

Mathematical Journal of Okayama University

Volume 33, Issue 1

1991

Article 13

JANUARY 1991

Global dimension and a question of Armendariz

Jae Keol Park*

Klaus W. Roggenkamp[†]

*Busan National University

[†]Busan National University

Copyright ©1991 by the authors. *Mathematical Journal of Okayama University* is produced by
The Berkeley Electronic Press (bepress). <http://escholarship.lib.okayama-u.ac.jp/mjou>

GLOBAL DIMENSION AND A QUESTION OF ARMENDARIZ

JAE KEOL PARK and KLAUS W. ROGGENKAMP

M. Auslander has shown that the global dimension of a ring Λ is bounded by the projective dimension of Λ/I for left ideals I of $\Lambda[\text{AU}]$. For noetherian rings satisfying a polynomial identity, Rainwater [R] restricted I to being a two sided maximal ideal. In this note we consider a somewhat dual statement. More precisely :

The aim is to give a positive answer to the following question of Armendariz in case of semiprimary rings and classical orders :

(1) Let Λ be a noetherian ring with a polynomial identity. If the injective dimension of all maximal two sided ideals is bounded by n , does n then also bound the global dimension of Λ ?

We shall prove the

(2) **Proposition.** (i) *If Λ is semiprimary, then the question (1) has a positive answer.*

(ii) *Let R be a Dedekind domain with the field of fractions K and Λ an R -order in a finite dimensional semisimple K -algebra A ; i. e., Λ is finitely generated over R as module. Then the question (1) has a positive answer.*

The **proof** is done in several steps :

Step 1 (for(ii)): Reduction to the case where R is complete. Since Λ is an R -order, we note that a two sided Λ -ideal \mathfrak{M} is maximal if and only if all its completions \mathfrak{M}_p at the maximal ideals p of R coincide with Λ_p , except for one, p_0 , where \mathfrak{M}_{p_0} is a maximal two sided ideal of Λ_{p_0} . Moreover, each such set of local data determines a unique maximal two sided ideal of Λ . In addition, if M and N are Λ -lattices ; i. e., left Λ -modules, which are finitely generated and projective over R , then

$$\text{Ext}_{\Lambda}^n(M, N) = \bigoplus_{p \in \max(R)} \text{Ext}_{\Lambda_p}^n(M_p, N_p)$$

where the subscript denotes the completion. Since every finitely generated module has a resolution by Λ -lattices, this formula also holds for finitely generated Λ -modules. Thus it is enough to prove the proposition in case R is complete. The importance of this is that in the complete situation we have the Krull-Schmidt theorem available.

Hence we assume from now on that R is complete and Λ is basic. In the semiprimary case the Krull-Schmidt theorem always holds, and projective covers exist.

Step 2: Assume that Λ is not local. Let $\Lambda = \bigoplus_{i=1}^m P_i$, where $\{P_i\}_{1 \leq i \leq m}$ are the indecomposable projective Λ -modules with $J_i = \text{rad}(P_i)$; note that $m > 1$. The maximal two sided ideals of Λ are then $\mathfrak{M}_i = \bigoplus_{j \neq i} P_j \oplus J_i$. According to the hypothesis,

$$0 = \text{Ext}_\Lambda^{n+1}(-, \mathfrak{M}_i) = \bigoplus_{j \neq i} \text{Ext}_\Lambda^{n+1}(-, P_j) \oplus \text{Ext}_\Lambda^{n+1}(-, J_i).$$

Since $m > 1$, we conclude

$$\text{Ext}_\Lambda^{n+1}(-, P_j) = 0 = \text{Ext}_\Lambda^{n+1}(-, J_i), \quad 1 \leq i \leq m.$$

Recall that given a short exact sequence of Λ -modules

$$(3) \quad 0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0,$$

we get functorially exact sequences

$$(4) \quad \begin{array}{ccccccc} \text{Ext}_\Lambda^n(-, X') & \rightarrow & \text{Ext}_\Lambda^n(-, X) & \rightarrow & \text{Ext}_\Lambda^n(-, X'') & \rightarrow & 0 \\ & & \text{Ext}_\Lambda^{n+1}(-, X') & \rightarrow & \text{Ext}_\Lambda^{n+1}(-, X) & & \end{array}$$

and

$$(5) \quad \begin{array}{ccccccc} \text{Ext}_\Lambda^n(X'', -) & \rightarrow & \text{Ext}_\Lambda^n(X, -) & \rightarrow & \text{Ext}_\Lambda^n(X', -) & \rightarrow & 0 \\ & & \text{Ext}_\Lambda^{n+1}(X'', -) & \rightarrow & \text{Ext}_\Lambda^{n+1}(X, -) & & \end{array}$$

