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NOTES ON BLOCKS OF GROUP CHARACTERS

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Introduction. We consider a group $\mathfrak{G}$ of finite order $g = p^a g'$, where p is a prime number and $(g', p) = 1$. Let $\Gamma = \Gamma(\mathfrak{G})$ denote the corresponding group ring formed with regard to an algebraic number field Ω which contains the g-th roots of unity. Let $K_1, K_2, \cdots\cdots, K_m$ be the classes of conjugate elements of $\mathfrak{G}$. Then Γ splits into a direct sum of m simple ideals Γ_i:

$$(1) \qquad \Gamma = \Gamma_1 \oplus \Gamma_2 \oplus \cdots\cdots \oplus \Gamma_m .$$

Denote the center of Γ by $\Lambda = \Lambda(\mathfrak{G})$. Corresponding to the decomposition (1) we have

$$(2) \qquad \Lambda = \Lambda_1 \oplus \Lambda_2 \oplus \cdots\cdots \oplus \Lambda_m ,$$

where each Λ_i is isomorphic to Ω.

Let $\mathfrak{o}$ be the ring of all integers of Ω and let $\mathfrak{p}$ be a prime ideal of $\mathfrak{o}$ dividing p. We denote by $\mathfrak{o}^*$ the ring of all $\mathfrak{p}$-integers of Ω, i.e., of all a/b, where a, b lie in $\mathfrak{o}$ and $(b, \mathfrak{p}) = \mathfrak{o}$. The ideal $\mathfrak{p}$ generates an ideal of $\mathfrak{o}^*$ which will be denoted by $\mathfrak{p}^*$. We then have

$$\Omega^* = \mathfrak{o}^*/\mathfrak{p}^* \cong \mathfrak{o}/\mathfrak{p}$$

for the residue class field. Let $\Gamma^* = \Gamma^*(\mathfrak{G})$ be the modular group ring of $\mathfrak{G}$ over Ω^* and let $\Lambda^* = \Lambda^*(\mathfrak{G})$ be its center.

In the present paper we study the structure of the center Λ^* and derive some results [1], [2] stated by R. Brauer without proofs. Some new results are also obtained. In section 1 certain ideals of Λ^* are defined. We determine the primitive idempotent elements of Λ^* in section 2[1]. Let

$$\Lambda^* = \Lambda_1^* \oplus \Lambda_2^* \oplus \cdots\cdots \oplus \Lambda_s^*$$

be the decomposition of Λ^* into indecomposable ideals Λ_σ^*. The ordinary irreducible characters χ_i of $\mathfrak{G}$ and the modular irreducible characters φ_κ of $\mathfrak{G}$ (for p) are distributed into s blocks $B_1, B_2, \cdots\cdots, B_s$, each χ_i and φ_κ belonging to exactly one block B_σ. In section 3 we investigate the properties of the defect group of a block B_σ.

1) The same result has been obtained by H. Nagao independently.

Section 4 deals with the elementary divisors of the Cartan matrix C_σ of B_σ.

1. The classes of conjugate elements $K_1, K_2, \cdots\cdots, K_m$ of $\mathfrak{G}$ form a basis of Λ. Here each class K_α is interpreted as the sum of all elements in K_α. We then have

$$(3) \qquad K_\alpha K_\beta = \sum_\gamma a_{\alpha\beta\gamma} K_\gamma,$$

where the $a_{\alpha\beta\gamma}$ are rational integers, $a_{\alpha\beta\gamma} \geqq 0$. Evidently $a_{\alpha\beta\gamma} = a_{\beta\alpha\gamma}$. Further we see easily that $\sum_\alpha a_{\alpha\beta\gamma} = g_\beta$, where g_β denotes the number of elements in K_β. The order of the normalizer $\mathfrak{N}(G_\alpha)$ of G_α in $\mathfrak{G}$ is given by $n_\alpha = g/g_\alpha$ for every element G_α in K_α. Let K_{α^*} denote the class which contains the elements reciprocal to those of K_α.

Lemma 1. $a_{\alpha\beta\gamma} = a_{\alpha^*\gamma\beta} n_\gamma / n_\beta$.

Proof. Let $G_\alpha^{(i)}$ $(i = 1, 2, \cdots\cdots, g_\alpha)$ be the elements in K_α and let G_β be a fixed element in K_β. The number of elements $G_\alpha^{(i)} G_\beta$ which lie in K_γ is equal to $a_{\alpha^*\gamma\beta}$. Hence

$$n_\beta K_\alpha K_\beta = K_\alpha (\sum_{G \text{ in } \mathfrak{G}} G^{-1} G_\beta G) = \sum_{G \text{ in } \mathfrak{G}} G^{-1} K_\alpha G_\beta G$$
$$= \sum_{G \text{ in } \mathfrak{G}} G^{-1} (\sum_{j=1}^{g_\alpha} G_\alpha^{(j)} G_\beta) G = \sum_\gamma a_{\alpha^*\gamma\beta} n_\gamma K_\gamma.$$

On the other hand, it follows from (3) that

$$n_\beta K_\alpha K_\beta = \sum_\gamma a_{\alpha\beta\gamma} n_\beta K_\gamma.$$

This proves our assertion.

We shall say that a group $\mathfrak{H}_\alpha$ of order p^{h_α} is the defect group [2] of a class K_α if $\mathfrak{H}_\alpha$ is a p-Sylow-subgroup of the normalizer of suitable elements in K_α. The exponent h_α is called the defect of K_α. If we consider conjugate subgroups of $\mathfrak{G}$ as not essentially different, then $\mathfrak{H}_\alpha$ is uniquely determined by K_α.

