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Note on compact manifolds with non-symmetric metric connections

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NOTE ON COMPACT MANIFOLDS WITH NON-SYMMETRIC METRIC CONNECTIONS

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Introduction. S. Bochner and K. Yano [3], [4][1] investigated grobal properties of compact manifolds with non-symmetric metric connections by means of pseudo-harmonic and pseudo-Killing tensor fields. The errors in [3], owing to the omission of torsion of the spaces, were corrected in [4]. T. Suguri [5] discussed also the spaces.

In this note, we shall give some remarks on spaces with non-symmetric metric connections with regards to the torsions of the spaces.

§1. Let S_n be an n-dimensional manifold on which there is given a positive definite metric

$$ds^2 = g_{ij}dx^i dx^j \qquad (i, j = 1, 2, \cdots\cdots, n)^{2)}$$

and a metric connection E^i_{jk} in local coordinates (x^i).

From the assumption, we have

$$(1) \qquad g_{ij\,|\,k} \equiv \frac{\partial g_{ij}}{\partial x^k} - g_{sj}E^s_{ik} - g_{is}E^s_{jk} = 0$$

where the solidus denotes covariant differentiation with respect to E^i_{jk}. From (1), we get

$$(2) \qquad \frac{\partial \sqrt{g}}{\partial x^i} = \sqrt{g}\, E^k_{ki}.$$

Define the torsion tensor of S_n by

$$(3) \qquad S_{ij}{}^k = \frac{1}{2}(E^k_{ij} - E^k_{ji}).$$

Now, for a scalar field φ on S_n, define an exterior form of degree $n-1$ by

$$(4) \qquad \Omega = g^{ih}(\varphi_{\,|\,h} + 2\varphi S_{hk}{}^k)\, d\sigma_i\,,$$

where

1) Numbers in brackets refer to the list of references at the end of the paper.
2) The summation convention of tensor analysis is used throughout.

Mathematical Journal of Okayama University, Vol. 4 [1954], Iss. 2, Art. 2

$$d\sigma_i = (-1)^{i+1}\sqrt{g}\, dx^1 \wedge \cdots\cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots\cdots \wedge dx^n .$$

We get by (1), (2), (3)

$$\begin{aligned}
&\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \{\sqrt{g}\, g^{ih}(\varphi_{|h} + 2\varphi S_{hk}{}^k)\} \\
&= E^r_{ri} g^{ih}(\varphi_{|h} + 2\varphi S_{hk}{}^k) \\
&\quad - (g^{rh} E^i_{ri} + g^{ir} E^h_{ri})(\varphi_{|h} + 2\varphi S_{hk}{}^k) \\
&\quad + g^{ih}\Big(\frac{\partial \varphi_{|h}}{\partial x^i} + 2\varphi_{|i} S_{hk}{}^k + 2\varphi \frac{\partial S_{hk}{}^k}{\partial x^i}\Big) \\
&= E^r_{ri} g^{ih}(\varphi_{|h} + 2\varphi S_{hk}{}^k) \\
&\quad - E^r_{ir} g^{ih}(\varphi_{|h} + 2\varphi S_{hk}{}^k) - g^{ir} E^h_{ri}(\varphi_{|h} + 2\varphi S_{hk}{}^k) \\
&\quad + g^{ih}(\varphi_{|h|i} + E^k_{hi}\varphi_{|k} + 2\varphi_{|i} S_{hk}{}^k + 2\varphi S_{hk}{}^k{}_{|i} + 2\varphi E^r_{hi} S_{rk}{}^k) \\
&= \Delta\varphi + 2\varphi g^{ih}(S_{hk}{}^k{}_{|i} - 2S_{hk}{}^k S_{ir}{}^r),
\end{aligned}$$

where we put

$$\Delta\varphi = g^{ih}\varphi_{|i|h} .$$

Accordingly, let D be a bounded domain on S_n with a regular boundary, then we have the following formula

$$(5) \qquad \int_D \Delta\varphi\, d\sigma = -2\int_D \varphi(S^i{}_k{}^k{}_{|i} - 2S^i{}_k{}^k S_{ir}{}^r)\, d\sigma + \int_{\partial D} \Omega,$$

where

$$d\sigma = \sqrt{g}\, dx^1 \wedge \cdots\cdots \wedge dx^n .$$

Especially, if S_n is compact, we have

$$(6) \qquad \int_{S_n} \Delta\varphi\, d\sigma = -2\int_{S_n} \varphi(S^i{}_k{}^k{}_{|i} - 2S^i{}_k{}^k S_{ir}{}^r)\, d\sigma .$$

