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Group rings with nilpotent unit groups

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GROUP RINGS WITH NILPOTENT UNIT GROUPS

Dedicated to Professor Keizo Asano on his 60th birthday

KAORU MOTOSE and HISAO TOMINAGA

In their paper [1], J. M. Bateman and D. B. Coleman stated the following: *Let F be a field, and G a finite group.* (a) *Let the group ring FG be semi-simple. Then the unit group of FG is nilpotent if and only if G is abelian.* (b) *Let the characteristic of F be a prime p dividing the order of G. Then the unit group of FG is nilpotent if and only if G is a nilpotent group such that the q-Sylow subgroup is abelian for every prime $q\neq p$.* Unfortunately, they used there an incorrect lemma, which should be corrected as follows:

Lemma 1. *Let S be a ring with 1, and N a nilpotent ideal of S. If S/N is commutative and $[N, S]=\{[x, y]=xy-yx \mid x\in N, y\in S\}$ is contained in N^2 then the unit group $S^\cdot$ of S is nilpotent. In particular, if S/N^2 is commutative then $S^\cdot$ is nilpotent.*

Proof. We define $(u, v)=u^{-1}v^{-1}uv$ for $u, v\in S^\cdot$, and inductively $(u_1, \cdots, u_n)=((u_1, \cdots, u_{n-1}), u_n)$ for $u_1, \cdots, u_n\in S^\cdot$. Then, we see by induction that for $n>1$

$(u_1, \cdots, u_n)-1=(u_1, \cdots, u_{n-1})^{-1}u_n^{-1}[(u_1, \cdots, u_{n-1})-1, u_n]\in N^{n-1}$. Since N is nilpotent, it follows that $S^\cdot$ is nilpotent.

Remark. Let $D=Q+Qi+Qj+Qij$ be the quaternion division algebra over the rational number field Q. We consider the ring $S=\{\begin{pmatrix} a & 0 \\ d & a\end{pmatrix} \mid d\in D, a\in C=Q+Qi\}$. Then, $N=\{\begin{pmatrix} 0 & 0 \\ d & 0\end{pmatrix} \mid d\in D\}$ is an ideal of S with $N^2=0$ and S/N is isomorphic to the field C. For an arbitrary integer n, we have $\begin{pmatrix} 1 & 0 \\ nj & 1\end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & i\end{pmatrix}\begin{pmatrix} 1 & 0 \\ nj & 1\end{pmatrix}^{-1}\begin{pmatrix} i & 0 \\ 0 & i\end{pmatrix}^{-1}=\begin{pmatrix} 1 & 0 \\ nj & 1\end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & i\end{pmatrix}\begin{pmatrix} 1 & 0 \\ -nj & 1\end{pmatrix}\begin{pmatrix} -i & 0 \\ 0 & -i\end{pmatrix}$ $=\begin{pmatrix} 1 & 0 \\ 2nj & 1\end{pmatrix}$, whence one will easily see that $S^\cdot$ is not nilpotent. This example shows that the assumption $[N, S]\subseteq N^2$ is indispensable in Lemma 1. Next, we shall claim that the converse of Lemma 1 is not true. Evidently the radical N of the ring $S=\{\begin{pmatrix} a & 0 \\ b & c\end{pmatrix} \mid a, b, c\in \mathrm{GF}(2)\}$ coincides with $\{\begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} | b \in \mathrm{GF}(2)\}$ and S/N is isomorphic to $\mathrm{GF}(2) \oplus \mathrm{GF}(2)$. Moreover, $S^\cdot = \{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\}$ is commutative and $[N, S] \neq 0 = N^2$.

Now, we shall prove the following:

Proposition. *Let S be a semi-primary ring with* 1 *such that the radical R is nilpotent and $S^* = S/R^2$ is commutative, and let G be a finite group. If* (1) *G is commutative or* (2) *S/R is of prime characteristic p and G is a nilpotent group such that the q-Sylow subgroup is commutative for every prime $q \neq p$, then the unit group of the group ring SG is nilpotent.*

