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## Abstract

A metric invariant $a_k$ is defined, and we have that $a_k(X) \leq a_k(S^n)$ holds in an Alexandrov space X with curvature $\geq 1$([So]). And the borderline case when $a_{2p-1}(X) = a_{2p-1}(S^n)$ and $a_k(S^n)$ are studied.

**KEYWORDS:** Metric Invariants, Alexandrov Spaces, Spheres, Borderline Cases

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# SOME METRIC INVARIANTS OF SPHERES AND ALEXANDROV SPACES II

Nobuyuki SOCHI

Abstract. A metric invariant $a_k$ is defined, and we have that $a_k(X) \leq a_k(S^n)$ holds in an Alexandrov space $X$ with curvature $\geq 1$([So]). And the borderline case when $a_{2p-1}(X) = a_{2p-1}(S^n)$ and $a_k(S^n)$ are studied.

## 1. Introduction

In the previous paper([So]) we introduced a metric invariant $a_k(X)$ for a compact metric space $X$, and studied the explicit value of $a_k(S^n)$ for an $n$-dimensional round sphere $S^n$ with radius 1. Furthermore, we studied the behavior of $a_k(X)$ for an Alexandrov space $X$ with curvature $\geq 1$. In the present paper we continue to study the above invariant $a_k$ and give answers to some problems that are conjectured in [So]. We begin with recalling the definition of $a_k$ and results obtained in [So]. The distance between $x, y \in X$ will be denoted by $dist(x, y)$.

**Definition 1.1.** *For a positive integer $k$, we define the metric invariant $a_k(X)$ of $X$ as follows:*

$$a_k(X) = \min_{x_1,\ldots,x_k \in X} \max_{x \in X} \frac{1}{k} \sum_{i=1}^{k} dist(x, x_i). \tag{1.1}$$

In the previous paper we were concerned with $a_k(S^n)$ and got the explicit value of $a_k(S^1)$. A $k$-tuple $(x_1, \cdots, x_k)$ of points $x_i (i = 1, \cdots, k)$ of $S^1$ located in counterclockwise order is called a configuration, where each $x_i$ is called a vertex of the configuration.

**Theorem 1.1.** *(1) For $k = 2p - 1$, we have*

$$a_k(S^1) = \frac{2p^2 - 2p + 1}{(2p-1)^2}\pi. \tag{1.2}$$

*$a_k(S^1)$ is realized if and only if a configuration $(x_1, \cdots, x_k)$ of $k$ points is equally spaced in $S^1$, and $\max_{x \in S^1}(1/k)\sum_{i=1}^{k} dist(x, x_i)$ is attained exactly at the antipodal points of $x_i (1 \leq i \leq k)$.*

*(2) For $k = 2p$, we have*

$$a_k(S^1) = \frac{1}{2}\pi. \tag{1.3}$$



 

*$a_k(S^1)$ is realized if and only if $(x_1, \cdots, x_{2p})$ consists of pairs of antipodal points, and in the case we have $(1/k)\sum_{i=1}^{k} dist(x, x_i) \equiv \pi/2$.*

Now in this paper we complete the following theorem for general dimension $n$.

**Theorem 1.2.** *(1) For $k = 2p-1$, we have*

$$a_k(S^n) = a_k(S^1) = \frac{2p^2 - 2p + 1}{(2p-1)^2}\pi. \tag{1.4}$$

*$a_{2p-1}(S^n)$ is realized if and only if $\{x_1, \cdots, x_{2p-1}\}$ is located on a great circle $S^1$ and gives an equally spaced configuration after rearranging the order of $\{x_1, \cdots, x_{2p-1}\}$, and $\max_{x \in S^n}(1/k)\sum_{i=1}^{k} dist(x, x_i)$ is attained exactly at the antipodal points of $x_i (1 \le i \le k)$.*
*(2) For $k = 2p$, we have*

$$a_k(S^n) = \frac{1}{2}\pi. \tag{1.5}$$

*Moreover, $a_k(S^n)$ is realized if and only if the set $\{x_1, \cdots, x_{2p}\}$ consists of pairs of antipodal points, and in the case we have $(1/k)\sum_{i=1}^{k} dist(x, x_i) \equiv \pi/2$.*

