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3-PRIMARY COMPONENTS OF STABLE HOMOTOPIES OF CP^∞

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1. Introduction

Let p be an odd prime, A the mod p Steenrod algebra, M and N the reduced cohomology groups of CP^∞ and $K(Z_p, 1)$, that is, $M = Z_p[y]/Z_p$, $M_k = Z_p[y^{p-1}]\cdot y^k$, $0<k\leqq p-2$, $M_0 = Z_p[y^{p-1}]/Z_p$, deg $(y)=2$; $N = (Z_p[\beta x]\otimes E(x))/Z_p$, where $E(x)$ is the exterior algebra with one generator x of degree 1 and β is Bochstein operator with $\beta\cdot x=\beta x$, N_k the left A-submodule of N and the Z_p-module generated by $x(\beta x)^{k+i(p-1)}$, $(\beta x)^{k+1+i(p-1)}$, $i\geqq 0$, $(p-2\geqq k\geqq 0)$. Then M and N are isomorphic to the direct sums of M_k and N_k $(0\leqq k\leqq p-2)$ as left A-modules.

The main purpose of this paper is to determine 3-primary components of stable homotopies of CP^∞ by Adams spectral sequence. Liulevicius [5] determined $\pi_i^S(CP^\infty; p)$, $i\leqq 12$ $(p=3)$, $i\leqq 6p-4$ $(p\geqq 5)$ and de Carvalho [4] determined $\pi_i^S(CP^\infty; 3)$, $i\leqq 17$. Theorem 2. 15. in Liulevicius [5] imply that $\pi_{2i}^S(CP^\infty)$, $i\geqq 1$, is isomorphic to the direct sum of Z and a torsion group and $\pi_{2i+1}^S(CP^\infty)$ is a torsion group.

We want to determine odd primary components of stable homotopies of CP^∞ and $K(Z_p, 1)$ in the paper to appear.

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2. $\mathrm{Ext}_A^0(M, Z_p)$

If $i=\sum_{u=0}^m i_u p^u$, $0\leqq i_u<p$, then the p-th set $[i]$ and the p-th number $\#[i]$ of i is defined by

$[i]=(i_0, i_1, \cdots, i_m, 0, 0, \cdots)$ (ordered set), $\#[i]=\sum_{u=0}^m i_u$.

If $j=\sum_{u=0}^n j_u p^u$, $0\leqq j_u<p$, then $[i]\geqq[j]$ means the condition: $\sum_{u=0}^t i_u\leqq\sum_{u=0}^t j_u$, $0\leqq t\leqq \max\{m, n\}$.

Proposition 2.1.

$A\cdot y^j = Z_p\{y^i;\, i\geqq j,\ [i]\geqq[j],\ \#[i]=\#[j]+c(p-1),\ c\geqq 0\}$.

Proof. Let e be an integer such that $e>j$, $[e]\geqq[j]$ and $\#[e]=\#[j]+c(p-1)$, for some $c\geqq 0$, and $e=\sum_{u=0}^q e_u p^u$, $0\leqq e_u<p$. Then for some a

and b, $e_u=j_u(u>a$ or $u<b)$, $e_a>j_a$, $e_b<j_b$, $a>b$; and for some f, $e_{b+1}>0$ (set $f=b+1$), or $e_f>0=e_{f-1}=\cdots=e_{b+1}$, $a\geqq f\geqq b+2$. We wish to determine e' such that $P^{e'}y^{r-(p-1)e'}=cy^{r'}$, $0\neq c\in Z_p$.

(2. 1) $a>f$.

(2. 2) $a=f$ and $e_a\geqq j_a+2$ (set $d=a-1$)

(2, 3) $a=f$ and $e_a=j_a+1$; $j_{a-1}=\cdots=j_{d+1}=p-1>j_d$ $(a-2\geqq d\geqq b)$, $j_{a-1}=\cdots=j_b=p-1$ (set $d=b$), or $j_{a-1}<p-1$, (set $d=a-1$).

In (2. 1) let $e'=p^{f-1}+\cdots+p^d$. In (2. 3) with $j_d>0$ and $d>b$, let $e'=p^{d-1}$. If otherwise, let $e'=p^d$.

Proposition 2.2. *We obtain the following direct sum decompositions*:

$M_k=\bar{A}\cdot M_k+Z_p\{y^{(k+1)p^n-1};\ n\geqq 0\}$. (*Replace with* $n>0$ *if* $k=0$.)

$M=\bar{A}\cdot M+Z_p\{y^{kp^n-1};\ 0<k<p,\ n\geqq 0,\ (k,n)\neq(1,0)\}$.

Let A-maps $f_1\colon \bar{A}\longrightarrow M_{p-2}$, $f_2\colon L_1=\ker f_1\longrightarrow M_0$, and $f_3\colon \bar{A}\longrightarrow N_{p-2}$ be such that $f_1(P^a)=(-1)^a y^{a(p-1)-1}$, $f_2(\beta P^a)=(-1)^a y^{a(p-1)}$, $f_3(P^a)=(-1)^a x(\beta x)^{a(p-1)-1}$, $f_3(\beta P^a)=(-1)^a(\beta x)^{a(p-1)}$, and f_1, f_2 and f_3 are trivial on other admissible monomials. Let $L_2=\ker f_2=\ker f_3$.

Let B be an algebra and L a left B-module, $x_1,\cdots,x_n$ are in L. Then we denote ay $B\{x_1,\cdots,x_n\}$ and $B\{\{x_1,\cdots,x_n\}\}$ the B-submodule and B-free module generated by $x_1,\cdots,x_n$, respectively. Sometimes we denote by $P^{a_1}\cdots P^{a_n}$ by $P(a_1,\cdots a_n)$, when $(a_1,\cdots,a_n)$ is complicated.