Proof. We consider the ring homorphism λ of $\mathfrak{S} = SG$ onto the group ring $\mathfrak{S}^* = S^*G$ deifined by $\sum_{\sigma \in G} s_\sigma \sigma \longmapsto \sum_{\sigma \in G} s_\sigma^* \sigma$ where s_σ^* is the residue class of $s_\sigma \in S$ modulo R^2. Evidently, RG is nilpotent and Ker $\lambda = R^2 G = (RG)^2$. If G is oommutative then $\mathfrak{S}/(RG)^2$ is isomorphic to the commutative ring $\mathfrak{S}^*$, and hence $\mathfrak{S}^\cdot$ is nilpotent by Lemma 1. It remains therefore to prove the case (2). Let $G = H \times P$ where P is a p-group and H an abelian group of order prime to p. By [3; Corollary 1], the respective radicals $\mathfrak{R}$ and $\mathfrak{R}^*$ of SP and S^*P are $\sum_{\rho \in P} S(\rho - 1) + RP$ and $\sum_{\rho \in P} S^*(\rho - 1) + (R/R^2)P$. Moreover, noting that $(\mathfrak{R}H)^2$ contains Ker λ and $\lambda((\mathfrak{R}H)^2) = (\mathfrak{R}^*H)^2$, we see that $\mathfrak{S}/(\mathfrak{R}H)^2$ is isomorphic to $\mathfrak{S}^*/(\mathfrak{R}^*H)^2$. As H is contained in the center of $\mathfrak{S}^*$ and $[\sigma, \tau] = [\sigma - 1, \tau - 1] \in (\mathfrak{R}^*H)^2$ for every $\sigma, \tau \in P$, it is easy to see that ($\mathfrak{S}^*/(\mathfrak{R}^*H)^2$ and hence) $\mathfrak{S}/(\mathfrak{R}H)^2$ is commutative. As was noted in the proof of [3; Corollary 1], $\mathfrak{R}^k$ is contained in RP for some k, which implies that $\mathfrak{R}H$ is nilpotent. Hence, again by Lemma 1, $\mathfrak{S}^\cdot$ is nilpotent.

As is well-known, the unit group of the complete $n \times n$ matrix ring D_n over a division ring D is not nilpotent for $n > 1$. Moreover, it is known that the unit group of a division ring D is not solvable if D is not commutative ([2] or [4]). Accordingly, we readily obtain

Lemma 2. *If the unit group of an artinian semi-simple ring S is nilpotent then S is commutative.*

Combining the proposition with Lemma 2, we can generalize somewhat the statement cited at the opening of this note.

Theorem. *Let S be an artinian semi-simple ring, and G a finite group. Then, the unit group of the group ring SG is nilpotent if and* *only if S is commutative and either* (1) *G is abelian or* (2) *S is of prime characteristic p and G is a nilpotent group such that the q-Sylow subgroup is commutative for every prime* $q \neq p$.

Proof. By the validity of our proposition, it suffices to prove the only if part. If S is simple and the characteristic of S does not divide the order of G then, as is well-known, SG is artinian semisimple. Hence, S and G must be commutative by Lemma 2. Next, if S is a simple ring of prime characteristic p dividing the order of G then by the fact noted just above S and every q-Sylow subgroup of G are commutative ($q \neq p$). Now, combining those above, we can readily complete our proof.

Although the converse of our proposition is not valid, we obtain the following:

Corollary. *Let S be a semi-primary ring with* 1, *and G a finite group. If the unit group of SG is nilpotent then the residue class ring* $\bar{S}$ *of S modulo its radical R is commutative and either* (1) *G is commutative or* (2) $\bar{S}$ *is of prime characteristic p and G is a nilpotent group such that the q-Sylow subgroup is commutative for each prime* $q \neq p$.

Proof. We consider the ring homomorphism μ of SG onto $\bar{S}G$ defined by $\sum_{\sigma \in G} s_\sigma \sigma \longmapsto \sum_{\sigma \in G} \bar{s}_\sigma \sigma$ where $\bar{s}_\sigma$ is the residue class of s_σ modulo R. As is well known, Ker $\mu = RG$ is contained in the radical of SG, and so the unit group of $\bar{S}G$ is nilpotent. Hence, the corollary is evident by our theorem.

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Added in proof. After the submission of this manuscript, the writers have learned that K. Eldridge has submitted a short paper that correct the error in [1]. Also P. B. Bhattacharya and S. K. Jain [Notices of Amer. Math. Soc. 16 (1969), 562] have presented a counterexample to the lemma of [1], provided another proof for the theorem of [1], and shown that if S is an artinian ring with 1 and G is a finite group such that the unit group of SG is nilpotent then SG satisfies a polynomial identity $(xy-yx)^n=0$. Indeed, the last is an easy consequence of our theorem.