1}$. Therefore, let us assume k to be non-zero. Consider the exact sequence

$$\begin{aligned} \widetilde{K}_o^{-2}(RP(4m+1+k)) &\xrightarrow{i_2} \widetilde{K}_o^{-2}(RP(4m)) \longrightarrow \widetilde{K}_o^{-1}(RP(4m+1+k)/RP(4m)) \\ &\longrightarrow \widetilde{K}_o^{-1}(RP(4m+1+k)) \xrightarrow{i_1} \widetilde{K}_o^{-1}(RP(4m)). \end{aligned}$$

By Lemma (5.1), i_1 and i_2 are non-zero, and $\widetilde{K}_o^{-1}(RP(4m)) = Z_2$, $\widetilde{K}_o^{-1}(RP(4m+1+k)) = Z_2$, $\widetilde{K}_o^{-2}(RP(4m)) = Z_2$ (m : odd) or $Z_2 + Z_2$ (m : even) and $\widetilde{K}_o^{-2}(RP(4m+1+k)) = Z_2$. Hence, we have

$$\widetilde{K}_o^{-1}(RP(4m+1+k)/RP(4m)) = \begin{cases} 0 & (m: \text{odd}), \\ Z_2 & (m: \text{even}). \end{cases}$$

By [3, Theorem 1],

$$i'_1: \widetilde{K}_o^{4m-3}(RP(4m+1+k)) \longrightarrow \widetilde{K}_o^{4m-3}(RP(4m))$$

is zero homomorphism. Therefore, considering the exact sequence of the pair, we can easily obtain the rest of this case.

6.2. $k \equiv 2 \pmod{8}$. Consider the exact sequence

$$\widetilde{K}_o^{-2}(RP(4m+1+k)) \xrightarrow{i_2} \widetilde{K}_o^{-2}(RP(4m)) \longrightarrow \widetilde{K}_o^{-1}(RP(4m+1+k)/RP(4m))$$

$$\longrightarrow \widetilde{K}_o^{-1}(RP(4m+1+k)) \xrightarrow{i_1} \widetilde{K}_o^{-1}(RP(4m)).$$

By Lemma (5. 1), i_1 and i_2 are non-zero homomorphisms. Moreover, $i_1(Z) = 0$ for $\widetilde{K}_o^{-1}(RP(4m+1+k)) = Z + Z_2$ by Lemma (5. 2). Hence, we have

$$\widetilde{K}_o^{-1}(RP(4m+1+k)/RP(4m)) = \begin{cases} Z + Z_2 & (m : \text{even}), \\ Z & (m : \text{odd}). \end{cases}$$

Considering the exact sequence of the pair, we have the rest of this case.

6. 3. $k \equiv 4 \pmod{8}$. Similar to the proof of 6. 1.

6. 4. $k \equiv 6 \pmod{8}$. Consider the exact sequence of the pair. In this case, the following homomorphisms

$$\begin{aligned} i_1 : \widetilde{K}_o^{-0}(RP(4m+1+k)) &\longrightarrow \widetilde{K}_o^{-0}(RP(4m)) && \text{for } m : \text{odd}, \\ i_1 : \widetilde{K}_o^{4m-3}(RP(4m+1+k)) &\longrightarrow \widetilde{K}_o^{4m-3}(RP(4m)) \end{aligned}$$

are zero by [3, Theorem 1]. And the image of the homomorphism

$$i_2 : \widetilde{K}_o^{-2}(RP(4m+1+k)) \longrightarrow \widetilde{K}_o^{-2}(RP(4m)) \quad \text{for } m : \text{even}$$

is Z_2 by Lemma (5. 1). Then we can easily obtain the results except for $\widetilde{K}_o^{4m-2}(RP(4m+1+k)/RP(4m))$ and it is also known that this is the group of order 8.

Next, consider the exact sequence

$$0 = \widetilde{K}_o^{4m-3}(S^{4m}) \longrightarrow \widetilde{K}_o^{4m-2}(RP(4m+1+k)/RP(4m)) \longrightarrow \widetilde{K}_o^{4m-2}(RP(4m+1+k)/RP(4m-1)).$$

By Proposition (2. 1), we have

$$\widetilde{K}_o^{4m-2}(RP(4m+1+k)/RP(4m-1)) = K_o^{-2}(RP(k+1)) = Z_2 + Z_2 + Z_2 + Z_2.$$

Therefore, we have $\widetilde{K}_o^{4m-2}(RP(4m+1+k)/RP(4m)) = Z_2 + Z_2 + Z_2$.