3. $\mathrm{Ext}_A^0(L_i,\ Z_p)$, $i=1,2$

Theorem 3.1. *We obtain the following* (*not necessarily direct sum*) *representations*:

$L_1=\bar{A}\cdot L_1+Z_p\{\beta;\ \beta P(p^n+\cdots+p+1),\ n>0;\ P(p^a,p^b),\ a>b\geqq 0\}$.

$L_2=\bar{A}\cdot L_2+Z_p\{\beta;\ P(p^a,p^b),\ a>b\geqq 0\}$.

Proof. The first half is proved by Propositions 3.3, 3.4, 3.5, 3.6, and the second half is proved by Propositions 3.3, 3.4, 3.5, replaced $\bar{A}\cdot L_1$ with $\bar{A}\cdot L_2$, and 3.7. The following formulas are used to prove these propositions.

Proposition 3.2.

(1) $a\geqq 2$,

$P(p^{n+1},\ (a-1)p^{n+1}+p^n)=(a-1)P(ap^{n+1}+p^n)+P(ap^{n+1},p^n)$,

(2) $m\geqq n\geqq 0$, $a\geqq 2$,

$P(p^m+\cdots+p^n,(a-1)p^m)=aP(ap^m+p^{m-1}+\cdots+p^n)+\sum_{i=n}^{m}\sum_t a_i(-1)^{t_1}(t_1-t_2,\cdots,t_{q-1}-t_q,t_q)P(ap^m+p^{m-1}+\cdots+p^i-p^{n-1}+t,\ p^{i-1}+\cdots+p^{n-1}-t)$, *where there is only the first summand in the left hand side in the case* $n=0$, $(t_1-t_2, \cdots, t_q)$ *stands for the polynomial coefficient,* $a_i=a$, $(i\neq n)$, $a_n=1$, *and* t *runs over the set of* t *satisfying the following condition*;

$t=0$ *or*

$t=t_1p^{n-2}+\cdots+t_qp^{n-q-1}$, $p>t_1\geqq\cdots\geqq t_q>0$, $n>p>0$. $\cdots(*)$

(*The following* (3) *and* (4) *are special cases of* (2) *and in them* t *runs over the same set as in* (2).)

(3) $P(p^n, (a-1)p^n)=aP(ap^n)+P(ap^n-p^{n-1}, p^{n-1})$
$+\sum_t(-1)^{t_1}(t_1-t_2, \cdots, t_{q-1}-t_q, t_q)P(ap^n-p^{n-1}+t, p^{n-1}-t)$, $a\geqq 2$.

(4) $e>m\geqq n$,
$P(p^m+\cdots+p^n, p^e+\cdots+p^{m+1})=P(p^e+\cdots+p^n)$
$+\sum_{i=n}^{m}\sum_t(-1)^{t_1}(t_1-t_2, \cdots, t_{q-1}-t_q, t_q)P(p^e+\cdots+p^i-p^{n-1}+t, p^{i-1}+\cdots$
$+p^{n-1}-t)$.

(5) $a>0, n>0$, $P(ap^{n+1}-p^n, ap^n)$
$=\sum_t(-1)^{1+t_1}(t_1-t_2, \cdots, t_{q-1}-t_q, t_q)P(ap^{n+1}-p^n+t, ap^n-t)$,

where t *runs over the set of* t *satisfying the condition*:

$t=t_1p^{n-1}+\cdots+t_qp^{n-q}$, $p>t_1>t_2\geqq\cdots\geqq t_q>0$; *or*
$t_1=\cdots=t_q=1$, $(n\geqq q>0)$.

(6) $P^1\beta P^a=a\beta P^{a+1}+P^{a+1}\beta$, $a>0$.

Proof. We prove the formula (3) here. The others are similar.

$$P(p^n, (a-1)p^n)=\sum_{t=0}^{p^{n-1}}(-1)^tQ_tP(ap^n-t, t),$$

where $Q_t\equiv\binom{p^{n-1}+(p-1)(p^{n-1}-t)-1}{p(p^{n-1}-t)}$ (mod p).

If $0<t<p^{n-1}$, then we can represent

$$p^{n-1}-t=t_1p^{m_1}+\cdots+t_qp^{m_q},\ n-2\geqq m_1>\cdots>m_q\geqq 0,\ 0<t_i<p.$$

We denote $m_0=n-1$. If there is r such that $m_{r-1}-2\geqq m_r\geqq m_{r+1}+2$, $q>r>0$, then

$$Q_t=\binom{\cdots+(t_r-1)p^{m_r+1}+(p-t_r)p^{m_r}+\cdots}{\cdots+\qquad t_rp^{m_r+1}\qquad+\cdots}\equiv 0 \pmod{p}.$$

If there is r such that $m_{r-1}-2\geqq m_r=m_{r+1}+1$, then

$$Q_t=\binom{\cdots+t_rp^{m_r+1}+(t_{r+1}-t_r)p^{m_r}+\cdots}{\cdots+\quad t_rp^{m_r+1}+t_{r+1}p^{m_r}\quad+\cdots}\equiv 0,$$

(or replaced $t_{r+1}-t_r$ with $t_{r+1}-t_r-1$) in the case $t_{r+1}\geqq t_r$, or

$$Q_t=\binom{\cdots+(t_r-1)p^{m_r+1}+(p+t_{r+1}-t_r)p^{m_r}+\cdots}{\cdots+\quad t_rp^{m_r+1}\quad+\quad t_{r+1}p^{m_r}\quad+\cdots}\equiv 0,$$

(or replaced $p+t_{r+1}-t_r$ with $p+t_{r+1}-t_r-1$) in the case $t_{r+1}\leqq t_r$. Therefore $Q_t\not\equiv 0 \pmod p$ implies the condition (*).