Theorem 1. *On a compact space S_n with a non-symmetric metric connection, in order that for any scalar field φ, we have*

$$\int_{S_n} \Delta\varphi\, d\sigma = 0,$$

it is necessary and sufficient that

$$(7) \qquad S \equiv S^i{}_k{}^k{}_{|i} - 2S^i{}_k{}^k S_{ir}{}^r = 0.$$

Let V_n be the Riemann space with line element

$$ds^2 = g_{ij}dx^i dx^j .$$

Then we have easily the relation

$$E^i_{jk} = \{^i_{jk}\} + S_{jk}{}^i - S^i{}_{jk} - S^i{}_{kj}, \tag{8}$$

where $\{^i_{jk}\}$'s are the Christoffel symbolds made by g_{ij}. Accordingly, we get

$$\begin{aligned} S &= S^i{}_k{}^k{}_{|i} - 2S^i{}_k{}^k S_{ir}{}^r \\ &= S^i{}_k{}^k{}_{,i} + S^h{}_k{}^k(S_{hi}{}^i - S^i{}_{hi} - S^i{}_{ih}) - 2S^i{}_k{}^k S_{ir}{}^r, \end{aligned}$$

that is

$$S = S^i{}_k{}^k{}_{,i},$$

where the comma denotes covariant differentiation of V_n.

For a given compact Riemann space with line element

$$ds^2 = g_{ij}dx^i dx^j,$$

if we have a tensor field on V_n, $S_{ij}{}^k = -S_{ji}{}^k \not\equiv 0$ such that every where $S \equiv S^i{}_k{}^k{}_{,i} = 0$, then we can obtain a space S_n with a non-symmetric metric connection on which for any scalar field φ, we have

$$\int_{S_n} \Delta\varphi \, d\sigma = 0.$$

If S_{ijk} is skew-symmetric, $S = 0$ always holds good.

Let φ_i be a covariant vector field on V_n and put

$$S_{ij}{}^k = \delta^k_i \psi_j - \delta^k_j \psi_i . \tag{10}$$

Then we have

$$\begin{aligned} S_{ik}{}^k &= -(n-1)\psi_i, \\ S &= -(n-1)\psi_{i,j} g^{ij}. \end{aligned}$$

Accordingly, in this case, (7) becomes

$$g^{ij}\psi_{i,j} = 0, \tag{11}$$

that is, the differential form $\psi_i dx^i$ of degree 1 is co-exact. In other words, the $n-1$-cochain corresponding to $\psi_i dx^i$ is a cocycle. According to de Rahm's theorem, there exists always a vector field ψ_i such that $\psi_i \not\equiv 0$, $g^{ij}\psi_{i,j} = 0$.

Thus we see that *for any compact Riemann space V_n, there exists a space S_n with a non-symmetric metric connection E^k_{ji} such that*

$$S_{ik}{}^k \not\equiv 0, \qquad g^{ij} S_{ik}{}^k{}_{,j} = 0.$$

§2. Let

$$\varphi = \frac{1}{p!} \varphi_{i_1 \cdots\cdots i_p} dx^{i_1} \wedge \cdots\cdots \wedge dx^{i_p},$$

be an exterior differential form of degree p on a compact space S_n with a non-symmetric metric connection, where $\varphi_{i_1 \cdots\cdots i_p}$ are a skew-symmetric tensor field over S_n. We define a pseudo-exterior differentiation $\hat{d}$ by

$$\hat{d}\varphi = \frac{1}{(p+1)!} \varphi_{i_1 \cdots\cdots i_{p+1}} dx^{i_1} \wedge \cdots\cdots \wedge dx^{i_{p+1}}, \tag{12}$$

where

$$\varphi_{i_1 \cdots\cdots i_{p+1}} = (-1)^p \left\{ \varphi_{i_1 \cdots\cdots i_p | i_{p+1}} - \sum_{s=1}^{p} \varphi_{i_1 \cdots\cdots i_{s-1} i_{p+1} i_{s+1} \cdots\cdots i_p | i_s} \right\}. \tag{13}$$

We define also a pseudo-codifferentiation $\hat{\delta}$ by

$$\hat{\delta}\varphi = \frac{-1}{(p-1)!} g^{jk} \varphi_{j i_1 \cdots\cdots i_{p-1} | k} dx^{i_1} \wedge \cdots\cdots \wedge dx^{i_{p-1}}. \tag{14}$$