Lemma 2. *Let ρ be a fixed rational integer such that $0 \leqq \rho \leqq a$. The classes K_β with $h_\beta \leqq \rho$ form a basis of an ideal $\mathfrak{Z}_\rho$ of the center Λ^* of the modular group ring Γ^*.*[1]

Proof. If $h_\beta < h_\gamma$, then $a_{\alpha\beta\gamma} \equiv 0 \pmod p$ by Lemma 1, whence for any class K_α

1) See [7], §4.

$$K_\alpha K_\beta \equiv \sum_\gamma a_{\alpha\beta\gamma} K_\gamma \qquad \pmod{p},$$

where the sum extends over all K_γ with $h_\gamma \leqq h_\beta$.

We have by Lemma 2 the following series:

$$(4) \qquad \Lambda^* = \mathfrak{Z}_a = \mathfrak{Z}_{a_0} \supset \mathfrak{Z}_{a_1} \supset \cdots\cdots \supset \mathfrak{Z}_{a_k} \supset 0 \qquad (0 \leqq a_k).$$

Lemma 3. *If no element of K_β lies in the centralizer $\mathfrak{C}(\mathfrak{H}_\gamma)$ of $\mathfrak{H}_\gamma$ in $\mathfrak{G}$, then $a_{\alpha\beta\gamma} \equiv 0 \pmod{p}$.*[1]

Assume that $a_{\alpha\beta\gamma} \not\equiv 0 \pmod{p}$ in (3). It follows from Lemma 3 that there exists an element in K_β which commutes with all elements of $\mathfrak{H}_\gamma$, and hence $\mathfrak{H}_\gamma \subseteq \mathfrak{H}_\beta$.[2] We then have

$$K_\alpha K_\beta \equiv \sum_\gamma a_{\alpha\beta\gamma} K_\gamma \qquad \pmod{p},$$

where the sum extends over all K_γ with $\mathfrak{H}_\gamma \subseteq \mathfrak{H}_\beta$. Thus we obtain the

Lemma 4. *Let K_β be a given class with the defect group $\mathfrak{H}_\beta$. The classes K_γ with $\mathfrak{H}_\gamma \subseteq \mathfrak{H}_\beta$ form a basis of an ideal $\mathfrak{Z}(\mathfrak{H}_\beta)$ of Λ^*.*

Let $\mathfrak{H}_1^{(a_i)}, \mathfrak{H}_2^{(a_i)}, \cdots\cdots, \mathfrak{H}_t^{(a_i)}$ be a system of defect groups of order p^{a_i} such that every defect group of order p^{a_i} is conjugate to exactly one $\mathfrak{H}_\nu^{(a_i)}$. We then see that

$$\mathfrak{Z}_{a_i} = \mathfrak{Z}(\mathfrak{H}_1^{(a_i)}) + \mathfrak{Z}(\mathfrak{H}_2^{(a_i)}) + \cdots\cdots + \mathfrak{Z}(\mathfrak{H}_t^{(a_i)}) + \mathfrak{Z}_{a_{i+1}}.$$

If we set

$$\mathfrak{Z}_{a_i}^{(\nu)} = \mathfrak{Z}(\mathfrak{H}_{\nu+1}^{(a_i)}) + \cdots\cdots + \mathfrak{Z}(\mathfrak{H}_t^{(a_i)}) + \mathfrak{Z}_{a_{i+1}},$$

then every $\mathfrak{Z}_{a_i}^{(\nu)}$ is an ideal of Λ^* and

$$(5) \qquad \mathfrak{Z}_{a_i} = \mathfrak{Z}_{a_i}^{(0)} \supset \mathfrak{Z}_{a_i}^{(1)} \supset \cdots\cdots \supset \mathfrak{Z}_{a_i}^{(t-1)} \supset \mathfrak{Z}_{a_{i+1}}.$$

Further if we set $(\mathfrak{Z}(\mathfrak{H}_\nu^{(a_i)}) + \mathfrak{Z}_{a_{i+1}})/\mathfrak{Z}_{a_{i+1}} = \mathfrak{M}_\nu$, then

$$\mathfrak{Z}_{a_i}/\mathfrak{Z}_{a_{i+1}} = \mathfrak{M}_1 \oplus \mathfrak{M}_2 \oplus \cdots\cdots \oplus \mathfrak{M}_t.$$

2. Every ordinary irreducible character χ_i of $\mathfrak{G}$ determines a character ω_i of Λ which is given by

$$(6) \qquad \omega_i(K_\alpha) = g_\alpha \chi_i(G_\alpha)/z_i,$$

where G_α is an element in K_α and z_i is the degree of χ_i. The

1) See [1], p. 112.
2) Strictly speaking, $\mathfrak{H}_\gamma$ is conjugate in $\mathfrak{G}$ to a subgroup of $\mathfrak{H}_\beta$.

modular characters ω^* of Λ^* are obtained by considering the different ω_i (mod $\mathfrak{p}$). As was shown in [6], two characters χ_i and χ_j belong to the same block if and only if for every class K_α

$$\omega_i(K_\alpha) \equiv \omega_j(K_\alpha) \qquad (\mathrm{mod}\ \mathfrak{p}).$$

As is well known, the primitive idempotent element e_i of Λ corresponding to the character χ_i is expressed in the form

$$(7) \qquad e_i = \frac{1}{g}\sum_{\alpha=1}^{m} z_i\chi_i(G_\alpha^{-1})K_\alpha .$$