Proposition 3.3. *If* $a\geqq b\geqq 2$, $b\not\equiv 0 \pmod p$, *then* $P(ap^{n+1}, bp^n)\in \bar{A}\cdot L_1$.

Proof. By Proposition 3.2 (1), (3),

$P(p^{n+1}, (a-1)p^{n+1}+p^n, (b-1)p^n)\equiv(a-1)P(ap^{n+1}+p^n, (b-1)p^n)$
$+P(ap^{n+1}, bp^n) \pmod{\bar{A}\cdot L_1}$.

It is reduced to Proposition 3.4. that the first summand of the right hand side is in $\bar{A}\cdot L_1$. Thus the proof is completed.

Proposition 3.4. *If* $a\geqq bp$, $a\not\equiv 0 \pmod p$, $b>0$, *then* $P(ap^n, bp^n)\in \bar{A}\cdot L_1$.

Proof. By induction on n. But this inductive hypothesis is of irregular type in the sense that if the proposition holds for $m\leqq n-2$, then it holds for n. So we must prove it for $n=0, 1$ at the first step. We point out that $a\geqq bp$, $a\not\equiv 0 \pmod p$ implies $a-1\geqq bp$.

By Proposition 3.2. (3)

$P(p^n, (a-1)p^n, bp^n)=aP(ap^n, bp^n)+P(ap^n-p^{n-1}, bp^n+p^{n-1})$
$+\sum_t e_t P(ap^n-p^{n-1}+t, bp^n+p^{n-1}-t) \pmod{\bar{A}\cdot L_1}$,

where $0\neq e_t\in Z_p$, and t runs over the set of t satisfying the condition (*). The second summand is not admissible only in the case $a=bp$, when we can make it a sum of admissible monomials by applying Proposition 3.2. (3) to it.

Proposition 3.5. *If* $a\not\equiv 0 \pmod p$, $a\neq 1$, $m>n$, *then* $P(ap^m, p^n)\in \bar{A}\cdot L_1$.

Proof. By Proposition 3.2. (2)

$P(p^m, (a-1)p^m, p^n)=aP(ap^m, p^n)+P(ap^m-p^{m-1}, p^{m-1}, p^n)$
$+\sum c_t P(ap^m-p^{m-1}+t, p^{m-1}-t, p^n)$,

where $c_t\in Z_p$. In the second summand $P(p^{m-1}, p^n)$ is not admissible only in the case $m-1=n$, when

$P(ap^{n+2}-p^n, p^n, p^n)\equiv 2P(ap^{n+2}-p^n, 2p^n)+\pmod{\bar{A}\cdot L_1}$

and is reduced to Proposition 3.4. The third summand is not admissible if $m=n+1$; $m=n+2$; $3\leqq m-n\leqq q+1$, $t_1=\cdots=t_{m-n-2}=p-1$, when

$P(ap^m-p^{m-1}+t,\ p^{m-1}-t,\ p^n)+cP(ap^m-p^{m-1}+t,\ p^{m-1}+p^n-t)$, $0\neq c\in Z_p$. The problem is reduced to Proposition 3.3. if $m=n+2$, $q=1$, $t_1=1$; $3\leqq m-n=q+1$, $t_1=\cdots=t_{q-1}=p-1, t_q=1$; and is reduced to Proposition 3.4. if otherwise.

Proposion 3.6. *$P^b\notin \bar{A}\cdot L_1$ implies $b=p^n+\cdots+p+1$, for some $n\geqq 0$.*

Proof. Let $b=ap^{n+1}+ip^n+p^{n-1}+\cdots+p+1$, $a\geqq 0$, $0<i<p$, $n\geqq 0$. Then $P^{p^n}\beta P^b=i\beta P^{p^n+b}$ (mod $\bar{A}\cdot L_1$).

Proposition 3.7. *$a\geqq pb+1, b>0$ implies $P^a\beta P^b\in\bar{A}\cdot L_2$.*

Proof. By Proposition 3.2. (4), we have $P^1\beta P^{a-1}P^b=(a-1)\beta P^aP^b+P^a\beta P^b$.

4. $\mathrm{Ext}_A^1(M, Z_p)$

We denote by $h_{n,k}$ the element in $\mathrm{Ext}_A^0(M_k, Z_p)$ and $\mathrm{Ext}_A^0(M, Z_p)$ corresponding to $y^{(k+1)p^n-1}\in M_k$. In particular we denote $h_{n,p-2}=\underline{h}_n$. We utilize the exact sequences (4.3) and (4.4) induced by (4.1) and (4.2) for determining $\mathrm{Ext}_A(M_k, Z_p)$, $k=0,\ p-2$;

$$(4.1)\quad 0\longrightarrow L_1\longrightarrow \bar{A}\xrightarrow{f_1} M_{p-2}\longrightarrow 0$$

$$(4.2)\quad 0\longrightarrow L_2\longrightarrow L_1\xrightarrow{f_2} M_0\longrightarrow 0$$

$$(4.3)\quad \cdots\longleftarrow \mathrm{Ext}_A^{s+1,t-2}(M_{p-2}, Z_p)\xleftarrow{\partial_s}\mathrm{Ext}_A^{s,t}(L_1, Z_p)\xleftarrow{I_s}\mathrm{Ext}_A^{s+1,t}(Z_p, Z_p)$$
$$\xleftarrow{F_s}\mathrm{Ext}_A^{s,t-2}(M_{p-2}, Z_p)\longleftarrow\cdots$$

$$(4.4)\quad \cdots\longleftarrow \mathrm{Ext}_A^{s+1,t-1}(M_0, Z_p)\xleftarrow{\partial_s}\mathrm{Ext}_A^{s,t}(L_2, Z_p)\xleftarrow{I_s}\mathrm{Ext}_A^{s,t}(L_1, Z_p)$$
$$\xleftarrow{F_s}\mathrm{Ext}_A^{s,t-1}(M_0, Z_p)\longleftarrow\cdots$$

We denote by α_0, $g_{a,b}$ the elements in $\mathrm{Ext}_A^0(L_1, Z_p)$ corresponding to β, $P(p^a, p^b)$ in L_1. Then in (4.3), $F_0(\underline{h}_n)=-h_n$, $I_0(\alpha_0)=\alpha_0$.