According to S. Bochner [3], if

$$\hat{d}\varphi = 0, \tag{15}$$

$$\hat{\delta}\varphi = 0, \tag{16}$$

then we call φ *pseudo-harmonic.*

Now, we define a *generalized Laplacian operator* on S_n by

$$\hat{\Delta} = -(\hat{d}\hat{\delta} + \hat{\delta}\hat{d}). \tag{17}$$

For any differential form of degree p on S_n, we have

$$\begin{aligned}
(\hat{\Delta}\varphi)_{i_1 \cdots\cdots i_p} &= -(\hat{d}\hat{\delta}\varphi)_{i_1 \cdots\cdots i_p} - (\hat{\delta}\hat{d}\varphi)_{i_1 \cdots\cdots i_p} \\
&= -\sum_{s=1}^{p} (-1)^{s-1} (\hat{\delta}\varphi)_{i_1 \cdots\cdots \hat{i}_s \cdots\cdots i_p | i_s} + g^{jk} (\hat{d}\varphi)_{j i_1 \cdots\cdots i_p | k} \\
&= \sum_{s=1}^{p} (-1)^{s-1} g^{jk} \varphi_{j i_1 \cdots\cdots \hat{i}_s \cdots\cdots i_p | k | i_s} + g^{jk} \varphi_{i_1 \cdots\cdots i_p | j | k} \\
&\quad + \sum_{s=1}^{p} (-1)^s g^{jk} \varphi_{j i_1 \cdots\cdots \hat{i}_s \cdots\cdots i_p | i_s | k} \\
&= g^{jk} \varphi_{i_1 \cdots\cdots i_p | j | k} + \sum_{s=1}^{p} g^{jk} (\varphi_{i_1 \cdots\cdots i_{s-1} j \cdots\cdots i_p | k | i_s} - \varphi_{i_1 \cdots\cdots i_{s-1} j \cdots\cdots i_p | i_s | k}),
\end{aligned}$$

where $\hat{i}_s$ denotes the omission of the index i_s.

Let

$$E_j{}^i{}_{kh} = \frac{\partial E^i_{jh}}{\partial x^k} - \frac{\partial E^i_{jk}}{\partial x^h} - E^r_{jk}E^i_{rh} + E^r_{jh}E^i_{rk},$$

$$E_{jk} = E_j{}^h{}_{kh}$$

be the components of the curvature tensor, Ricci tensor of S_n respectively. Then, by means of a well known formula, we can write the above equation as follows.

$$\begin{aligned}
(\hat{\Delta}\varphi)_{i_1\cdots\cdots i_p} &= g^{jk}\varphi_{i_1\cdots\cdots i_p|j|k} \\
&+ \sum_{s=1}^{p} g^{jk}(E_j{}^h{}_{ki_s}\varphi_{i_1\cdots\cdots i_{s-1}h\cdots\cdots i_p} \\
&+ \sum_{t<s} E_{i_t}{}^h{}_{ki_s}\varphi_{i_1\cdots\cdots i_{t-1}h\cdots\cdots i_{s-1}j\cdots\cdots i_p} \\
&+ \sum_{s<t} E_{i_t}{}^h{}_{ki_s}\varphi_{i_1\cdots\cdots i_{s-1}j\cdots\cdots i_{t-1}h\cdots\cdots i_p} \\
&- 2S_{ki_s}{}^h\varphi_{i_1\cdots\cdots i_{s-1}j\cdots\cdots i_p|h}) \\
&= g^{jk}\varphi_{j_1\cdots\cdots i_p|j|k} + \sum_{s=1}^{p} E^h{}_{i_s}\varphi_{i_1\cdots\cdots i_{s-1}h\cdots\cdots i_p} \\
&+ \sum_{s<t}(E^h{}_{i_s i_t}{}^j - E^h{}_{i_t i_s}{}^j)\varphi_{i_1\cdots\cdots i_{s-1}h\cdots\cdots i_{t-1}j\cdots\cdots i_p} \\
&+ 2\sum_{s=1}^{p} S_{i_s}{}^{jh}\varphi_{i_1\cdots\cdots i_{s-1}j\cdots\cdots i_p|h}.
\end{aligned} \tag{18}$$

Accordingly, for a pseudo-harmonic tensor field φ, we have

$$\begin{aligned}
&g^{jk}\varphi_{i_1\cdots\cdots j_p|j|k} + \sum_{s=1}^{p} E^h{}_{i_s}\varphi_{i_1\cdots\cdots i_{s-1}h\cdots\cdots i_p} \\
&+ \sum_{s<t}(E^h{}_{i_s i_t}{}^k - E^h{}_{i_t i_s}{}^k)\varphi_{i_1\cdots\cdots i_{s-1}h\cdots\cdots i_{t-1}k\cdots\cdots i_p} \\
&+ 2\sum_{s=1}^{p} S_{i_s}{}^{hk}\varphi_{i_1\cdots\cdots i_{s-1}h\cdots\cdots i_p|k} = 0.
\end{aligned} \tag{19}$$

If S_n is a compact space such that $S_{ik}{}^k = 0$, especially a compact Riemann space, in order that φ be pseudo-harmonic, it is necessary and sufficient that (19) *hold good for* φ [4].