We set

$$E_\sigma = \sum_i{}' e_i = \frac{1}{g}\sum_{\alpha=1}^{m}(\sum_i{}' z_i\chi_i(G_\alpha^{-1}))K_\alpha ,$$

where the sum extends over those i for which the χ_i belong to a block B_σ. If we set

$$b_\alpha = \frac{1}{g}\sum_i{}' z_i\chi_i(G_\alpha^{-1}),$$

then $b_\alpha = 0$ for any p-singular class K_α [5]. We may assume that $K_1, K_2, \cdots\cdots, K_{m^*}$ are the p-regular classes of $\mathfrak{G}$. We then have

$$(8) \qquad E_\sigma = \sum_{\alpha=1}^{m^*} b_\alpha K_\alpha = \frac{1}{g}\sum_{\alpha=1}^{m^*}(\sum_i{}' z_i\chi_i(V_\alpha^{-1}))K_\alpha ,$$

where V_α is an element in K_α ($\alpha = 1, 2, \cdots\cdots, m^*$). Denote by η_κ the character of the indecomposable constituent of the regular representation of $\mathfrak{G}$ corresponding to φ_κ and by u_κ its degree. Since

$$\sum_i{}' z_i\chi_i(V_\alpha^{-1}) = \sum_\kappa{}' u_\kappa\varphi_\kappa(V_\alpha^{-1}),$$

we see that the b_α ($\alpha = 1, 2, \cdots\cdots, m^*$) are $\mathfrak{p}$-integers of Ω. Observe that $u_\kappa \equiv 0$ (mod p^a) for every κ. Since $\omega_i(E_\sigma) = \sum_{\alpha=1}^{m^*} b_\alpha\omega_i(K_\alpha) = 1$ for any character χ_i in B_σ, we have

$$(9) \qquad \sum_{\alpha=1}^{m^*} b_\alpha^*\omega_i^*(K_\alpha) = 1,$$

where $b_\alpha^* = b_\alpha$ (mod $\mathfrak{p}$). This implies that there exists a koefficient b_α such that $b_\alpha^* \neq 0$. If we set $E_\sigma^* = E_\sigma$ (mod $\mathfrak{p}$), then we see by the above discussion that $E_\alpha^* \neq 0$. Evidently

$$E_\sigma^* = (E_\sigma^*)^2, \qquad E_\sigma^* E_\tau^* = 0 \quad (\sigma \neq \tau)$$

and hence s primitive idempotent elements of Λ^* are given by E_σ^* $(\sigma = 1, 2, \cdots\cdots, s)$.

Theorem 1. *Every block B_σ contains an indecomposable character η_κ of degree $u_\kappa \not\equiv 0 \pmod{p^{a+1}}$.*

Proof. Suppose that $u_\kappa \equiv 0 \pmod{p^{a+1}}$ for all η_κ in B_σ. Then $b_\alpha \equiv 0 \pmod{\mathfrak{p}}$ for $\alpha = 1, 2, \cdots\cdots, m^*$. This gives a contradiction.

We have for any χ_j outside of B_σ

$$\omega_j(E_\sigma) = \sum_{\alpha=1}^{m^*} b_\alpha \omega_j(K_\alpha) \equiv 0 \pmod{\mathfrak{p}}. \tag{10}$$

We then obtain by (9)

Theorem 2. *Two characters χ_i and χ_j belong to the same block if and only if $\omega_i(K_\alpha) \equiv \omega_j(K_\alpha) \pmod{\mathfrak{p}}$ for all p-regular classes K_α.*

Lemma 5. *Let V be a fixed p-regular element of $\mathfrak{G}$. If $\chi_i(V) \equiv 0 \pmod{\mathfrak{p}}$ for all χ_i in B_σ, then $\varphi_\kappa(V) \equiv 0 \pmod{\mathfrak{p}}$ for all φ_κ in B_σ.*

Proof. Denote by y_σ the number of modular characters φ_κ in B_σ. Our assertion follows immediately from the fact that the decomposition matrix D_σ of B_σ has the rank y_σ when it is considered mod $\mathfrak{p}$ [**6**].

Let p^d be the highest power of p dividing one of the number g/z_i with χ_i in B_σ. The exponent d is called the defect of B_σ. In the following we consider a block B_σ of defect d. Since $\omega_i(K_\alpha) = g_\alpha \chi_i(V_\alpha)/z_i = g\chi_i(V_\alpha)/n_\alpha z_i$ are algebraic integers, we have for all p-regular classes K_α with $h_\alpha > d$ and for all χ_i in B_σ

$$\chi_i(V_\alpha) \equiv 0 \pmod{\mathfrak{p}}.$$

Hence it follows from Lemma 5 that $b_\alpha^* = 0$ for all p-regular classes K_α with $h_\alpha > d$. On the other hand, we have $\omega_i(K_\alpha) \equiv 0 \pmod{\mathfrak{p}}$ for all p-regular classes K_α with $h_\alpha < d$ and for all χ_i in B_σ. Consequently we have for any χ_i in B_σ

$$\sum_\alpha b_\alpha^* \omega_i^*(K_\alpha) = 1, \tag{11}$$

where the sum extends over all p-regular classes K_α of defect d. This implies that two characters χ_i and χ_j belonging to blocks of defect d appear in the same block if and only if $\omega_i(K_\alpha) \equiv \omega_j(K_\alpha)$ $\pmod{\mathfrak{p}}$ for all p-regular classes of defect d.

It follows from (11) that there exists a p-regular class K_γ of defect d such that

$$b_\gamma^* \neq 0, \qquad \omega_i^*(K_\gamma) \neq 0 \tag{12}$$

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for any χ_i in B_σ. We have by (12) the

Lemma 6. *A character χ_i belongs to a block of defect d if and only if $\omega_i(K_\alpha) \equiv 0 \pmod{\mathfrak{p}}$ for all p-regular classes K_α with $h_\alpha < d$ and $\omega_i(K_\gamma) \not\equiv 0 \pmod{\mathfrak{p}}$ for at least one p-regular class K_γ of defect d.*

We see further that if $\omega_i(K_\alpha) \not\equiv 0 \pmod{\mathfrak{p}}$ for a p-regular class K_α, then χ_i belongs to a block of defect $d \leqq h_\alpha$.