Theorem 1.2(2) is already proved in section 4 of [So]. Theorem 1.2(1) for $k = 3$ is also proved in section 3 of [So]. We give a proof of Theorem 1.2(1) for all $k = 2p-1$ in section 2 of this paper. In our proof another metric invariant $k$-extent $xt_k(X)$ introduced by Grove and Markvorsen plays an important role. Recall that for an integer $k \ge 2$, $xt_k(X)$ is defined as follows:

$$xt_k(X) = \max_{x_1, \cdots, x_k \in X} \binom{k}{2}^{-1} \sum_{i<j} dist(x_i, x_j). \tag{1.6}$$

Now we give some results of [G-M] on $xt_k(S^n)$ which are necessary for our proof.

**Theorem 1.3** (Grove-Markvorsen)**.** *For all $n \ge 1$ and $k \ge 2$ we have*

$$xt_k(S^n) = xt_k(S^1) = \pi \Big/ \left(2 - \left[\frac{k+1}{2}\right]^{-1}\right). \tag{1.7}$$

*Those points that realize $xt_k(S^n)$ all lie on a great circle except for antipodal pairs.*

More specifically, $xt_k(S^1)$ is given as follows:

$$xt_{2p}(S^1) = \frac{p}{2p-1}\pi, \tag{1.8}$$

$$xt_{2p-1}(S^1) = \frac{p}{2p-1}\pi. \tag{1.9}$$

$xt_{2p}(S^1)$ is realized if and only if $\{x_1, \cdots, x_{2p}\}$ consists of pairs of antipodal points.

Now we explain the separation condition as follows. A configuration $(x_1, \cdots, x_k)$ in $S^1$ which dose not contain any pair of antipodal points is said to satisfy the separation condition if the following is satisfied: For any $x_i$ the line through $x_i$ and the origin separates $(x_1, \cdots, x_k) \setminus \{x_i\}$ into sets of equal cardinality. Note that $k$ is odd in this case. Then $xt_{2p-1}(S^1)$ is realized if and only if $\{x_1, \cdots, x_{2p-1}\}$ consists of pairs of antipodal points together with a configuration of points satisfying the separation condition.

For $k = 2p - 1$, $a_k(S^n)$ is related to $xt_k(S^n)$ by an inequality

$$a_k(S^n) \geq \pi - 2k^{-2}\binom{k}{2} xt_k(S^n) = \frac{2p^2 - 2p + 1}{(2p-1)^2}\pi, \tag{1.10}$$

as is shown in section 2.

Our proof of Theorem 1.2 depends also on a theorem of K. Kiyohara in [K] : For an equally spaced configuration $(x_1, \cdots, x_n)$ of a great circle $S^1$ of $S^2$, a function $f_{x_1,\cdots,x_k}(x) = \sum_{i=1}^{k} dist(x, x_i)$ on $S^2$ attains its maximum when and only when $x = \bar{x}_l (l = 1, \cdots, k)$.

Let $X$ be an Alexandrov space with curvature $\geq 1$ and we want to compare $a_k(X)$ with $a_k(S^n)$. In the previous paper we got the following theorem by using the generalized Toponogov comparison theorem([Pe]).

**Theorem 1.4.** *Let $X$ be an $n$-dimensional Alexandrov space with curvature $\geq 1$, then we have*

$$a_k(X) \leq a_k(S^n). \tag{1.11}$$

*Especially, we have*

$$a_{2p}(X) \leq a_{2p}(S^n) = \frac{\pi}{2}. \tag{1.12}$$

In the case where equality holds in (1.11) for $k = 3$ we showed that $X$ is isometric to a double spherical suspension, where the generalized Toponogov comparison theorem played an important role(see section 5 of [So]). In the present paper we improve Theorem 1.4 and show the following theorem.