For the next proposition we introduce some notion. In general, let A be an algebra over a field, M a left A-module and $\bar{M}$ the K-submodule of M determined by $M=\bar{M}+\bar{A}\cdot M$ (direct sum).

Definition. Let $R(M)=\ker(A\otimes\bar{M}\longrightarrow M)$ and $R(M)=\bar{A}\cdot R(M)+\bar{R}(M)$ (direct sum), then generators in $\bar{R}(M)$ are called *basic relations* and we have

$$\mathrm{Tor}_1^A(K, M)\cong\bar{R}(M),\quad \mathrm{Ext}_A^1(M, K)\cong\bar{R}(M)^*$$

Proposition 4.1. *$P(p^e+\cdots+p^n)y^{(p-1)p^{e+1}-1}=0$ is an basic relation in M.*

Proof. We have only to prove the following:

(1) There is no Adem relation such that

$P^a\beta P^b=c\beta P(p^m+\cdots+p+1)+\cdots,\ 0\neq c\in Z_p,\ a\leqq pb.$

(2) $P^aP^b=cP(p^m+\cdots+p^n)+\cdots$ (*Adem retion*), $m>n$, implies $a=p^m+\cdots+p^i$, $m\geqq i>n$, for some i.

(3) We have the following three equalities:

$P(p^e+\cdots+p^i-p^{n-1}+t,\ p^{i-1}+\cdots+p^{n-1}-t)y^{(p-1)p^{e+1}-1}=0$

(for t satisfying the condition (*) in Proposition 3.2. (2))

$P(p^e+\cdots+p^n)y^{(p-1)p^{e+1}-1}\neq 0,$

$P(p^{m_r}+\cdots+p^n,\ p^{m_{r-1}}+\cdots+p^{m_r+1},\ \cdots,\ p^{m_1}+\cdots+p^{m_2+1},$

$p^e+\cdots p^{m_1+1})y^{(p-1)p^{e+1}-1}\neq 0,\ (e>m_1>\cdots>m_r\geqq n,\ r\geqq 1).$

For the proof of (1) and (2), express $a=a'p^{m+1}+a''p^m$, $0\leqq a''<p$, $a'\geqq 0$ and $a=a'p^{m+1}+ip^m+p^{m-1}+\cdots+p+1$, $2\leqq i<p$, $a'\geqq 0$, $m<$n.

We denote by e_n in $\mathrm{Ext}_A^1(M_{p-2}, Z_p)$ corresponding to the basic relation in Proposition 4.1. In particular $e_1=\underline{h}_0\alpha_0$.

Proposition 4.2. ([6]). *Indecomposable elements in* $\mathrm{Ext}_A^2(Z_p, Z_p)$ *are in the following*:

(*Massey product*)		(*representative in the cobar construction*)
$-\langle h_{i+1}, h_i, h_i\rangle$	$\ni\mu_i\ni$	$[\xi_2^{p^i}\mid\xi_1^{p^i}]+1/2[\xi_1^{p^{i+1}}\mid\xi_1^{2p^i}]$
$-\langle h_{i+1}, h_{i+1}, h_i\rangle$	$\ni\nu_i\ni$	$[\xi_1^{p^{i+1}}\mid\xi_2^{p^i}]+1/2[\xi_1^{2p^{i+1}}\mid\xi_1^{p^i}]$
$-\langle h_0, h_0, \alpha_0\rangle$	$\ni\rho\ni$	$[\xi_1\mid\tau_1]+1/2[\xi_1^2\mid\tau_0]$
	$\lambda_i\ni$	$\sum_{j=1}^{p-1}\frac{1}{p}\binom{p}{j}[\xi_1^{p^i(p-j)}\mid\xi_1^{p^ij}]$

In case $p=3$,

$-2\langle h_i, h_i, h_i\rangle\ \ni\lambda_i\ni\ [\xi_1^{2\cdot 3^i}\mid\xi_1^{3^i}]+[\xi_1^{3^i}\mid\xi_1^{2\cdot 3^i}]$

Proposition 4.3. *In* $\mathrm{Ext}_A(M_k, Z_p)$, $0\leqq k\leqq p-2$, $h_{n,k}\alpha_0\neq 0$, $n>0$, $=0$, $n=0$: $h_{n,k}h_i\neq 0$, $i>n$, $i\leqq n-2$: $h_{n,k}h_i\neq h_{i,k}h_n$, $i\leqq n-2$.

Theorem 4.4. *A basis for* $\mathrm{Ext}_A^1(M_{p-2}, Z_p)$ *is*

generator	*degree*	*range of indices*
$\underline{h}_i h_j$	$2(p-1)(p^i+p^j)-2$	$0\leqq i<j,\ i-2\geqq j\geqq 0$
$\underline{h}_i\alpha_0$	$2(p-1)p^i-1$	$i\geqq 0$
e_i	$2p^{i+1}-3$	$i>0$
$\underline{\rho}_2$	$4p-5$	
$\underline{\lambda}_i$	$2p^{i+1}(p-1)-2$	$i\geqq 0$
$\underline{\mu}_i$	$2p^i(p-1)(p+2)-2$	$i\geqq 0$
$\underline{\nu}_i$	$2p^i(p-1)(2p+1)-2$	$i\geqq 0$

Proposition 4.5. *The following elements in* $\mathrm{Ext}_A^3(M, Z_p)$ *form a linearly independent set;*