We consider a character χ_i of degree $z_i \equiv 0 \pmod{p^a}$. We set

$$e_i = \frac{1}{g}\sum_{\alpha=1}^{m} z_i\chi_i(G_\alpha^{-1})K_\alpha = \sum_{\alpha=1}^{m} a_\alpha K_\alpha .$$

Then $a_\alpha \equiv 0 \pmod{\mathfrak{p}}$ for all K_α with $h_\alpha > 0$ since $\chi_i(G_\alpha^{-1}) \equiv 0 \pmod{\mathfrak{p}}$ for G_α in these classes. Hence

$$\sum_\alpha a_\alpha \omega_i(K_\alpha) \equiv 1 \pmod{\mathfrak{p}}, \tag{13}$$

where the sum extends over all K_α of defect 0. Thus we see that there exists a class K_ρ of defect 0 such that $\omega_i(K_\rho) \not\equiv 0 \pmod{\mathfrak{p}}$. Since any class of defect 0 is p-regular, we have by Lemma 6 the following

Lemma 7. *A character χ_i of degree $z_i \equiv 0 \pmod{p^a}$ belongs to a block of defect 0.*

Theorem 3. *Let B be a set of ordinary characters of $\mathfrak{G}$ such that $\sum_{\chi_i \text{ in } B} \chi_i(V)\chi_i(S) = 0$ for any p-regular element V and for any p-singular element S. Then B is a collection of blocks of $\mathfrak{G}$.*[1]

Proof. Denote by B'_σ the set of characters χ_i which lie in both B and B_σ. We then have [**6**, Theorem 6]

$$\sum{}' \chi_i(V)\chi_i(S) = 0,$$

where the sum extends over all χ_i in B'_α. We shall prove that if B'_σ is not empty, then $B'_\sigma = B$, namely, B contains all χ_i in B_σ. For a fixed p-regular element V, we consider a generalized character

$$\theta_V(G) = \sum{}' \chi_i(V)\chi_i(G),$$

where the sum extends over all χ_i in B'_σ. Applying Theorem 17 [**4**] to $\theta_V(G)$, we have $\theta_V(G) = \sum_\kappa s_\kappa(V)\eta_\kappa(G)$. Since the $\chi_i(V)$ are algebraic integers, the $s_\kappa(V)$ are also algebraic integers.[2] This implies that

1) The converse of the theorem is also true. See Theorem VIII [**5**].
2) Cf. the proof of second half of Theorem 17 [**4**].

$$\frac{1}{g}\theta_V(1) = \frac{1}{g}\sum' z_i\chi_i(V) = \frac{1}{g}\sum_\kappa u_\kappa s_\kappa(V)$$

is a $\mathfrak{p}$-integer for any p-regular element V and so

$$E'_\sigma = \frac{1}{g}\sum_{\alpha=1}^{m^*}(\sum' z_i\chi_i(V_\alpha^{-1}))K_\alpha \qquad (\text{mod } \mathfrak{p})$$

is an idempotent element of Λ^*. Since $\omega_j(E'_\alpha) = 1$ for χ_j in B'_σ, $E'_\sigma \not\equiv 0$ (mod $\mathfrak{p}$). Suppose that a character χ_k in B_σ does not appear in B'_σ. Then $\omega_k(E'_\sigma) = 0$. On the other hand, we have $\omega_k(E'_\sigma) \equiv 1$ (mod $\mathfrak{p}$) since $\omega_k(K_\alpha) \equiv \omega_j(K_\alpha)$ (mod $\mathfrak{p}$). This gives a contradiction. Hence if B contains a character χ_i in B_σ, then all characters in B_σ appear in B.

3. Let $\mathfrak{H}$ be any subgroup of $\mathfrak{G}$ and let its order be p^h, $h \geqq 0$. Denote by $\mathfrak{C}(\mathfrak{H})$ the centralizer of $\mathfrak{H}$ in $\mathfrak{G}$ and by $\mathfrak{N}(\mathfrak{H})$ the normalizer of $\mathfrak{H}$ in $\mathfrak{G}$. Let $\mathfrak{N}$ be a subgroup such that

$$\mathfrak{H}\mathfrak{C}(\mathfrak{H}) \subseteq \mathfrak{N} \subseteq \mathfrak{N}(\mathfrak{H}). \tag{14}$$

If K_α^0 is the part of K which lies in $\mathfrak{C}(\mathfrak{H})$, then either $K_\alpha^0 = 0$ or K_α^0 is a sum of complete classes of $\mathfrak{N}$. As was shown in [**2**], we have from (3)

$$K_\alpha^0 K_\beta^0 \equiv \sum_\gamma a_{\alpha\beta\gamma}K_\gamma^0 \qquad (\text{mod } p). \tag{15}$$

Hence the classes K_α with $K_\alpha^0 = 0$ form a basis of an ideal T^* of Λ^*. On the other hand, the $K_\alpha^0 \neq 0$ can be considered as the basis of a subring R^* of the center $\Lambda^*(\mathfrak{N})$ of the modular group ring $\Gamma^*(\mathfrak{N})$ of $\mathfrak{N}$. (15) implies

$$R^* \cong \Lambda^*/T^*. \tag{16}$$

Let E_σ^* be a primitive idempotent element of Λ^* corresponding to B_σ. Suppose that $E_\sigma^* \notin T^*$ and let $\tilde{E}_\sigma^*$ be the element of R^* corresponding to E_σ^* (mod T^*) in (16). Then $\tilde{E}_\sigma^*$ is a sum of primitive idempotent elements of $\Lambda^*(\mathfrak{N})$. We denote by $\tilde{B}^{(\sigma)}$ the collection of blocks of $\Lambda^*(\mathfrak{N})$ determined by $\tilde{E}_\sigma^*$. If a block $\tilde{B}_\tau$ of $\Lambda^*(\mathfrak{N})$ is contained in $\tilde{B}^{(\sigma)}$, then we say that $\tilde{B}_\tau$ determines the block B_σ of Λ^*. We have for χ_i in B_σ and $\tilde{\chi}_\rho$ in $\tilde{B}_\tau$

$$\omega_i(K_\alpha) \equiv \sum_\mu \tilde{\omega}_\rho(\tilde{K}_\mu) \qquad (\text{mod } \mathfrak{p}), \tag{17}$$

where $\tilde{K}_\mu$ ranges over all classes of $\mathfrak{N}$ which lie in K_α and whose

