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ON q-REGULAR INJECTIVE STRUCTURES IN mod-R

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Let R be a ring with identity and F an "*idempotent topologizing*" of right ideals. Then Bourbaki [1] (see also, Gabriel [2]) has shown that there exists a left exact functor $F: \mathcal{M}od\text{-}R \ni M \longmapsto F(M) \in \mathcal{M}od\text{-}R$. $F(M)$ is called a *generalized localization* of M by F, which is nothing but a *rational completion* in the sense of [3]. On the other hand, given a torsion radical, Maranda [3] has defined a quotient ring and module by a special regular injective structure determined by the torsion radical.

In this note, we shall define a *q-regular injective structure* and show that in a commutative ring every localization determines uniquely a q-regular injective structure with functorial description.

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Let r be a radical functor of the category of all right R-modules $\mathcal{M}od\text{-}R$. We shall define the following classes of right R-modules:

$\mathcal{S}_r = \{A:$ right ideal of $R \mid r(R/A) = R/A\}$.

$\widetilde{\mathfrak{Q}}_r = \{M \in \mathcal{M}od\text{-}R \mid M$ is injectve with respect to the canonical inclusion of right ideals in $\mathcal{S}_r$ in $R\}$,

$\mathfrak{Q}_r = \{M \in \mathcal{M}od\text{-}R \mid M \in \widetilde{Q}_r$ and $r(M) = 0\}$,

$\mathfrak{A}_r = \{M \in \mathcal{M}od\text{-}R \mid r(M) = 0\}$.

Definition 1. Let $\mathfrak{B}(\mathfrak{S}, \mathfrak{Q})$ be a regular injective structure and r a radical functor. We shall say that $\mathfrak{B}(\mathfrak{S}, \mathfrak{Q})$ is *defined* by r if $\mathfrak{Q}$ is a subclass of $\mathfrak{A}_r$.

For a regular injective structure $\mathfrak{B}(\mathfrak{S}, \mathfrak{Q})$ of $\mathcal{M}od\text{-}R$, let Cat$(\mathfrak{Q})$ be the full subcategory of $\mathcal{M}od\text{-}R$ with object class $\mathfrak{Q}$. Then we have a functor

$$G: \mathcal{M}od\text{-}R \longrightarrow \text{Cat}(\mathfrak{Q})$$

and a natural transformation

$$\kappa: I_{\mathcal{M}od\text{-}R} \longrightarrow G$$

such that the R-homomorphism $\kappa_M: M \longrightarrow G(M)$ is in $\mathfrak{S}$ for all $M \in \mathcal{M}od\text{-}R$.

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(G, κ) will be called a *functorial description* of $\mathfrak{B}(\mathfrak{S}, \mathfrak{Q})$ following [3; p. 107].

In [3], Maranda proved the following:

Theorem ([3; Th. 6.2]). If r is a torsion radical of $\mathscr{M}od$-R, then $\mathfrak{Q}_r$ is the class of injectives of the coarsest regular injective structure $\mathfrak{B}_r(\mathfrak{S}_r, \mathfrak{Q}_r)$ of $\mathscr{M}od$-R defined by the radical r.

From this fact, if $\mathfrak{B}(\mathfrak{S}, \mathfrak{Q})$ is a regular injective structure defined by a torsion radical r, $\mathfrak{Q}$ contains $\mathfrak{Q}_r$.

Definition 2. A regular injective structure $\mathfrak{B}(\mathfrak{S}, \mathfrak{Q})$ is called a *q-regular injective strncture* defined by a torsion radical t if $\tilde{\mathfrak{Q}}_t \supseteq \mathfrak{Q} \supseteq \mathfrak{Q}_t$ and has a functorial description (G_t, κ). Two q-regular injective structures $\mathfrak{B}_t(\mathfrak{S}_t, \mathfrak{Q}_t)$ and $\mathfrak{B}'_t(\mathfrak{S}'_t, \mathfrak{Q}'_t)$ defined by t are *equivalent* if there is a natural equivalence $\eta : G_t \longrightarrow G'_t$ such that $\eta_M \kappa_M = \kappa'_M$ for all $M \in \mathscr{M}od$-R, where (G_t, κ) and (G'_t, κ') are functorial descriptions of $\mathfrak{B}_t(\mathfrak{S}_t, \mathfrak{Q}_t)$ and $\mathfrak{B}'_t(\mathfrak{S}'_t, \mathfrak{Q}'_t)$ respectively.