**Theorem 1.5.** *Let $X$ be an $n$-dimensional Alexandrov space with curvature $\geq 1$. Suppose $a_{2p-1}(X) = a_{2p-1}(S^n)$. Then $X$ is isometric to the unit sphere $S^n$.*

## 2. Proof of Theorem1.2(1)

In this section we give a proof of Theorem 1.2(1). First we will show for $k = 2p - 1$

$$a_k(S^n) \geq \frac{2p^2 - 2p + 1}{(2p-1)^2}\pi. \tag{2.1}$$

In the proof we apply a result of Grove-Markvorsen on $k$-extent $xt_k(S^n)$ of the sphere(see Theorem 1.3). Recall that for $k = 2p - 1$ the $k$-extent $xt_k(S^n) = 1/\binom{k}{2} \max_{y_1,\cdots,y_k \in S^n} \sum_{i<j} dist(y_i, y_j)$ of the sphere is equal to $p(p-1)\pi/\binom{2p-1}{2} = p\pi/(2p-1)$. Let $x_1, \cdots, x_k$ be points on $S^n$ that realize $a_k(S^n)$, i.e., $a_k(S^n) = \max_{x\in S^n} 1/k \sum_{i=1}^k dist(x, x_i)$. We denote by $\bar{x}$ the antipodal point of $x \in S^n$. Then for any $\bar{x}_j (1 \leq j \leq k)$ we have

$$ka_k(S^n) \geq \sum_{i=1}^k dist(x_i, \bar{x}_j). \tag{2.2}$$

Adding (2.2) with respect to $j$ we obtain

$$\begin{aligned} k^2 a_k(S^n) &\geq \sum_{j=1}^k \sum_{i=1}^k dist(x_i, \bar{x}_j) \\ &= k^2\pi - 2\sum_{i<j} dist(x_i, x_j) \\ &\geq k^2\pi - 2 \max_{y_1,\cdots,y_k\in S^n} \sum_{i<j} dist(y_i, y_j) \\ &= k^2\pi - 2\binom{k}{2} xt_k(S^n) = (2p^2 - 2p + 1)\pi. \end{aligned} \tag{2.3}$$

Hence we have (2.1). To complete the proof of Theorem 1.2(1) we have to show the following assertions:

(1) $a_k(S^n) = (2p^2 - 2p + 1)\pi/(2p-1)^2$.
(2) If $\{x_1, \cdots, x_k\}$ realizes $a_k(S^n)$, then $\{x_1, \cdots, x_k\}$ is located on a great circle $S^1$ and a configuration obtained by rearranging $\{x_1, \cdots, x_k\}$ is equally spaced on $S^1$.
(3) For an equally spaced configuration $(x_1, \cdots, x_k)$, $f_{x_1,\cdots,x_k}(x) = \sum_{i=1}^k dist(x, x_i), x \in S^n$ takes the maximum exactly at $x = \bar{x}_j$ for all $j$.

To show these assertions we need the following theorem.

**Theorem 2.1.** *Let $(x_1, \cdots, x_k)$ be an equally spaced configuration on a great circle $S^1$ in $S^n$. The function $f(x) = \sum_{i=1}^k dist(x, x_i)$ takes its maximum at $x \in S^n$ when and only when $x = \bar{x}_l (1 \leq l \leq k = 2p - 1)$.*

Indeed, making use of Theorem 2.1 we may show (1), (2), and (3) as follows. First consider a configuration $(x_1, \cdots, x_k)$ as in Theorem 2.1, then we have $\max_{x \in S^n} f_{x_1,\cdots,x_k} = \sum_{i=1}^{k} dist(x_i, \bar{x}_l) = (2p^2 - 2p + 1)\pi/(2p-1)$. It follows that $a_k(S^n) \leq (2p^2 - 2p + 1)\pi/(2p-1)^2$ and the assertion (1) holds. Next we show the assertion (2). Suppose $\{x_1, \cdots, x_k\}$ realizes $a_k(S^n)$, i.e., $\max_{x \in S^n} f_{x_1,\cdots,x_k} = (2p^2 - 2p + 1)\pi/(2p-1)$. Then equality holds in the above inequalities (2.3), $\{x_1, \cdots, x_k\}$ also realizes $xt_k(S^n)$, i.e., they are lying on a great circle $S^1$ except for some antipodal pairs by Theorem 1.3. Note, however, that $a_{2p-1}(S^n)$ cannot be realized if $\{x_1, \cdots, x_k\}$ contains an antipodal pair. Indeed, suppose that $\{x_1, \cdots, x_{2m-1}\}(1 \leq m < p)$ is on $S^1$ and $\{x_{2m}, \cdots, x_{2p-1}\}$ consists of $(p-m)$ antipodal pairs. Then we have