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elements belong to the centralizer $\mathfrak{C}(\mathfrak{H})$ of $\mathfrak{H}$. If K_α belongs to T^*, then

$$\omega_i(K_\alpha) \equiv 0 \pmod{\mathfrak{p}}. \tag{18}$$

Lemma 8. *If $E_\sigma^* \in T^*$, then there is a class K_α in T^* such that $\omega_i(K_\alpha) \not\equiv 0 \pmod{\mathfrak{p}}$ for χ_i in B_σ, and conversely.*

As was shown in section 2, there is a p-regular class K_γ of defect d such that $b_\gamma^* \neq 0$ and $\omega_i^*(K_\gamma) \neq 0$ for χ_i in B_σ of defect d. Let $\mathfrak{H}$ be a subgroup of $\mathfrak{G}$ which is not conjugate to a subgroup of the defect group $\mathfrak{H}_\gamma$ of K_γ and let p^h be its order. Choose $\mathfrak{N}$ in (14) as the normalizer $\mathfrak{N}(\mathfrak{H})$ of $\mathfrak{H}$. Our assumption implies that $K_\gamma^0 = 0$ and hence K_γ lies in T^*. Since $\omega_i(K_\gamma) \not\equiv 0 \pmod{\mathfrak{p}}$, it follows from Lemma 8 that $E_\sigma^* \in T^*$. Consequently $b_\alpha^* = 0$ for any class K_α outside of T^*. We then have

Theorem 4. *Let $E_\sigma^* = \sum_{\alpha=1}^{m^*} b_\alpha^* K_\alpha$ be a primitive idempotent element of A^* corresponding to a block B_σ and let $b_\gamma^* \neq 0$, $\omega_i^*(K_\gamma) \neq 0$ for χ_i in B_σ. If $b_\alpha^* \neq 0$, then $\mathfrak{H}_\alpha \subseteq \mathfrak{H}_\gamma$.*

The defect group $\mathfrak{H}_\gamma$ of K_γ in Theorem 4 is called the defect group of the block B_σ. Theorem 4 implies that the defect group of B_σ is uniquely determined by B_σ if we consider conjugate subgroups of $\mathfrak{G}$ as not essentially different. The defect group of B_σ will be denoted by $\mathfrak{D}_\sigma$. It follows that $A_\sigma^* = A^* E_\sigma^* \subseteq \mathfrak{Z}(\mathfrak{D}_\sigma)$.

Corollary 1. *Let B_σ be a block of defect d with the defect group $\mathfrak{D}$. Then $\sum_\alpha b_\alpha \omega_i(K_\alpha) \equiv 1 \pmod{\mathfrak{p}}$ for χ_i in B_α, where the sum extends over all p-regular classes K_α with $\mathfrak{H}_\alpha = \mathfrak{D}$.*

Corollary 2. *Two characters χ_i and χ_j belonging to blocks with the defect group $\mathfrak{D}$ appear in the same block if and only if $\omega_i(K_\alpha) \equiv \omega_j(K_\alpha) \pmod{\mathfrak{p}}$ for all p-regular classes K_α with $\mathfrak{H}_\alpha = \mathfrak{D}$.*

It follows from (18) that if $\omega_i(K_\alpha) \not\equiv 0 \pmod{\mathfrak{p}}$ for χ_i in B_σ with the defect group $\mathfrak{D}$, then $\mathfrak{D} \subseteq \mathfrak{H}_\alpha$.

Lemma 9. *If $\mathfrak{G}$ contains a normal subgroup $\mathfrak{H}$ of order p^d, $d > 0$, then all blocks of $\mathfrak{G}$ have at least the defect d.*

Proof. Since every block B_σ of $\mathfrak{G}$ contains at least one character of $\mathfrak{G}/\mathfrak{H}$, our assertion is proved readily.

Theorem 5. *The defect group $\mathfrak{D}$ of a block B_σ is a maximal normal p-subgroup of the normalizer $\mathfrak{N}(\mathfrak{D})$ of $\mathfrak{D}$ in $\mathfrak{G}$.*

Proof. Choose $\mathfrak{N}$ in (14) as the normalizer $\mathfrak{N}(\mathfrak{D})$. Since there exists a p-regular class K_γ with the defect group $\mathfrak{H}_\gamma = \mathfrak{D}$ such that

$\omega_i(K_\gamma) \not\equiv 0 \pmod{\mathfrak{p}}$ for χ_i in B_σ and since K_γ contains only one class $\tilde{K}_\mu$ of $\mathfrak{N}(\mathfrak{D})$ which consists of elements of $\mathfrak{C}(\mathfrak{D})$, we have by (17)

$$\omega_i(K_\gamma) \equiv \tilde{\omega}_\rho(\tilde{K}_\mu) \not\equiv 0 \qquad \pmod{\mathfrak{p}}$$

for any $\tilde{\chi}_\rho$ in a block $\tilde{B}_\tau$ of $\mathfrak{N}(\mathfrak{D})$ corresponding to B_σ. Hence it follows from Lemmas 6 and 9 that the defect group of $\tilde{B}_\tau$ is $\mathfrak{D}$. We then see by Lemma 9 that $\mathfrak{D}$ is a maximal normal p-subgroup of $\mathfrak{N}(\mathfrak{D})$.