In the following, we shall investigate the same problem without any restriction for $S_{ij}{}^k$.

For any two exterior differential forms φ, ψ of degree p and a bounded domain D with a regular boundary, we define an *inner product* of φ and ψ on D by

$$(\varphi, \psi)_D = \frac{1}{p!}\int_D \varphi^{i_1\cdots\cdots i_p}\psi_{i_1\cdots\cdots i_p}\, d\sigma. \tag{20}$$

If S_n is compact and $D = S_n$, we simply write

$$(\varphi, \psi)_{S_n} = (\varphi, \psi).$$

Let φ and ψ be exterior differential forms of degree $p-1$ and p respectively. Define a differential form of degree 1 by

$$\varphi \,\lrcorner\, \psi = \frac{1}{(p-1)!} \varphi^{i_1 \cdots\cdots i_{p-1}} \psi_{i i_1 \cdots\cdots i_{p-1}} dx^i. \tag{21}$$

We call $\varphi \,\lrcorner\, \psi$ the *left inner product* of φ and ψ, and we define analogously $\varphi \,\lrcorner\, \psi$ for φ and ψ of degrees p and $q\,(p \leqq q)$.

Then, we have

$$\frac{(\partial \sqrt{g}\,(\varphi \,\lrcorner\, \psi)^i)}{\sqrt{g}\;\partial x^i} = (\varphi \,\lrcorner\, \psi)^i{}_{|i} - 2S_{ik}{}^k(\varphi \,\lrcorner\, \psi)^i$$
$$= \frac{1}{(p-1)!}(\varphi_{i_1 \cdots\cdots i_{p-1}|i} \psi^{i i_1 \cdots\cdots i_{p-1}} + \varphi_{i_1 \cdots\cdots i_{p-1}} \psi^{i i_1 \cdots\cdots i_{p-1}}{}_{|i}$$
$$- 2S_{ik}{}^k(\varphi \,\lrcorner\, \psi)^i$$
$$= \frac{1}{p!}(\hat{d}\varphi)_{i_1 \cdots\cdots i_p} \psi^{i_1 \cdots\cdots i_p} - \frac{1}{(p-1)!}\varphi_{i_1 \cdots\cdots i_{p-1}}(\hat{\delta}\psi)^{i_1 \cdots\cdots i_{p-1}}$$
$$- 2S_{ik}{}^k(\varphi \,\lrcorner\, \psi)^i.$$

Hence we have

$$(\hat{d}\varphi, \psi)_D - (\varphi, \hat{\delta}\psi)_D = 2\int_D S^{ik}{}_k(\varphi \,\lrcorner\, \psi)_i d\sigma + \int_{\partial D}(\varphi \,\lrcorner\, \psi)^i d\sigma_i.$$

Using a differential form π of degree 1 defined by

$$\pi = S_{ik}{}^k dx^i, \tag{22}$$

we obtain the following formula

$$(\hat{d}\varphi, \psi)_D - (\varphi, \hat{\delta}\psi)_D = 2(\pi, \varphi \,\lrcorner\, \psi)_D + \int_{\partial D}(\varphi \,\lrcorner\, \psi)^i d\sigma_i \tag{23}$$

For any exterior differential form φ of degree p, we obtain from (23)

$$(\hat{d}\varphi, \hat{d}\varphi)_D - (\varphi, \hat{\delta}\hat{d}\varphi)_D = 2(\pi, \varphi \,\lrcorner\, \hat{d}\varphi)_D + \int_{\partial D}(\varphi \,\lrcorner\, \hat{d}\varphi)^i d\sigma_i$$
$$(\hat{d}\hat{\delta}\varphi, \varphi)_D - (\hat{\delta}\varphi, \hat{\delta}\varphi)_D = 2(\pi, \hat{\delta}\varphi \,\lrcorner\, \varphi)_D + \int_{\partial D}(\hat{\delta}\varphi \,\lrcorner\, \varphi)^i d\sigma_i.$$

Hence we have the formula

$$(\hat{d}\varphi, \hat{d}\varphi)_D + (\hat{\delta}\varphi, \hat{\delta}\varphi)_D + (\varphi, \hat{\Delta}\varphi)_D$$
$$= 2(\pi, \varphi \lrcorner \hat{d}\varphi - \hat{\delta}\varphi \lrcorner \varphi)_D + \int_{\partial D} (\varphi \lrcorner \hat{d}\varphi - \hat{\delta}\varphi \lrcorner \varphi)^i d\sigma_i .$$