Throughout the following we consider a q-regular injective structure $\mathfrak{B}(\mathfrak{S}, \mathfrak{Q})$ with functorial description (G_t, κ) defined by a torsion radical t. Here $G_t(R)$ has a ring structure and M $(M \in \mathfrak{Q})$ has a right $G_t(R)$-module structure [3; p. 113—114].

Now let t be a torsion radical. Then $\mathscr{S}_t$ satisfies the condition G1-G4 in [3; p. 119]. Therefore $\mathscr{S}_t$ is an idempotent topologizing. For any right R-module M, the generalized localization $F_t(M)$ of M by $\mathscr{S}_t$ is constructed as follows: Let A, B be in $\mathscr{S}_t$ such that $A \subseteq B$. For any $M \in \mathscr{M}od$-R, the canonical injection $j : A \longrightarrow B$ defines a homomorphism of commutative groups

$$u_{A,B} = \mathrm{Hom}_R(j, 1_M) : \mathrm{Hom}_R(B, M) \longrightarrow \mathrm{Hom}_R(A, M).$$

Since $\mathscr{S}_t$ is an ordered set with respect to the inclusion $\supseteq$, the $u_{A,B}$ define an inductive system of commutative groups, hence we define

$$M_{(F)} = \varinjlim \mathrm{Hom}_R(A, M).$$

Let $h : M \longrightarrow M_{(F)}$ be the canonical homomorphism. We consider the canonical homomorphism

$$j'_M : M \longrightarrow M/h^{-1}(0) \longrightarrow (M/h^{-1}(0))_{(F)},$$

and set $F_t(M) = (M/h^{-1}(0))_{(F)}$. Then $j^t : I_{\mathscr{M}od\text{-}R} \longrightarrow F_t$ is a natural transformation. Note that $h^{-1}(0) = \{m \in M \mid (0 : m)_r \in \mathscr{S}_t\}$, where $(0 : m)_r$

is the right annihilator of m. Hence $h^{-1}(0)$ is the torsion submodule $t(M)$ of M with respect to t.

Lemma. *Let M be a right R-module. Then $F_t(M)$ is contained in Q_t.*

Proof. For any $A \in \mathscr{S}_t$, let $i_A : A \longrightarrow R$ be the canonical injection and $f: A \longrightarrow F_t(M)$ be any homomorphism in $\mathscr{M}od$-R. Then we have $F_t(A)=F_t(R)$ [1; p. 160] and $F_t(M)=F_t(F_t(M))$ [1; p. 161]. Since $j^t_R i_A = j^t_A$, we have the following commutative diagram

$$\begin{array}{ccccc} 0 \longrightarrow A & \xrightarrow{i_A} & R & \xrightarrow{j^t_R} & F_t(R)=F_t(A) \\ f\big\downarrow & & & \swarrow F_t(f) & \\ F_t(M)=F_t(F_t(M)) & & & & \end{array}$$

Hence we have $F_t(M) \in \widetilde{\mathfrak{Q}}_t$, and so $F_t(M)$ is in $\mathfrak{Q}_t$ [1; p. 160].

Proposition 1. *There is a natural transformation*

$$\chi : G_t \longrightarrow F_t.$$

Proof. Let M be any R-module. By the above lemma, there is a unique homomorphism χ_M: $G_t(M) \longrightarrow F_t(M)$ in $\mathscr{M}od$-R such that $\chi_M \kappa_M = j^t_M$. We must show that χ is a natural transformation. For any $f: M \longrightarrow N$ in $\mathscr{M}od$-R, we have

$$F_t(f)\chi_M\kappa_M = F_t(f)j^t_M = j^t_N f, \quad \chi_N G_t(f)\kappa_M = \chi_N\kappa_N f = j^t_N f.$$

Then by the uniqueness, we obtain $F_t(f)\chi_M = \chi_N G_t(f)$. This shows that χ is a natural transformation.