Let $\mathfrak{H}$ be a normal subgroup of $\mathfrak{G}$ and let its order be p^h, $h > 0$. We choose $\mathfrak{N}$ in (14) now as the normalizer $\mathfrak{N}(\mathfrak{H}) = \mathfrak{G}$. Since $\mathfrak{C}(\mathfrak{H})$ is a normal subgroup of $\mathfrak{G}$, if $K_\alpha^0 \neq 0$, then $K_\alpha^0 = K_\alpha$. The classes K_α such that $K_\alpha^0 \neq 0$ form a basis of a subring R^* of Λ^*. We then have

$$\text{(19)} \qquad \Lambda^* = R^* + T^*, \qquad R^* \cong \Lambda^*/T^*.$$

Since the defect group of every block B_σ of $\mathfrak{G}$ contains $\mathfrak{H}$,[1] no E_σ^* lies in T^* and hence $E_\sigma^* \in R^*$. Consequently T^* is contained in the radical of Λ^*. This, combined with Theorem 4, yields the

Lemma 10. *Let $\mathfrak{H}$ be a normal p-subgroup of $\mathfrak{G}$ and let B_σ be a block of $\mathfrak{G}$ with the defect group $\mathfrak{H}$. Then*

$$E_\sigma^* = \sum_\alpha b_\alpha^* K_\alpha,$$

where the sum extends over the p-regular classes K_α with $\mathfrak{H}_\alpha = \mathfrak{H}$.

Now we can prove the following

Theorem 6. *$\mathfrak{G}$ possesses r blocks of defect d with the defect group $\mathfrak{D}$ if and only if $\mathfrak{N}(\mathfrak{D})$ possesses r blocks of defect d (with the defect group $\mathfrak{D}$).*

Theorem 7. *If $\mathfrak{G}$ contains a normal p-subgroup $\mathfrak{H}$ and if the centralizer $\mathfrak{C}(\mathfrak{H})$ of $\mathfrak{H}$ in $\mathfrak{G}$ is also a p-group, then $\mathfrak{G}$ possesses only one block.*[2]

Proof. The subring R^* of Λ^* in (19) can be considered as the subring of the center $\Lambda^*(\mathfrak{C}(\mathfrak{H}))$ of $\Gamma^*(\mathfrak{C}(\mathfrak{H}))$. Hence, by our hypothesis, R^* contains only one primitive idempotent element. Since any primitive idempotent element of Λ^* is contained in R^*, we see that Λ^* is completely primary.

4. We arrange $\varphi_\kappa(V_\alpha)$, $\eta_\kappa(V_\alpha)$ in matrix form

1) See Lemma 1 [3].
2) This is an improvement of Lemma 2 [3].

$$\Phi = (\varphi_\kappa(V_\alpha)), \qquad H = (\eta_\kappa(V_\alpha))$$

(κ row index, α column index; $\kappa, \alpha = 1, 2, \cdots\cdots, m^*$). We have by [6]

$$(20) \qquad |\Phi| \not\equiv 0 \qquad (\mathrm{mod}\ \mathfrak{p}).$$

We denote by $\bar{\Phi}'$ the transepose of $\bar{\Phi} = (\varphi_\kappa(V_\alpha^{-1}))$. Then

$$\bar{\Phi}'H = (n_\alpha \delta_{\alpha\beta}) = T.$$

We set $Y = HT^{-1} = (\eta_\kappa(V_\alpha)/n_\alpha)$, where the $\eta_\kappa(V_\alpha)/n_\alpha$ are $\mathfrak{p}$-integers [5, Theorem V]. Since $\bar{\Phi}'Y = I$, we have by (20)

$$(21) \qquad |Y| \not\equiv 0 \qquad (\mathrm{mod}\ \mathfrak{p}).$$

If the block B_σ contains y_σ modular characters φ_κ, then we can choose a minor $|\Phi_\sigma|$ of degree y_σ containing y_σ rows of Φ corresponding to B_σ such that $|\Phi_\sigma| \not\equiv 0$ (mod $\mathfrak{p}$). It can be shown that it is possible to make this selection of y_σ columns for each block B_σ in such a manner that every column appears for one and only one block. Hence we may assume without restriction that

$$(22) \qquad \Phi = \begin{pmatrix} \Phi_1 & & & * \\ & \Phi_2 & & \\ & & \ddots & \\ * & & & \Phi_s \end{pmatrix}, \qquad |\Phi_\sigma| \not\equiv 0 \qquad (\mathrm{mod}\ \mathfrak{p}).$$

In what follows we shall denote by $K_{\sigma,1}, K_{\sigma,2}, \cdots\cdots, K_{\sigma,y_\sigma}$ the p-regular classes of $\mathfrak{G}$ associated with B_σ by the preceding construction. We set

$$Y_\sigma = (\eta_\kappa(V_{\sigma,\alpha})/n_{\sigma,\alpha}).$$

We then have

$$(23) \qquad |Y_\sigma| = |\Phi_\sigma|\,|C_\sigma| / \prod_{\alpha=1}^{y_\sigma} n_{\alpha,\sigma},$$

where C_σ is the Cartan matrix of B_σ. Since $|Y_\sigma|$ is $\mathfrak{p}$-integer and $|C_\sigma|$ is a power of p, it follows from (22) and (23) that $|C_\sigma| \geqq \prod_{\alpha=1}^{y_\sigma} p^{h_{\alpha,\sigma}}$. On the other hand, we have

$$|C| = \prod_\sigma |C_\sigma| = \prod_\sigma \left(\prod_{\alpha=1}^{y_\sigma} p^{h_{\sigma,\alpha}}\right).$$