Let S_n be compact, putting $D = S_n$ we get

$$(24) \qquad (\hat{d}\varphi, \hat{d}\varphi) + (\hat{\delta}\varphi, \hat{\delta}\varphi) + (\varphi. \hat{\Delta}\varphi) = 2(\pi, \varphi \lrcorner \hat{d}\varphi - \hat{\delta}\varphi \lrcorner \varphi).$$

From (24), we see that on a compact S_n, the system of equations

$$(19) \qquad \hat{\Delta}\varphi = 0,$$

$$(25) \qquad (\pi, \varphi \lrcorner \hat{d}\varphi - \hat{\delta}\varphi \lrcorner \varphi) = 0$$

is equivalent to the one

$$(15) \qquad \hat{d}\varphi = 0,$$

$$(16) \qquad \hat{\delta}\varphi = 0.$$

Thus we have a conclusion.

Theorem 2. *On a compact space S_n with a non-symmetric metric connection, in order that a exterior differential form φ be pseudo-harmonic, it is sufficient that*

$$(26) \qquad \hat{\Delta}\varphi = 0, \quad S_{ik}{}^k (\varphi \lrcorner \hat{d}\varphi - \hat{\delta}\varphi \lrcorner \varphi)^i = 0.$$

§3. In this section, we shall deduce some grobal results from (23), (24) on a compact space S_n.

Let S_n be compact and φ be any exterior differential form of degree p. By (12), (14), (21) we have

$$(\varphi \lrcorner \hat{d}\varphi - \hat{\delta}\varphi \lrcorner \varphi)_i$$
$$= \frac{1}{p!} \varphi^{i_1 \cdots\cdots i_p} (\varphi_{i_1 \cdots\cdots i_p | i} - \sum_{s=1}^{p} (-1)^{s-1} \varphi_{i i_1 \cdots\cdots \hat{i}_s \cdots\cdots i_p | i_s})$$
$$+ \frac{1}{(p-1)!} \varphi^{j i_1 \cdots\cdots i_{p-1}}{}_{|j} \varphi_{i i_1 \cdots\cdots i_{p-1}}$$
$$= \frac{1}{2} (\varphi \lrcorner \varphi)_{|i} + \frac{1}{(p-1)!} (\varphi^{j i_1 \cdots\cdots i_{p-1}}{}_{|j} \varphi_{i i_1 \cdots\cdots i_{p-1}}$$
$$- \varphi^{j i_1 \cdots\cdots i_{p-1}} \varphi_{i i_1 \cdots\cdots i_{p-1} | j}).$$

Hence we have

$$2(\pi, \varphi \lrcorner \hat{d}\varphi - \hat{\delta}\varphi \lrcorner \varphi) = (\pi, \hat{d}(\varphi \lrcorner \varphi))$$
$$+ \frac{2}{(p-1)!} \int \varphi^{j i_1 \cdots\cdots i_{p-1}}{}_{|j} \varphi_{i i_1 \cdots\cdots i_{p-1}} S^{ik}{}_k \, d\sigma$$

$$- \frac{2}{(p-1)!}\int \varphi^{ji_1\cdots\cdots i_{p-1}}\varphi_{ii_1\cdots\cdots i_{p-1}|j}S^{ik}{}_k\, d\sigma$$

$$= (\pi,\ \hat{d}(\varphi \rfloor \varphi)) + 2(-1)^p(\hat{\delta}\varphi,\ \pi \rfloor \varphi)$$

$$- \frac{2}{(p-1)!}\int \varphi^{ji_1\cdots\cdots i_{p-1}}(\varphi_{ii_1\cdots\cdots i_{p-1}}S^{ik}{}_k)_{|j}\, d\sigma$$

$$+ \frac{2}{(p-1)!}\int \varphi^{ji_1\cdots\cdots i_{p-1}}\varphi_{ii_1\cdots\cdots i_{p-1}}S^{ik}{}_{k|j}\, d\sigma$$

$$= (\pi,\ \hat{d}(\varphi \rfloor \varphi)) + 2(-1)^p(\hat{\delta}\varphi,\ \pi \rfloor \varphi)$$

$$+ 2(-1)^p(\varphi,\ \hat{d}(\pi \rfloor \varphi)) + \frac{2}{(p-1)!}\int \varphi^{ji_1\cdots\cdots i_{p-1}}\varphi_{ii_1\cdots\cdots i_{p-1}}S^{ik}{}_{k|j}\, d\sigma.$$