$$\begin{aligned}
\max_{x \in S^n} f_{x_1,\cdots,x_{2p-1}}(x) &\geq \max_{x \in S^1} f_{x_1,\cdots,x_{2p-1}}(x) \\
&\geq (p-m)\pi + (2m-1)a_{2m-1}(S^1) \\
&= (p-m)\pi + \frac{2m^2 - 2m + 1}{2m-1}\pi \\
&= \frac{(2p-1)m + 1 - p}{2m-1}\pi \\
&> \frac{2p^2 - 2p + 1}{2p-1}\pi.
\end{aligned}$$

It follows that $\{x_1, \cdots, x_k\}$ is located on a great circle $S^1$. We may also assume that $(x_1, \cdots, x_k)$ is a configuration in $S^1$. Then $f_{x_1,\cdots,x_k}(x)$ attains the maximum $ka_k(S^1) = \frac{2p^2-2p+1}{2p-1}\pi$ at $x = \bar{x}_j \in S^1$ by (2.3). By restricting $f_{x_1,\cdots,x_k}$ to $S^1$ we see that $(x_1, \cdots, x_k)$ is equally spaced in $S^1$ by Theorem 1.1. Then assertion (3) also follows from Theorem 2.1.

Now we give some remarks about a proof of Theorem 2.1. It suffices to consider the case $n = 2$ to prove Theorem 2.1. By virtue of Theorem 1.1 we need only to show that $f_{x_1,\cdots,x_k}(x)$ cannot take a maximum at a point $x \in S^2 \backslash S^1$. For $k = 3$ we gave a proof of the theorem by showing that $f_{x_1,\cdots,x_k}(x)$ admits no critical points in $S^2 \backslash S^1$(see section 3 of [So]). But for $k = 2p-1 > 3$ the behavior of critical points of $f_{x_1,\cdots,x_k}(x)$ is rather complicated and it is not so clear whether the above approach works for general $k = 2p-1$. Then K. Kiyohara gave a simple and ingenious proof of theorem 2.1 which will be presented in the appendix.

## 3. PROOF OF THEOREM 1.5

Let $X$ be an $n$-dimensional Alexandrov space with curvature $\geq 1$. Recall that we have an inequality

$$a_{2p-1}(X) \leq a_{2p-1}(S^n) = \frac{2p^2 - 2p + 1}{(2p-1)^2}\pi \tag{3.1}$$

by Theorem 1.2 and Theorem 1.4. In this section we show that $X$ is isometric to the round sphere $S^n$ of radius 1 when equality holds in (3.1). First we recall the notion of the spherical suspension and the spherical join([B-G-P]).

**Definition 3.1.** *The spherical suspension of a metric space $Y$ is the quotient space*

$$\Sigma_1 Y = Y \times [0, \pi] / \sim, \tag{3.2}$$

*where the equivalence relation $\sim$ is given by $(x_1, a_1) \sim (x_2, a_2) \Leftrightarrow x_1 = x_2, 0 < a_1 = a_2 < \pi$ or $a_1 = a_2 = 0$ or $a_1 = a_2 = \pi$, and is equipped with the canonical metric*

$$\cos dist(\hat{x}_1, \hat{x}_2) = \cos a_1 \cos a_2 + \sin a_1 \sin a_2 \cos dist(x_1, x_2), \tag{3.3}$$

*where we set $\hat{x}_1 = (x_1, a_1)$, $\hat{x}_2 = (x_2, a_2)$.*

**Definition 3.2.** *The spherical join of $X$ and $Y$ is defined as*

$$X * Y = X \times Y \times [0, \pi/2] \,/ \sim, \tag{3.4}$$

*where $(x_1, y_1, a_1) \sim (x_2, y_2, a_2) \Longleftrightarrow x_1 = x_2, y_1 = y_2, 0 < a_1 = a_2 < \pi/2$ or $a_1 = a_2 = 0$, $x_1 = x_2$ or $a_1 = a_2 = \pi/2$, $y_1 = y_2$, and is equipped with the canonical metric*

$$\begin{aligned} &\cos dist((x_1, y_1, a_1), (x_2, y_2, a_2)) \\ &= \cos a_1 \cos a_2 \cos dist(x_1, x_2) + \sin a_1 \sin a_2 \cos dist(y_1, y_2). \end{aligned} \tag{3.5}$$