Applying this to the exact sequence

$$0 \rightarrow J_i \rightarrow P_i \rightarrow S_i \rightarrow 0,$$

where S_i is the associated simple module, we conclude

$$\text{Ext}_\Lambda^{n+1}(-, S_i) = 0$$

and by induction—using (4)—we get

$$\text{Ext}_\Lambda^{n+1}(-, L) = 0$$

for every finitely generated artinian Λ -module L . (Since Λ/\mathfrak{M} is finitely generated artinian for every maximal two sided ideal \mathfrak{M} , we could quote a result of Rainwater [R] to conclude that the global dimension of Λ is bounded by n ; however, the arguments below give a very short direct proof.) Now let M be a Λ -lattice, and let π be a parameter of R . Then the exact sequence

$$0 \rightarrow M \xrightarrow{\pi} M \rightarrow M/\pi \cdot M \rightarrow 0,$$

where $\cdot \pi$ is multiplication by π , gives rise to the exact sequence (cf. (4))

$$\text{Ext}_\Lambda^n(-, M) \xrightarrow{\mu} \text{Ext}_\Lambda^n(-, M) \rightarrow \text{Ext}_\Lambda^{n+1}(-, M/\pi \cdot M);$$

however, $M/\pi \cdot M$ is artinian and finitely generated, and so the map μ , which is induced from $\cdot \pi$ is surjective. Since π is a central element, μ is still multiplication by π , which generates the radical of R . But for each finitely generated Λ -module X , $\text{Ext}_\Lambda^n(X, M)$ is finitely generated over R . Thus Nakayama's lemma implies $\text{Ext}_\Lambda^n(X, M) = 0$, and so $\text{Ext}_\Lambda^n(-, M) = 0$. If now Y is an arbitrary finitely generated left Λ -module, then we have an exact sequence

$$0 \rightarrow t(Y) \rightarrow Y \rightarrow Y/t(Y) \rightarrow 0,$$

where $t(Y)$ is the R -torsion submodule of Y and $Y/t(Y)$ is a Λ -lattice. Again the sequence (4) implies that for every finitely generated left Λ -module Y ,

$$\text{Ext}_\Lambda^{n+1}(-, Y) = 0$$

on finitely generated modules. This implies that Λ has global dimension bounded by n . In fact, the global dimension of any ring is bounded by the projective dimension of the finitely generated modules, and for a noetherian ring the syzygies of finitely generated modules are finitely generated, thus the above formula guarantees that the global dimension is bounded by n .

Assume now that Λ is semiprimary. In that case $\Lambda/\text{rad}(\Lambda)$ is semisimple artinian and $\text{rad}(\Lambda)$ is nilpotent; consequently every finitely generated left Λ -module has finite Loewy length. The above argument has shown that $\text{Ext}_\Lambda^{n+1}(-, S_i) = 0$ for every simple module S_i . But then $\text{Ext}_\Lambda^{n+1}(-, \Lambda/\text{rad}(\Lambda)) = 0$, and quoting a result of Eilenberg [E, Theorem 12] we conclude $\text{gl.dim.}(\Lambda) \leq n$.

Step 3: Λ is local semiprimary. Let E be the injective envelope of the unique simple Λ -module. The radical of Λ now is the unique maximal two sided ideal, which has injective dimension n . So we get a minimal injective resolution

$$0 \rightarrow \text{rad}(\Lambda) \rightarrow E_1 \rightarrow \dots \rightarrow E_{n-1} \xrightarrow{\alpha} E_n \xrightarrow{\beta} E_{n+1} \rightarrow 0,$$

where $\{E_i\}_{1 \leq i \leq n+1}$ are injective Λ -modules. The natural map $\text{Im}(\alpha) \rightarrow E_n$ is an essential monomorphism and hence $\text{Soc}(E_n)$, the socle of E_n , is contained in $\text{Im}(\alpha) = \text{Ker}(\beta)$. Thus we obtain a factorization of β as

$$E_n \rightarrow E_n/\text{Soc}(E_n) \rightarrow E_{n-1}.$$

An argument with the Loewy lengths now shows that this can not happen. This also proves that for a local semiprimary ring the only modules of finite injective dimension are the injective ones.