Hence $|C_\sigma| = \prod_{\alpha=1}^{y_\sigma} p^{h_{\sigma,\alpha}}$. This implies $|Y_\sigma| \not\equiv 0 \pmod{\mathfrak{p}}$. If we set $\Phi'_\sigma Y_\sigma = Q_\sigma$, then $|Q_\sigma| \not\equiv 0 \pmod{\mathfrak{p}}$ and

(24) $$\Phi'_\sigma C_\sigma \Phi_\sigma = Q_\sigma T_\sigma,$$

where $T_\sigma = (n_{\sigma,\alpha}\delta_{\alpha\beta})$. If we work in the ring $\mathfrak{o}^*$ of $\mathfrak{p}$-integers of Ω, we obtain by (24) the following

Theorem 8. *Let $K_{\sigma,1}, K_{\sigma,2}, \cdots\cdots, K_{\sigma,v_\sigma}$ be the p-regular classes of $\mathfrak{G}$ associated with the block B_σ. Then the elementary divisors of C_σ are the powers of p with the exponents $h_{\sigma,\alpha}$ $(\alpha = 1, 2, \cdots\cdots, y_\sigma)$.*

We see easily that our theorem is identical with [**1**, Theorem 2].

Now we set

$$M = \begin{pmatrix} Y_1 & & & 0 \\ & Y_2 & & \\ & & \ddots & \\ 0 & & & Y_s \end{pmatrix}.$$

Then

$$S = \bar{\Phi}'M = \left(\frac{1}{n_{\sigma,\alpha}} \sum_{\varphi_\kappa \text{ in } B_\sigma} \varphi_\kappa(V_{\tau,\beta}^{-1})\eta_\kappa(V_{\sigma,\alpha})\right)$$
$$= \left(\frac{1}{n_{\sigma,\alpha}} \sum_{\chi_i \text{ in } B_\sigma} \chi_i(V_{\tau,\beta}^{-1})\chi_i(V_{\sigma,\alpha})\right) = (s(\tau, \beta;\ \sigma, \alpha)),$$

where each row is characterized by a pair of indices τ, β and each column is characterized by a pair of indices σ, α. Since $|Y_\sigma| \not\equiv 0$ (mod $\mathfrak{p}$), we have

(25) $$|S| \not\equiv 0 \pmod{\mathfrak{p}}.$$

By the simple computation we see that

(26) $$K_{\sigma,\alpha}E_\sigma = \sum_{\tau,\beta}\left(\frac{1}{n_{\sigma,\alpha}} \sum_{\chi_i \text{ in } B_\sigma} \chi_i(V_{\sigma,\alpha})\chi_i(V_{\tau,\beta}^{-1})\right)K_{\tau,\beta}$$
$$= \sum_{\tau,\beta} s(\tau, \beta;\ \sigma, \alpha)K_{\tau,\beta}.$$

Let $\mathfrak{D}$ be the defect group of B_σ. Since $E_\sigma^* \in \mathfrak{Z}(\mathfrak{D})$, if $\mathfrak{H}_{\tau,\beta} \not\subseteq \mathfrak{D}$, then $s(\tau, \beta;\ \sigma, \alpha) \equiv 0$ (mod $\mathfrak{p}$). We see also that $s(\tau, \beta;\ \sigma, \alpha) \equiv 0$ (mod $\mathfrak{p}$) if $\mathfrak{H}_{\tau,\beta} \not\subseteq \mathfrak{H}_{\sigma,\alpha}$. It follows from (26) that

(27) $$(K_{1,1}E_1, K_{1,2}E_1, \cdots\cdots, K_{s,v_s}E_s) = (K_{1,1}, K_{1,2}, \cdots\cdots, K_{s,v_s})S.$$

(25) implies that $\{K_{\sigma,\alpha}E_\sigma^*\}$ are linearly independent. If

$$K_{1,1}E_1;\ K_{1,2}E_1,\ \cdots\cdots,\ K_{1,y_1}E_1,\ K_{2,1}E_2,\ \cdots\cdots,\ K_{s,y_s}E_s$$

are taken in a suitable order corresponding to (4), we have by the above argument

$$P^{-1}SP \equiv \begin{pmatrix} W_a & & & 0 \\ & W_{a_1} & & \\ & & \ddots & \\ * & & & W_{a_k} \end{pmatrix} \pmod{\mathfrak{p}},$$

where P denotes a suitable permutation matrix. $K_{\tau,\beta}$ and $K_{\sigma,\alpha}$ range over only the p-regular classes of defect a_i in $W_{a_i} = (s^*(\tau, \beta;\ \sigma, \alpha))$, where $s^*(\tau, \beta;\ \sigma, \alpha) = s(\tau, \beta;\ \sigma, \alpha) \pmod{\mathfrak{p}}$. (25) yields

(28) $$|W_{a_i}| \neq 0.$$

Moreover we may assume by (5) that

$$W_{a_i} = \begin{pmatrix} \Delta_1 & & & 0 \\ & \Delta_2 & & \\ & & \ddots & \\ 0 & & & \Delta_t \end{pmatrix},$$

where $K_{\tau,\beta}$ and $K_{\sigma,\alpha}$ range over only the p-regular classes with the defect group $\mathfrak{H}_\nu^{(a_i)}$ in $\Delta_\nu = (s^*(\tau, \beta:\ \sigma, \alpha))$. Hence

(29) $$|\Delta_\nu| \neq 0.$$

Consequently we have the

Lemma 11. *There exists at least one class $K_{\tau,\beta}$ with $\mathfrak{H}_{\tau,\beta} = \mathfrak{H}_{\sigma,\alpha}$ such that $s^*(\tau, \beta;\ \sigma, \alpha) \neq 0$ in (26).*