$\underline{h}_i h_j h_k$, $i+2\leqq j\leqq k-2$, $(\underline{h}_j h_i h_k, \underline{h}_k h_i h_j)$, $\underline{h}_i\mu_j$, $i\neq j$, $j-1$, $j+2$ $(\underline{\mu}_j h_i)$, $\underline{h}_i\nu_j$, $i\neq j\pm 1$, $j+2(\underline{\nu}_j h_i)$, $\underline{h}_i\rho_2$, $i\geqq 2$, $(\underline{\rho}_2 h_i)$, $\underline{h}_i h_j\alpha_0$, $0\neq i\leqq j-2$, $(\underline{h}_j h_i\alpha_0)$, $\underline{\nu}_i\alpha_0$, $i\neq 0$, $\underline{\mu}_i\alpha_0$, $i\neq 0$, $\underline{\lambda}_i\alpha_0$, $\underline{h}_i\alpha_0^2$, $i\neq 0$. *Here so is the set, for example, replaced* $\underline{h}_i h_j h_k$ *with* $\underline{h}_j h_i h_k$ *or* $\underline{h}_k h_i h_j$ *in the parentheses.*

Proof. By the similar result on $\mathrm{Ext}_A^3(Z_p, Z_p)$ by Liulevicius [6].

5. A formula on the Steenrod algebra

Theorem 5.1. (1) $u\geqq 0$,

$$\bar{A}=A\{\beta;\ P^{p^i},\ i\neq u;\ P(p^{u+1}, p^u)\}+Z_p\{\beta^e P^b;\ e=0,1,\ b=\textstyle\sum_{t=0}^{u} b_t p^t,\ p>b_u \geqq\cdots\geqq b_0\geqq e\geqq 0,\ b_u>0\}.$$

$$\bar{A}=A\{P^{p^i}, i\geqq 0;\ P^1\beta\}+Z_p\{\beta\}.$$

(2) *If we denote by* A *the* mod 2 *Steenrod algebra only in this part, then for* $u\geqq 0$

$$\bar{A}=A\{Sq^{2^i}, i\neq u;\ Sq(2^{u+1}, 1^u)\}+Z_p\{Sq^b;\ b=2^u+\cdots+2^v,\ u\geqq v\geqq 0\}.$$

We denote by K the first summand of the right hand side of (1). All congruences mean "mod K". The proof of this theorem is by Proposition 5.3.

Lemma 5.2. *Let* $b=\sum b_t p^t$, $0\leqq b_t<p$. $P^{p^i}P^b=0\cdot P^{p^i+b}+\cdots$ (*Adem relation*) *if and only if* (1) $b_i=\cdots=b_m<b_{m-1}$, $i>m>0$, (2) $b_i+1=b_{i-1}=\cdots=b_m>b_{m-1}$, $i>m>0$, (3) $b_i+1=b_{i-1}=\cdots=b_0$, (4) $i=0$, $b_0=p-1$, *or* (5) $b_i=p-1$, $b_{i-1}=\cdots=b_0$.

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For b and e in Theorem 5.1, we can construct $\beta^e P^b$ by Lemma 5.2: $c\beta^e P(1, \cdots, 1, p, \cdots, p, \cdots, p^u, \cdots, p^u)$, where $0 \neq c \in Z_p$, and p_i are b_i-fold.

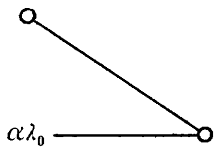
Proposition 5.3. *Let b be as in Theorem* 5.1. (1) $b_{m+1} < b_m$ *for some m implies* $P^b \equiv 0$. (2) $P^{p^{u+1}} P^b \equiv 0$, $P^{p^{u+1}} \beta P^b \equiv 0$. (3) *When we ask all (not necessarily admissible) monomials if they are in K, we can omit $P^{p^i} P^b$ and $P^{p^i} \beta P^b$ $(i \geqq u+2)$.*

Proof. (1) By Lemma 5.2. $P^b \equiv cP(p^m, \cdots, p^m, p^{m+1}, \cdots, p^{m+1}, \cdots, p^u, \cdots, p^u, b') \equiv 0$, $0 \neq c \in Z_p$, where they are (b_m+1)-, b_{m+1}-, $\cdots$, b_{u-1}-, (b_u-1)-fold, $b' = p^u - p^m + \sum_{t-0}^{m-1} b_t p^t$, and $b_u \geqq \cdots \geqq b_m < b_{m-1}$. (2) It is sufficient to prove the following three congruences: $P(p^{u+1}, b) \equiv c_1 P(p^{u+1}+b) P(p^{u+1}+b-p^u, p^u)$, $0 \equiv P(b-p^u, p^{u+1}, p^u) \equiv c_2 P(p^{u+1}+b)$, $0 \equiv P(b, p^{u+1}) + c_3 P(p^{u+1}+b)$, $0 \neq c_j \in Z_p$.

6. Tables

We can determine $\mathrm{Ext}_A(M, Z_3)$ by determining $\mathrm{Ext}_A(Z_3, Z_3)$ and $\mathrm{Ext}_A(L_i, Z_p)$, $i=1, 2$. We denote by $x_{i,j}$ the generators in $\mathrm{Ext}_A(Z_3, Z_3)$ in j-th order of total degree in irreducible generators of cohomolodical dimension $i(\geqq 3)$, except $x_{7,2} = x_{3,1}x_{4,1}$, $x_{8,3} = x_{3,1}x_{5,1}$ and $x_{8,4} = x_{3,1}x_{5,2}$. The partners of α in $\mathrm{Ext}_A(\ \ , Z_3)$ mean $\alpha\lambda_0^i$, $\alpha\lambda_0^i\alpha_0$, $\alpha\lambda_0^i\rho$, $\alpha\lambda_0^i h_0$, $i \geqq 0$ and α', if exists (which is the generator satisfying $\alpha\alpha_0 = \alpha' h_0$).