By virtue of (7), (23), the last side of the equation above is written as

$$= (\hat{\delta}\pi,\ \varphi \rfloor \varphi) + 2(\pi,\ (\varphi \rfloor \varphi)\pi)$$

$$+ 2(-1)^p\{(\hat{\delta}\varphi,\ \pi \rfloor \varphi) + (\varphi,\ \hat{d}(\pi \rfloor \varphi))\}$$

$$+ \frac{2}{(p-1)!}\int \varphi^{ji_1\cdots\cdots i_{p-1}}\varphi_{ii_1\cdots\cdots i_{p-1}}S^{ik}{}_{k|j}\, d\sigma$$

$$= -(S,\ \varphi \rfloor \varphi) + 4(-1)^p\{(\hat{\delta}\varphi,\ \pi \rfloor \varphi) + (\pi,\ (\pi \rfloor \varphi) \rfloor \varphi)\}$$

$$+ \frac{2}{(p-1)!}\int \varphi^{ji_1\cdots\cdots i_{p-1}}\varphi_{ii_1\cdots\cdots i_{p-1}}S^{ik}{}_{k|j}\, d\sigma,$$

that is

$$2(\pi,\ \varphi \rfloor \hat{d}\varphi - \hat{\delta}\varphi \rfloor \varphi)$$

$$= -(S,\ \varphi \rfloor \varphi) + 4(-1)^p(\hat{\delta}\varphi,\ \pi \rfloor \varphi)$$

$$+ \frac{2}{(p-1)!}\int \varphi^{ji_1\cdots\cdots i_{p-1}}\varphi_{ii_1\cdots\cdots i_{p-1}}S^{ik}{}_{k|j}\, d\sigma$$

$$+ \frac{(-1)^p 4}{(p-1)!}\int S_{ik}{}^k S^{jh}{}_h \varphi_{i_1\cdots\cdots i_{p-1}j}\varphi^{ii_1\cdots\cdots i_{p-1}}\, d\sigma.$$

Define a symmetric tensor of order 2 by

$$S_{ij} = \frac{1}{2}(S_{ik}{}^k{}_{|j} + S_{jk}{}^k{}_{|i}) - 2S_{ik}{}^k S_{jh}{}^h. \tag{27}$$

Then, we have easily

$$S = g^{ij}S_{ij}. \tag{28}$$

Making use of S_{ij}, we obtain a formula on a compact S_n as

$$
\begin{aligned}
&(\hat{d}\varphi,\ \hat{d}\varphi)+(\hat{\delta}\varphi,\ \hat{\delta}\varphi)+(\varphi,\ \hat{\Delta}\varphi) \\
(29)\qquad &= -(S,\ \varphi \lrcorner\, \varphi)+(-1)^p 4(\hat{\delta}\varphi,\ \pi \lrcorner\, \varphi) \\
&\quad + \frac{2}{(p-1)!}\int S_{ij}\varphi^{i i_1 \cdots\cdots i_{p-1}}\varphi^{j}{}_{i_1 \cdots\cdots i_{p-1}}\, d\sigma .
\end{aligned}
$$

Accordingly we obtain from (29) the theorem.

Theorem 3. *On any compact space S_n with a non-symmetric metric connection, for any pseudo-harmonic field φ, we have*

$$
(30)\qquad \frac{1}{p!}\int S\varphi^{i_1\cdots\cdots i_p}\varphi_{i_1\cdots\cdots i_p}\, d\sigma = \frac{2}{(p-1)!}\int S_{ij}\varphi^{i i_1\cdots\cdots i_{p-1}}\varphi^{j}{}_{i_1\cdots\cdots i_{p-1}}\, d\sigma .
$$

Define a symmetric tensor of order 2 by

$$
\begin{aligned}
L_{ij} &= S_{ij} - \frac{1}{2p} g_{ij} S \\
(31)\qquad &= \frac{1}{2}(S_{ik}{}^{k}{}_{|j} - S_{ik}{}^{k}{}_{|i}) - 2S_{ik}{}^{k}S_{jh}{}^{h} - \frac{1}{2p} g_{ij}(S^{hk}{}_{k|h} - 2S^{hk}{}_{k}S_{hr}{}^{r}).
\end{aligned}
$$

Then (30) is written as

$$
(30')\qquad \int_{S_n} L_{ij}\varphi^{i i_1\cdots\cdots i_{p-1}}\varphi^{j}{}_{i_1\cdots\cdots i_{p-1}}\, d\sigma = 0.
$$

Since we have

$$
L = g^{ij}L_{ij} = \left(1-\frac{n}{2p}\right)S,
$$

if *$n \neq 2p$ and L_{ij} is positive definite or negative definite, then there exists no pseudo-harmonic field of degree p on S_n. If L_{ij} is positive semi-definite or negative semi-definite, then any pseudo-harmonic field φ of degree p must satisfy*

$$
(32)\qquad L_{ij}\varphi^{i i_1\cdots\cdots i_{p-1}}\varphi^{j}{}_{i_1\cdots\cdots i_{p-1}} = 0.
$$

In the theory of Bochner and Yano [3], [4], the argument in the existence of pseudo-harmonic tensor fields holds good for the spaces such that $S=0$. But, for the spaces S_n such that $S \neq 0$, in order to perform the anologous argument to the case $S=0$, we can also make use of the tensor L_{ij}.