This completes the table (2).

7. $\widetilde{K}_o^i(RP(4m+3+k)/RP(4m+2)) \quad (k : \text{even})$

In the same way as §6, we have the rest parts of the table (4) except for $\widetilde{K}_o^{4n+2}(RP(4m+5)/RP(4m+2)) = 0$ or Z_2 . On the other hand the consideration of the E_2 -terms of the spectral sequence of K_o -theory for

$RP(4m+5)/RP(4m+2)$ leads that $\widetilde{K}_0^{4m+2}(RP(4m+5)/RP(4m+2))$ has at least two elements. Hence, we have $\widetilde{K}_0^{4m+2}(RP(4m+5)/RP(4m+2)) = Z_2$.

This completes the table (4).

8. $\widetilde{K}_0^t(RP(4m+2+k)/RP(4m+1))$

8. 1. $k \equiv 0 \pmod{8}$. If $k = 0$, since $RP(4m+2)/RP(4m+1) \approx S^{4m+2}$, the results are obvious. Therefore, let us assume k to be non-zero.

In case of $m \neq 0$, considering the exact sequence of the pair, we have the results except for $\widetilde{K}_0^{4m+2}(RP(4m+2+k)/RP(4m+1)) = Z + Z_2$ or Z . Consider the exact sequence of the triple $(RP(8t+4m+2), RP(4m+2), RP(4m+1))$, then we have $\widetilde{K}_0^{4m+2}(RP(4m+2+k)/RP(4m+1)) = Z + Z_2$.

In case of $m = 0$, considering the exact sequence of the pair, we have the results except for $\widetilde{K}_0^{-6}(RP(k+2)/RP(1))$ and $\widetilde{K}_0^{-2}(RP(k+2)/RP(1))$, and it is also known that these are Z or $Z + Z_2$. Consider the exact sequence of the triple $(RP(8t+2), RP(2), RP(1))$, then we have $\widetilde{K}_0^{-6}(RP(k+2)/RP(1)) = Z + Z_2$, and $\widetilde{K}_0^{-2}(RP(k+2)/RP(1)) = Z$.

8. 2. $k \equiv 1 \pmod{8}$. Considering the exact sequence of the pair, we have the results except for $\widetilde{K}_0^{4m+2}(RP(4m+2+k)/RP(4m+1))$ and $\widetilde{K}_0^{-2}(RP(k+2)/RP(1))$, and it is also known that the ranks of these groups are 1. Consider the exact sequence of the triple $(RP(4m+2+k), RP(4m+2), RP(4m+1))$, then we have

$$\widetilde{K}_0^{4m+2}(RP(4m+2+k)/RP(4m+1)) = \begin{cases} Z + Z_2 + Z_2 & \text{if } k \neq 1, \\ Z + Z_2 & \text{if } k = 1, \end{cases}$$

and

$$\widetilde{K}_0^{-2}(RP(k+2)/RP(1)) = Z.$$

8. 3. $k \equiv 3 \pmod{8}$. Noticing that the following homomorphisms

$$\begin{aligned} i : \widetilde{K}_0^{4m-3}(RP(4m+2+k)) &\longrightarrow \widetilde{K}_0^{4m-3}(RP(4m+1)) \\ i : \widetilde{K}_0^{4m+1}(RP(4m+2+k)) &\longrightarrow \widetilde{K}_0^{4m+1}(RP(4m+1)) \end{aligned}$$

are zero by [3, Theorem 1], we can easily obtain the results.

8. 4. The rest is similar to the above. This completes the table (3).

REFERENCES

- [1] J.F. ADAMS: Vector fields on spheres, *Ann. of Math.* **75** (1962), 603—622.
- [2] M.F. ATIYAH, R. Bott and A. Sapiro: Clifford modules, *Topology* **3** (1964), 3—38.
- [3] M. FUJII; *Ko*-groups of projective spaces, *Osaka J. Math.* **4** (1967), 141—149.
- [4] D. HUSEMOLLER: *Fiber Bundles*, McGraw-Hill, 1966.
- [5] H. TODA: Order of the identity class of a suspension space, *Ann. of Math.* **78** (1963) 300—325.

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(Received August 21, 1972)