Horizontal and slanting segments mean "multiplied by α_0 and h_0", respectively, in the tables. Some generators are missing in the table of $\mathrm{Ext}_A(Z_3, Z_3)$ in May [8] and Table 2. is different from May's. $\mathrm{Ext}_A^{s,t}(L_2, Z_3)$, $t-s \leqq 52$, is generated by α_0, $g_{1,0}$, $g_{2,0}$, $b_{1,i}$ $(1 \leqq i \leqq 4)$, $b_{1,1}h_1$, $b_{1,4}\alpha_0$, $b_{1,4}h_0$, and $g_{2,1}\alpha_0^i (0 \leqq i \leqq 2)$, where $g_{i,j}\alpha_0^r$ and $b_{i,j}\alpha_0^r$ are of same digree as $g_{i,j}\alpha_0^r$ and $a_{i,j}\alpha_0^r$ in $\mathrm{Ext}_A(L_1, Z_3)$ and the images of those by the homomorphism induced by the inclusion $L_2 \longrightarrow L_1$. The upper element (small circle) in the form of the right figure in our tables means the generator $\alpha\rho$ in $\mathrm{Ext}_A(\ \ , Z_3)$, if α is a generator in $\mathrm{Ext}_A(\ \ , Z_3)$.



We only show some typical cases to determine the differentials d_r in the Adams spectral sequence.

(1) By factorizations by products: $d_2(\underline{\mu}_0 h_0) = d_2(\underline{h}_0 \mu_0) = \underline{h}_0(d_2\mu_0) = \underline{h}_0\lambda_0\rho = \underline{\lambda}_0\rho h_0$. If $d_2\underline{\mu}_0 = 0$, then $d_2(\underline{\mu}_0 h_0) = 0$, which contradicts the above result. Therefore $d_2\underline{\mu}_0 = \underline{\lambda}_0\rho$. If $d_2\underline{h}_1 = 0$, then $d_2(\underline{h}_1\lambda_0) = 0$, which contradicts the above result, since $\underline{h}_1\lambda_0 = \underline{\mu}_0 h_0$. (Remark: We determined $d_2\underline{h}_1 = \underline{\lambda}_0\alpha_0$ by

purely algebraic method, but Liulevicius [5] did by the definition of d_r on the secondary cohomology operation.) This method is the strongest tool to determine d_r for generators come from $\mathrm{Ext}_A(Z_p, Z_p)$ such as partners of $\underline{\lambda}_0$, $\underline{x}_{5,2}$, $\underline{x}_{8,2}$, $\underline{x}_{11,2}$, $\underline{x}_{14,2}$, $\underline{x}_{3,1}\lambda_0$, $\underline{x}_{8,4}$, $\underline{h}_0 x_{3,1}$, $\underline{h}_0 x_{9,1}$, $\underline{\lambda}_1$, $\underline{\mu}_0$, $\underline{\nu}_0$, etc.

(2) By factorizations by Massey products: $d_2 a_{3,1} = d_2 \langle \underline{x_{3,1}\alpha_0}, h_0, \alpha_0 \rangle = \langle d_2 \underline{x_{3,1}\alpha_0}, h_0, \alpha_0 \rangle = \langle \underline{x}_{5,1} h_0, h_0, \alpha_0 \rangle = \underline{x}_{5,1} \langle h_0, h_0, \alpha_0 \rangle = \underline{x}_{5,1}\rho = \underline{h}_0 x_{3,1}\alpha_0^3$. $d_2 e_{1,1}\alpha_0^3 = d_2 \langle a_{3,1}, h_0, \alpha_0 \rangle = \langle \underline{h}_0 x_{3,1}\alpha_0^3, h_0, \alpha_0 \rangle = \underline{\rho} x_{3,1}\alpha_0^3 = a_{1,1}\alpha_0^5$. Therefore $d_2 e_{1,1} = a_{1,1}\alpha_0^2$.

(3) By secondary cohomology operations: $d_2 \underline{e}_1 = \underline{e}_0 h_1 \alpha_0$. See de Carvalho [4].

Table 1.

i	2	3	4	5	6	7	8	9	10	11	12	13
$\pi_i^S(CP^\infty; 3)$	Z	0	Z	0	Z	0	Z	Z_3	Z	0	$Z+Z_3$	Z_9

14	15	16	17	18	19	20	21	22	23	24	25
Z	Z_3	Z	Z_3	Z	Z_3+Z_3	Z	Z_3	$Z+Z_3$	Z_3	Z	Z_9+Z_3

26	27	28	29	30	31	32	33	34	35
Z	(Z_9)	Z	$Z_9+Z_3+Z_3$	Z	(Z_3)	Z	(Z_3)	Z	$Z_3+Z_3+Z_3$

36	37	38	39	40	41
$Z+Z_3+Z_3$	$Z_3+Z_3+Z_{3^i}$ $(i=3, 4)$	$Z+Z_3+Z_3$	$Z_3+Z_3+Z_{3^i}$ $(i=3, 4, 5)$	$Z+Z_3$	$Z+Z_{3^i}$ $(3\leq i\leq 7)$

42	44	46	48	50
$(Z+Z_3)$	$(Z+Z_3+Z_3)$	$Z+Z_3$	Z	Z

In Table 1., (Z_9) and so on mean the groups which is strongly imagined by the analogy of the algebraic structure; $Z_3+Z_3+Z_{3^i}$ $(i=3, 4)$ means either $Z_3+Z_3+Z_{3^3}$ or $Z_3+Z_3+Z_{3^4}$.