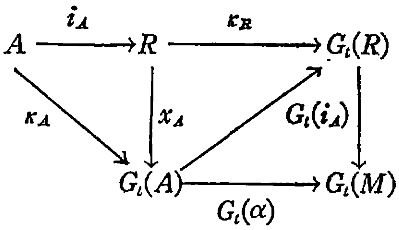
Proposition 2. *Let i_A: $A \longrightarrow R$ be the canonical inclusion, where $A \in \mathscr{S}_t$. If $G_t(i_A)$ is an epimorphism for all $A \in \mathscr{S}_t$, then there exists a natural transformation*

$$\theta_M : M_{(F)} \longrightarrow G_t(M),$$

where $M \in \mathscr{M}od$-R.

Proof. Let $A \in \mathscr{S}_t$, $M \in \mathscr{M}od$-R, and α: $A \longrightarrow M$. Then there is a homomorphism $x_A : R \longrightarrow G_t(A)$ such that $\kappa_A = x_A i_A$.

Now let us consider the following diagram:



Then there exists a unique homomorphism

$$\varphi_\alpha:\ G_t(R) \longrightarrow G_t(M)$$

such that $G_t(\alpha)x_A = \varphi_\alpha \kappa_R$. Thus we have

$$G_t(\alpha)\kappa_A = G_t(\alpha)x_A i_A = \varphi_\alpha \kappa_R i_A = \varphi_\alpha G_t(i_A)\kappa_A,$$

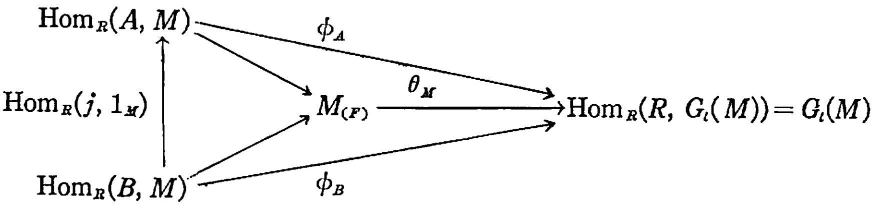
and so $G_t(\alpha) = \varphi_\alpha G_t(i_A)$. Since $G_t(i_A)$ is an epimorphism, φ_α is uniquely determined by α. Hence we have a homomorphism

$$\phi_A:\ \mathrm{Hom}_R(A, M) \ni \alpha \longmapsto \varphi_\alpha \kappa_R \in \mathrm{Hom}_R(R, G_t(M)).$$

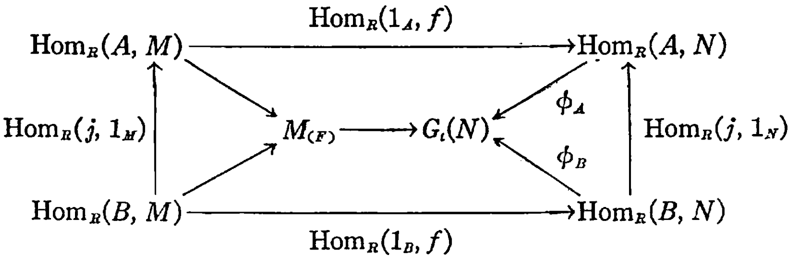
On the other hand, for the canonical inclusion $j: A \longrightarrow B (A, B \in \mathscr{S}_t)$, and for any $\beta: B \longrightarrow M$, we have

$$\begin{aligned} G_t(\beta)G_t(j)\kappa_A &= G_t(\beta j)\kappa_A = \varphi_{\beta j} G_t(i_A)\kappa_A \\ &= G_t(\beta)\kappa_B j = \varphi_\beta G_t(i_A)\kappa_A, \end{aligned}$$

whence it follows $\varphi_{\beta j} = \varphi_\beta$. From this fact, we can obtain a unique homomorphism $\theta_M: M_{(F)} \longrightarrow G_t(M)$ such that the diagram



is commutative. Similary, for any $f: M \longrightarrow N$ in $\mathscr{Mod}$-R, we have a unique homomorphism $M_{(F)} \longrightarrow G_t(N)$ such that the diagram



is commutative. Therefore θ is a natural transformation.