Step 4 : An R -order Λ has also injective lattices ; i.e., Λ -lattices, which are injective with respect to the category of left Λ -lattices. They are the modules $Q_i^* = \text{Hom}_R(Q_i, R)$, where Q_i are the indecomposable projective right Λ -modules. For a Λ -lattice M we write $\text{LExt}_\Lambda^n(-, M)$ for the functor $\text{Ext}_\Lambda^n(-, M)$ restricted to the category of Λ -lattices. Let now Λ be a local R -order, where R is a complete Dedekind domain. Then arguments similar to the ones above show

$$\text{Ext}_\Lambda^{n+1}(-, M) = 0 \text{ iff } \text{LExt}_\Lambda^n(-, M) = 0.$$

Since Λ is a local order, it has a unique indecomposable injective left Λ -lattice $E = \text{Hom}_R(\Lambda, R)$, and if $\text{rad}(\Lambda)$ has injective dimension bounded by n , then

$$0 = \text{Ext}_\Lambda^{n+1}(-, \text{rad}(\Lambda)) = \text{LExt}_\Lambda^n(-, \text{rad}(\Lambda)),$$

and so we have a minimal injective resolution in the category of left Λ -lattices

$$(6) \quad 0 \rightarrow \text{rad}(\Lambda) \rightarrow E^{(s_1)} \rightarrow \dots \rightarrow E^{(s_{n-1})} \rightarrow E^{(s_n)} \rightarrow 0, \quad s_i \in \mathbf{N}.$$

Applying the exact functor $\text{Hom}_R(-, R)$, we get a minimal projective resolution for $\text{Hom}_R(\text{rad}(\Lambda), R)$, which ends at the left hand side as

$$0 \rightarrow \Lambda^{(s_n)} \xrightarrow{\beta} \Lambda^{(s_{n-1})} \rightarrow \dots.$$

Since this is part of a minimal projective resolution, the map β factorizes via $\text{rad}(\Lambda)^{(s_{n-1})}$. Since $\text{rad}(\Lambda/\pi \cdot \Lambda) = \text{rad}(\Lambda)/\pi \cdot \Lambda$, and since reduction modulo π is exact, we get a monomorphism

$$\beta^\wedge : (\Lambda/\pi \cdot \Lambda)^{(s_n)} \rightarrow (\Lambda/\pi \cdot \Lambda)^{(s_{n-1})},$$

which factorizes via $\text{rad}(\Lambda/\pi \cdot \Lambda)^{(s_{n-1})}$. Now an argument as above with the Loewy lengths shows that this is impossible. Hence there can not be any Λ -lattice of finite injective dimension. *This completes the proof of the proposition.*

Remarks. 1) The fact that Λ is an R -order in a semisimple K -algebra is only used to pass from the global to the local situation : The ext-formula linking global and local extensions of lattices.

2) The arguments in Step 2 are totally general for rings, where the

Krull-Schmidt theorem is valid for finitely generated modules, thanks to Rainwater's argument [R].

3) That for a local perfect ring finitely generated modules of finite injective dimension must be injective should be a general fact; however, we were not able to prove this.

Acknowledgements. This note arose during the first author's visit in Stuttgart in January 1990, and he greatly appreciates the support from the Universität Stuttgart and from the KOSEF. The second author greatly appreciates the support of KOSEF and DFG.

REFERENCES

- [A] E. P. ARMENDARIZ : Private communication, 1988.
- [AU] M. AUSLANDER : See J. ROTMAN : An Introduction to Homological Algebra, Academic Press, New York, 1979, p. 237.
- [CE] H. CARTAN and S. EILENBERG : Homological Algebra, Princeton Univ. Press, 1956.
- [E] S. EILENBERG : Homological dimension and syzygies, Ann. Math. **64** (1956), 328–336.
- [G] K. R. GOODEARL : Ring Theory — Nonsingular rings and modules, Marcel Dekker, New York, 1976.
- [PR] J. K. PARK and K. W. ROGGENKAMP : Hereditary rings with a polynomial identity, Comm. Algebra **18** (1990), 3223–3243.
- [R] J. RAINWATER : Global dimension of fully bounded noetherian rings, Comm. Algebra **15** (1987), 2143–2156.
- [RM] I. REINER : Maximal Orders, Academic Press, New York, 1975.
- [RH] K. W. ROGGENKAMP and V. HUBER-DYSON : Lattices over Orders I, Lecture Notes Math. **115**, Springer-Verlag, Berlin-Heidelberg-New York, 1970.
- [RO] K. W. ROGGENKAMP : Lattices over Orders II, Lecture Notes Math. **142**, Springer-Verlag, Berlin-Heidelberg-New York, 1970.

DEPARTMENT OF MATHEMATICS
 BUSAN NATIONAL UNIVERSITY
 BUSAN 609-735, KOREA

MATHEMATISCHES INSTITUT B
 UNIVERSITÄT STUTTGART
 STUTTGART, GERMANY

(Received December 11, 1990)