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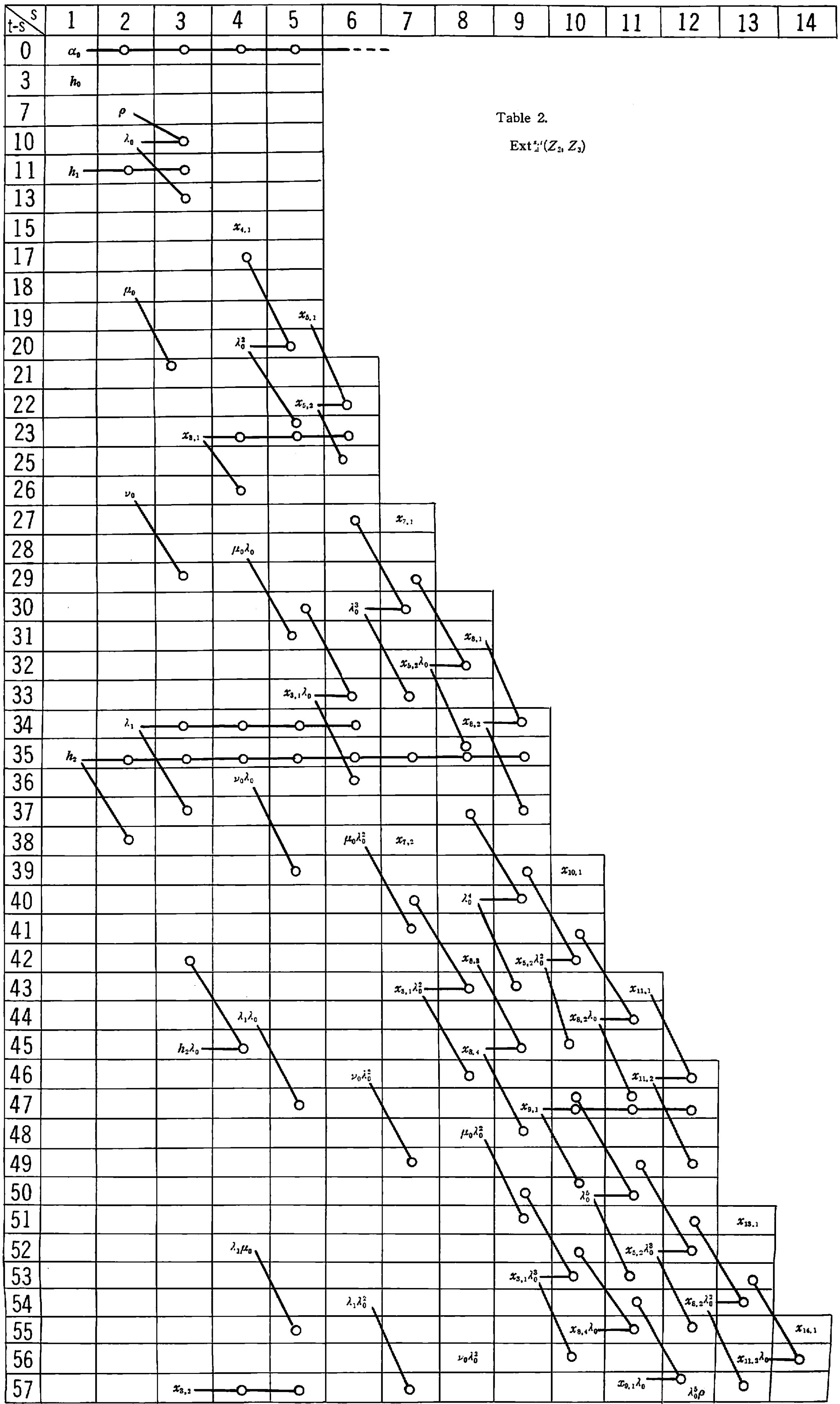