Corollary 1. *Let M be a right R-module. If $t(M)=0$, or if $M_{(F)}=F_t(M)$, then*

$$\theta_M : M_{(F)} \longrightarrow G_t(M)$$

is an isomorphism.

Proof. If $t(M)=0$, then $M_{(F)}=F_t(M)$. Therefore, if we identify M and $\mathrm{Hom}_R(R, M)$, then we have $\theta_M j'_M=\kappa_M$. By the uniqueness and Proposition 1, it is easy to see that $\theta_M \chi_M=1_{G_t(M)}$ and $\chi_M \theta_M=1_{M_{(F)}}$.

Corollary 2. *Let $i_A: A \longrightarrow R$ be the canonical inclusion $(A \in \mathscr{S}_t)$. Suppose $G_t(i_A)$ is an epimorphism for all $A \in \mathscr{S}_t$.*

(1) *If $M_{(F)}=F_t(M)$ for all M, then χ_R: $G_t(R) \longrightarrow F_t(R)$ is a unique ring isomorphism such that $\chi_R \kappa_R=j'_R$, and hence the functor G_t and F_t are naturally equivalent.*

(2) *If $G_t(M)=G_t(M/t(M))$ for all M, then $\chi: G_t \longrightarrow F_t$ is a natural equivalence.*

Proof. (1): By [3; Th. 5. 3], it is clear that χ_R is a unique ring isomorphism such that $\chi_R \kappa_R=j'_R$. Therefore $G_t(M)$ is a right $F_t(R)$-module for all $M \in \mathscr{Mod}$-R. The latter statement is clear by Proposition 2 and Corollary 1.

(2): Since $t(\overline{M})=0$, where $\overline{M}=M/t(M)$, we have an isomorphism $G_t(\overline{M}) \cong F_t(\overline{M})$ by Corollary 1, and $G_t(M)=G_t(\overline{M})$ and $F_t(M)=F_t(\overline{M})$. Hence χ_M is an isomorphism.

Theorem. *Let M be any module in $\mathscr{Mod}$-R. Suppose $G_t(i_A)$ is an epimorphism for all $A \in \mathscr{S}_t$. If $M_{(F)}=F_t(M)$, or if $G_t(M/t(M))$, then $\mathfrak{B}_t(\mathfrak{S}_t, \mathfrak{Q}_t)$ is the only q-regular injective structure defined by t up to equivalence. In particular, $\{G_t(M) \mid M \in \mathscr{Mod}\text{-}R\}=\mathfrak{Q}_t=\{F_t(M) \mid M \in \mathscr{Mod}\text{-}R\}$.*

Finally, we consider the commutative case:

R: a commutative ring with identity,

S: a multiplicatively closed subset of R,

$S(M)=\{m \in M \mid ms=0$ for some $s \in S\}$, for any $M \in \mathscr{Mod}$-R.

Then $S(*)$ is a torsion radical of $\mathscr{Mod}$-R. Therefore, we can define $\mathscr{S}_S$, $\tilde{\mathfrak{Q}}_S$, $\mathfrak{Q}_S$, $\mathfrak{A}_S$. In this case, $F_t(M)$ is the localization $S^{-1}M$ and $F_t(M)=M_{(F)}$

[1 ; p. 162] and $j_M^S = i_M^S$; $M \ni m \longmapsto m/s \in S^{-1}M$ for all $M \in \mathcal{M}od\text{-}R$. Therefore, in a commutative ring, q-regular injective structure cited above is characterized by the functorial description (G_S, κ).

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