§4. Nextly, we shall investigate the same problem for pseudo-Killing tensor fields. According to Bochner and Yano [4], we call a skew-symmetric tensor $\varphi_{i_1\cdots\cdots i_p}$ *pseudo-Killing* if

Mathematical Journal of Okayama University, Vol. 4 [1954], Iss. 2, Art. 2

$$\text{(33)} \qquad \varphi_{i_1\cdots\cdots i_p|j} = -\varphi_{i_1\cdots\cdots i_{s-1}ji_{s-1}\cdots\cdots i_p|i_s}, \qquad s = 1, \cdots\cdots, p.$$

Let $\varphi_{i_1\cdots\cdots i_p}$ be a pseudo-Killing tensor field of degree p on S_n, then for the exterior differential form of degree p

$$\varphi = \frac{1}{p!}\varphi_{i_1\cdots\cdots i_p}dx^{i_1}\wedge\cdots\cdots\wedge dx^{i_p},$$

we have clearly from (33)

$$\hat{\delta}\varphi = 0.$$

Accordingly we have

$$\begin{aligned}(\hat{\Delta}\varphi)_{i_1\cdots\cdots i_p} &= -(\hat{\delta}\hat{d}\varphi)_{i_1\cdots\cdots i_p} \\ &= g^{jk}\left\{\varphi_{i_1\cdots\cdots i_p|j} - \sum_{s=1}^{p}\varphi_{i_1\cdots\cdots i_{s-1}j\cdots\cdots i_p|i_s}\right\}_{|k} \\ &= (p+1)g^{jk}\varphi_{i_1\cdots\cdots i_p|j|k}.\end{aligned}$$

Define a *linear operator* K defined for any exterior differential form ψ of degree p by

$$\begin{aligned}(K\psi)_{i_1\cdots\cdots i_p} &= g^{jk}\psi_{i_1\cdots\cdots i_p|j|k} - \frac{1}{p}\sum_{s}E^{h}{}_{i_s}\psi_{i_1\cdots\cdots i_{s-1}h\cdots\cdots i_p} \\ &\quad - \frac{1}{p}\sum_{s<t}(E^{h}{}_{i_s i_t}{}^{j} - E^{h}{}_{i_t i_s}{}^{j})\psi_{i_1\cdots\cdots i_{s-1}h\cdots\cdots i_{t-1}j\cdots\cdots i_p} \qquad \text{(34)} \\ &\quad - \frac{2}{p}\sum_{s}S_{i_s}{}^{jh}\psi_{i_1\cdots\cdots i_{s-1}j\cdots\cdots i_p|h}.\end{aligned}$$

Then, for a pseudo-Killing tensor φ of degree p, we have by (18) and the equation above

$$K\varphi = 0.$$

By means of (29) and (34), for any field φ on compact space S_n, we have

$$\begin{aligned}&(\hat{d}\varphi, \hat{d}\varphi) + (\hat{\delta}\varphi, \hat{\delta}\varphi) - p(\varphi, K\varphi) + \frac{(p+1)}{p!}\int g^{jk}\varphi_{i_1\cdots\cdots i_p|j|k}\varphi^{i_1\cdots\cdots i_p}d\sigma \\ &= (-1)^p4(\hat{\delta}\varphi, \pi \rfloor \varphi) + \frac{2}{(p-1)!}\int L_i{}^{j}\varphi^{ii_1\cdots\cdots i_{p-1}}\varphi_{ji_1\cdots\cdots i_{p-1}}d\sigma.\end{aligned}$$

The left hand side of the above equation is written as

$$\begin{aligned}&(\hat{d}\varphi, \hat{d}\varphi) + (\hat{\delta}\varphi, \hat{\delta}\varphi) - p(\varphi, K\varphi) \\ &\quad + \frac{p+1}{2\cdot p!}\int\{\Delta(\varphi_{i_1\cdots\cdots i_p}\varphi^{i_1\cdots\cdots i_p}) - 2\varphi_{i_1\cdots\cdots i_p|j}\varphi^{i_1\cdots\cdots i_1|j}\}d\sigma.\end{aligned}$$