Table 2.
$\mathrm{Ext}_A^{s,t}(Z_3, Z_3)$

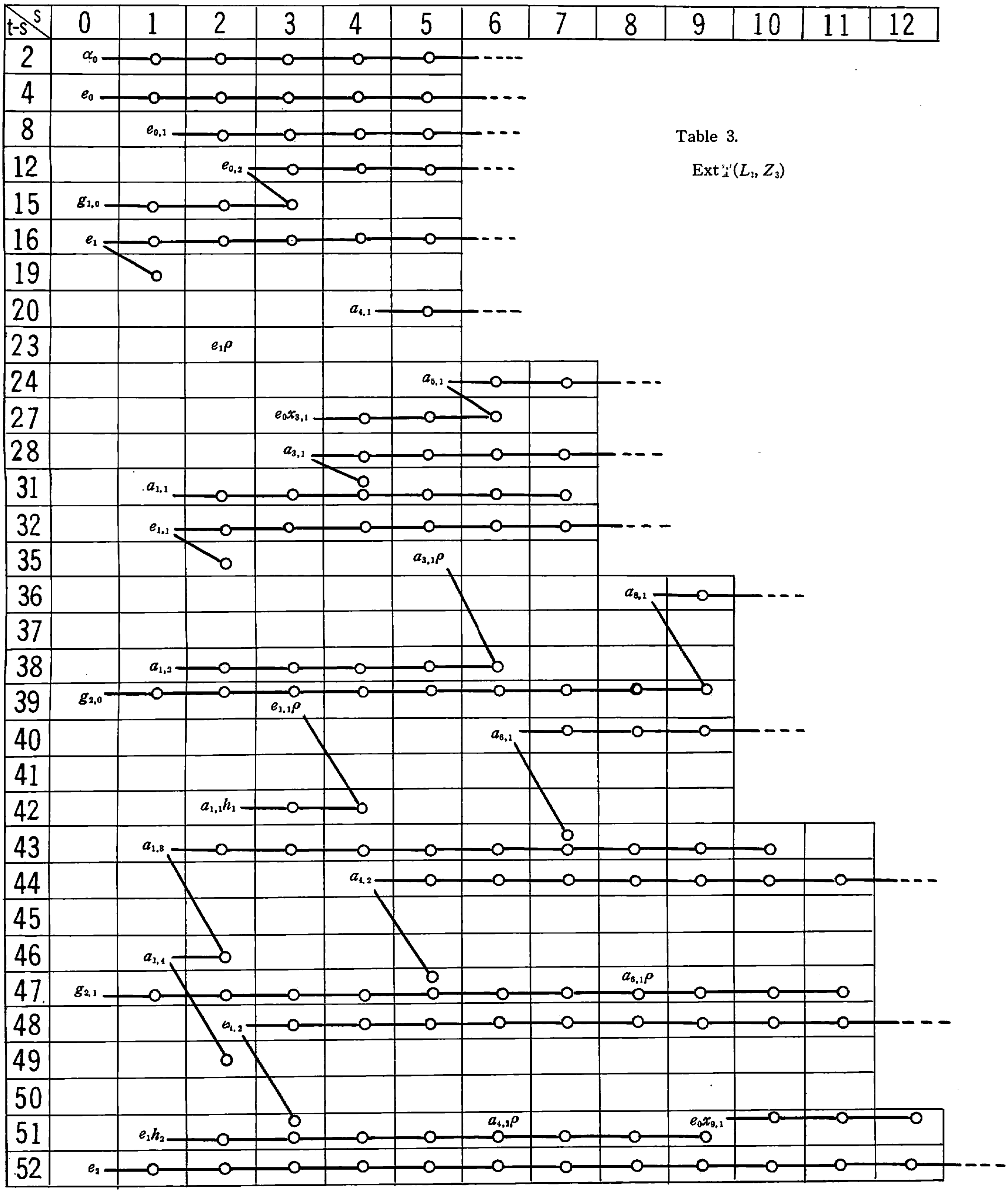

Table 3.

$Ext(L, Z_3)$

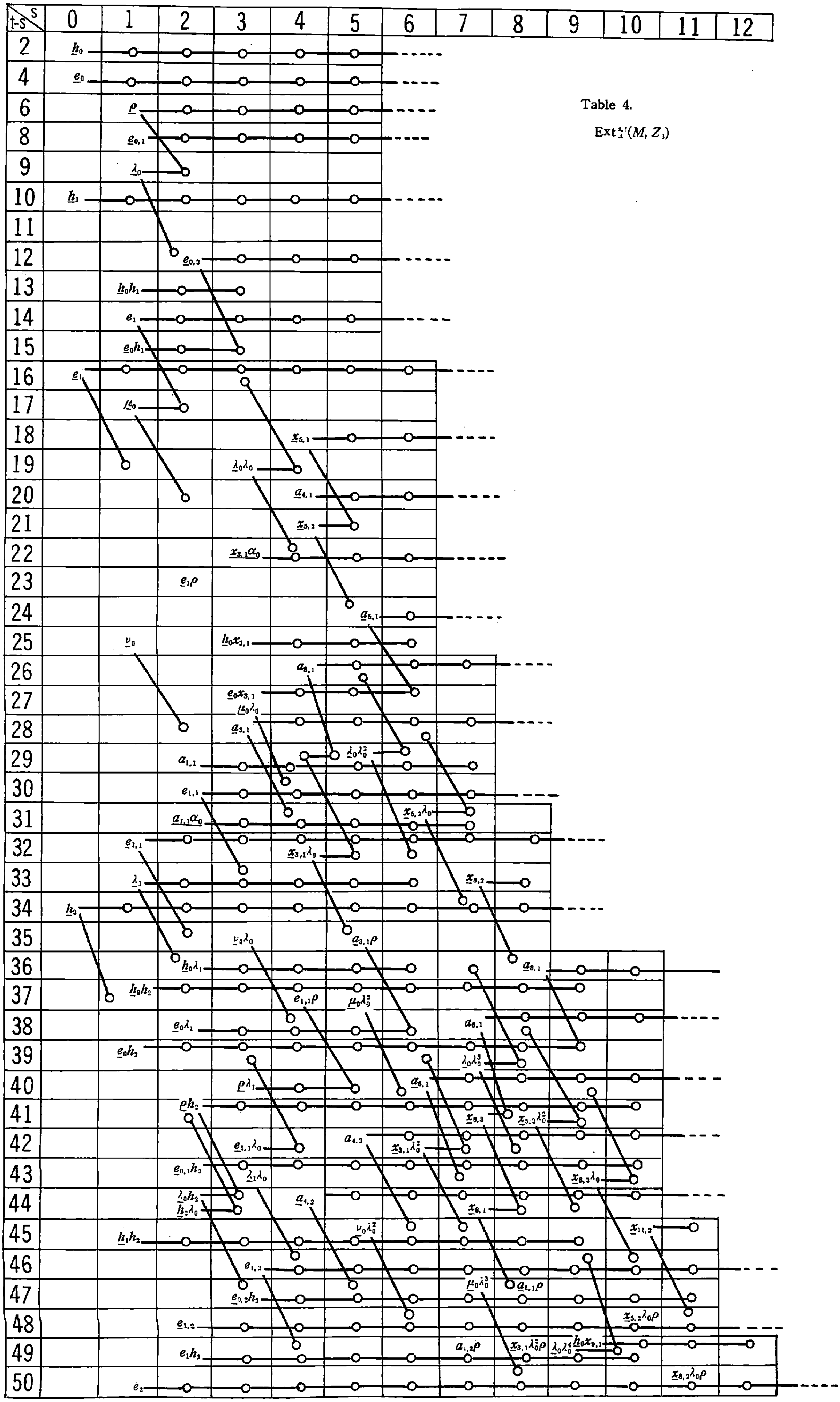

Table 4.

$\mathrm{Ext}_A^{s,t}(M, Z_3)$