Theorem 9. *Let $K_{\sigma,\alpha}$ $(\alpha = 1, 2, \cdots\cdots, y_\sigma)$ be the p-regular classes of $\mathfrak{G}$ associated with a block B_σ of defect d with the defect group $\mathfrak{D}$. Then $\mathfrak{H}_{\sigma,\alpha} \subseteq \mathfrak{D}$ $(\alpha = 1, 2, \cdots\cdots, y_\sigma)$ and there exists exactly one class $K_{\sigma,\alpha}$ with $\mathfrak{H}_{\sigma,\alpha} = \mathfrak{D}$.*

Proof. Lemma 11 implies $\mathfrak{H}_{\sigma,\alpha} \subseteq \mathfrak{D}$. It follows from (27) that E_σ^* is expressed as a linear combination of $K_{\sigma,\alpha}E_\sigma^*$ $(\alpha = 1, 2, \cdots\cdots, y_\sigma)$. Hence there exists at least one class, say, $K_{\sigma,1}$ with $\mathfrak{H}_{\sigma,1} = \mathfrak{D}$. Suppose that $\mathfrak{H}_{\sigma,2} = \mathfrak{D}$. Then $\chi_i(V_{\sigma,1}) \equiv \chi_i(V_{\sigma,2}) \equiv 0 \pmod{\mathfrak{p}}$ for χ_i in B_σ whose degree z_i is divisible by p^{a-d+1}. Let χ_j and χ_l be two characters in B_σ such that $z_j \not\equiv 0$, $z_l \not\equiv 0 \pmod{p^{a-d+1}}$. Since $\omega_j(V_{\sigma,1}) \equiv \omega_l(V_{\sigma,1}) \pmod{\mathfrak{p}}$, we have $\chi_j(V_{\sigma,1}) \equiv \frac{z_j}{z_l}\chi_l(V_{\sigma,1}) \pmod{\mathfrak{p}}$. Similarly, $\chi_j(V_{\sigma,2}) \equiv$

$\frac{z_j}{z_i}\chi_i(V_{\sigma,2})$ (mod $\mathfrak{p}$). We set $Z_\sigma = (\chi_i(V_{\sigma,\alpha}))$, where row index i ranges over all χ_i in B_σ. It follows by the above argument that Z_σ has the rank $r < y_\sigma$ when it is considered mod $\mathfrak{p}$. But this gives a contradiction and hence the theorem is proved.

Corollary 1. *Let C_σ be the Cartan matrix of a block B_σ of defect d. C_σ has one elementary divisor p^d while all other elementary divisors of C_σ are powers of p with exponents smaller than d.*

Corollary 2. *If there exist k p-regular classes K_α in $\mathfrak{G}$ with $\mathfrak{H}_\alpha = \mathfrak{D}$, then $\mathfrak{G}$ possesses at most k blocks with the defect group $\mathfrak{D}$.*

If B_σ is a block of defect 0, then B_σ consists of exactly one ordinary character χ_i and one modular character φ_κ. Moreover $\chi_i(V) = \varphi_\kappa(V)$ for any p-regular element V. Since χ_i with $z_i \equiv 0 \pmod{p^a}$ belongs to a block of defect 0, χ_i forms a block B_σ of its own.

Theorem 10. *Let $K_{\sigma,\alpha}$ $(\alpha = 1, 2, \cdots\cdots, y_\sigma)$ be the p-regular classes of $\mathfrak{G}$ associated with a block B_σ with the defect group $\mathfrak{D}$ and let $r_{\sigma,\rho\nu}$ be the number of classes $K_{\sigma,\alpha}$ with $\mathfrak{H}_{\sigma,\alpha} = \mathfrak{H}_\nu^{(\rho)}$ $(\rho = a_i)$. Then $r_{\sigma,\rho\nu}$ depends only the subgroup $\mathfrak{H}_\nu^{(\rho)}$ and the block B_σ.*

Proof. Let $K'_{\sigma,\alpha}$ $(\alpha = 1, 2, \cdots\cdots, y_\sigma)$ be a second set of p-regular classes of $\mathfrak{G}$ associated with B_σ and let $r'_{\sigma,\rho\nu}$ be the number of classes $K'_{\sigma,\alpha}$ with $\mathfrak{H}'_{\sigma,\alpha} = \mathfrak{H}_\nu^{(\rho)}$. We have for $K_{\sigma,\alpha}$ with $\mathfrak{H}_{\sigma,\alpha} = \mathfrak{H}_\nu^{(\rho)}$

$$K_{\sigma,\alpha}E_\sigma^* = \sum_\beta t_{\alpha\beta}K'_{\sigma,\beta}E_\sigma^*. \tag{30}$$

Here the sum extends over only those $K'_{\sigma,\beta}E_\sigma^*$ with $\mathfrak{H}'_{\sigma,\beta} \subseteq \mathfrak{H}_\nu^{(\rho)}$, since $K_{\sigma,\alpha}E_\sigma^* \in \mathfrak{Z}(\mathfrak{H}_\nu^{(\rho)})$. Moreover there exists at least one $K'_{\sigma,\beta}$ with the defect group $\mathfrak{H}_\nu^{(\rho)}$ such that $t_{\alpha\beta} \neq 0$. Suppose that $r_{\sigma,\rho\nu} > r'_{\sigma,\rho\nu}$. Then we can conclude that the $r_{\sigma,\rho\nu}$ $K_{\sigma,\alpha}E_\sigma^*$ are linearly dependent (mod $\mathfrak{Z}_{\rho-1}$) and hence $|\Delta_\nu| = 0$. This contradicts (29), so that $r_{\sigma,\rho\nu} \leqq r'_{\sigma,\rho\nu}$. Similarly, we have $r_{\sigma,\rho\nu} \geqq r'_{\sigma,\rho\nu}$ and hence $r_{\sigma,\rho\nu} = r'_{\sigma,\rho\nu}$.

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