Hence we have by (6) an equation

$$(\hat{d}\varphi, \hat{d}\varphi) + (\hat{\delta}\varphi, \hat{\delta}\varphi) - p(\varphi, K\varphi) - \frac{p+1}{p!}\int \varphi_{i_1\cdots\cdots i_p|j}\varphi^{i_1\cdots\cdots i_{p-1}|j} d\sigma$$
$$= (-1)^p 4(\hat{\delta}\varphi, \pi \lrcorner \varphi) + \frac{2}{(p-1)!}\int M_i^{\ j}\varphi^{ii_1\cdots\cdots i_{p-1}}\varphi_{ji_1\cdots\cdots i_{p-1}} d\sigma.$$

where we put

$$M_{ij} = L_{ij} + \frac{p+1}{2p}Sg_{ij} = S_{ij} + \frac{1}{2}Sg_{ij}. \tag{35}$$

Since we have

$$(\hat{d}\varphi)_{i_1\cdots\cdots i_p j}(\hat{d}\varphi)^{i_1\cdots\cdots i_p j} - (p+1)^2\varphi_{i_1\cdots\cdots i_p|j}\varphi^{i_1\cdots\cdots i_p|j}$$
$$= (\varphi_{i_1\cdots\cdots i_p|j} - \sum_{s=1}^{p}\varphi_{i_1\cdots\cdots i_{s-1}j\cdots\cdots i_p|s})(\varphi^{i_1\cdots\cdots i_p|j} - \sum_{t=1}^{p}\varphi^{i_1\cdots\cdots i_{t-1}j\cdots\cdots i_p|i_t})$$
$$- (p+1)^2\varphi_{i_1\cdots\cdots i_p|j}\varphi^{i_1\cdots\cdots i_p|j}$$
$$= -\frac{p(p+1)}{2}(\varphi_{i_1\cdots\cdots i_{p-1}i|j} + \varphi_{i_1\cdots\cdots i_{p-1}j|i})(\varphi^{i_1\cdots\cdots i_{p-1}i|j} + \varphi^{i_1\cdots\cdots i_{p-1}j|i}),$$

we obtain the relation

$$(\hat{d}\varphi, \hat{d}\varphi) - \frac{p+1}{p!}\int \varphi_{i_1\cdots\cdots i_p|j}\varphi^{i_1\cdots\cdots i_p|j} d\sigma$$
$$= -\frac{1}{2\cdot(p-1)!}\int (\varphi_{i_1\cdots\cdots i_{p-1}i|j} + \varphi_{i_1\cdots\cdots i_{p-1}j|i})(\varphi^{i_1\cdots\cdots i_{p-1}i|j} + \varphi^{i_1\cdots\cdots i_{p-1}j|i}) d\sigma.$$

Thus we obtain a formula from the last equation

$$\begin{aligned}(\hat{\delta}\varphi, \hat{\delta}\varphi) &- p(\varphi, K\varphi) \\ &- \frac{1}{2\cdot(p-1)!}\int (\varphi_{i_1\cdots\cdots i_{p-1}j|k} + \varphi_{i_1\cdots\cdots i_{p-1}k|j})(\varphi^{i_1\cdots\cdots i_{p-1}j|k} + \varphi^{i_1\cdots\cdots i_{p-1}k|j}) d\sigma \\ = (-1)^p & 4(\hat{\delta}\varphi, \pi \lrcorner \varphi) \\ &+ \frac{2}{(p-1)!}\int M_j^{\ k}\varphi^{ji_1\cdots\cdots i_{p-1}}\varphi_{ki_1\cdots\cdots i_{p-1}} d\sigma.\end{aligned} \tag{36}$$

If φ is pseudo-Killing, we have from (36)

$$\int M_j^{\ k}\varphi^{ji_1\cdots\cdots i_{p-1}}\varphi_{ki_1\cdots\cdots i_{p-1}} d\sigma = 0$$

since (33), $\hat{\delta}\varphi = 0$ and $K\varphi = 0$.

Mathematical Journal of Okayama University, Vol. 4 [1954], Iss. 2, Art. 2

Thus, we obtain the following theorem.

Theorem 4. *On a compact space S_n with a non-symmetric metric connection, for any pseudo-Killing tensor field $\varphi_{i_1 \cdots\cdots i_p}$, we have*

$$\int M_j^{\ k} \varphi^{j i_1 \cdots\cdots i_{p-1}} \varphi_{k i_1 \cdots\cdots i_p} d\sigma = 0. \tag{37}$$

On the equivalent conditions (Bochner and Yano [4], Theorem 14), we obtain easily from (36) the following theorem.

Theorem 5. *On a compact space S_n with a non-symmetric metric connection, in order that a skew-symmetric tensor field $\varphi_{i_1 \cdots\cdots i_p}$ be pseudo-Killing, it is necessary and sufficient that*

$$\hat{\delta}\varphi = 0, \quad K\varphi = 0$$

and (37) *hold good.*

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