Further, we define $\Sigma_k Y = \Sigma_{k-1}(\Sigma_1 Y)$ to be the $k$-times repeated spherical suspension. Then for an Alexandrov space $X$ with curvature $\geq 1$ we have $X = \Sigma_k Y$ for some Alexandrov space $Y$ with curvature $\geq 1$ if and only if $S^{k-1}$ is isometrically embedded in $X$([G-W]). Hence the $k$-times repeated spherical suspension $\Sigma_k Y$ is isometric to the spherical join $S^{k-1} * Y$.

In the previous paper, in the case of $k = 3$ we showed that $X$ is isometric to $\Sigma_2 Z$ for some Alexandrov space $Z$ with curvature $\geq 1$. First we show that $X$ is isometric to the spherical suspension $\Sigma_1 Y$ in the same manner as in the case of $k = 3$ for completeness.

**Lemma 3.1.** *Let $X$ be an $n$-dimensional Alexandrov space with curvature $\geq 1$. Suppose $a_{2p-1}(X) = a_{2p-1}(S^n) = \frac{2p^2-2p+1}{(2p-1)^2}\pi$. Then $X$ is isometric to the spherical suspension $\Sigma_1 Y$, where $Y$ is an $(n-1)$-dimensional Alexandrov space with curvature $\geq 1$.*

*Proof.* By the maximal diameter theorem([G-P2]), it suffices to show that $diam X$ is equal to $\pi$. Let $\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_{2p-1}$ be points on $S^n$ that realize $a_{2p-1}(S^n)$. We may assume that $(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_{2p-1})$ is an equally spaced configuration in a great circle $S^1$. Take a point $\tilde{p} \in S^n$ different from the antipodals of $\tilde{x}_i (i = 1, 2, \cdots, 2p-1)$. Take a regular point $p \in X$. We denote by $S_p$ the space of directions of $X$ at $p$. Then $\Sigma_1 S_p$ is isometric to $S^n$, and we identify $\Sigma_1 S_p$ (resp. $S_p$) with $S^n = \Sigma_1 S_{\tilde{p}}$ (resp. $S_{\tilde{p}} = S^{n-1}$). Let $x_i \in X$ be a point $\exp_p \bar{c}_{v_i}(dist(\tilde{p}, \tilde{x}_i)) = c_{v_i}(dist(\tilde{p}, \tilde{x}_i))$, where $\bar{c}_{v_i}$ is a minimal geodesic in $S^n$ emanating from $\tilde{p}$ to $\tilde{x}_i$ with initial direction $v_i \in S_{\tilde{p}} = S_p = S^{n-1} (i = 1, 2, \cdots, 2p-1)$ and $c_{v_i}$ is a quasigeodesic in $X$ emanating from $p$ with initial direction $v_i$(see section 5 of [So]). Take a point $x_0 \in X$ such that

$$
\begin{aligned}
a_{2p-1}(x_1, x_2, \cdots, x_{2p-1}) :&= \max_{x \in X} \frac{1}{2p-1} \sum_{i=1}^{2p-1} dist(x, x_i) \\
&= \frac{1}{2p-1} \sum_{i=1}^{2p-1} dist(x_0, x_i).
\end{aligned}
$$

Let $\gamma_0 : [0, dist(p, x_0)] \longrightarrow X$ be a minimal geodesic from $p$ to $x_0$, and set $\tilde{x}_0 = \exp_{\tilde{p}}^{S^n}(dist(p, x_0)\dot{\gamma_0}(0))$. Then by the generalized Toponogov comparison theorem for $\triangle p x_i x_0$ and $\triangle \tilde{p} \tilde{x}_i \tilde{x}_0$([Pe], see also [So]), we have

$$dist(x_0, x_i) \leq dist(\tilde{x}_0, \tilde{x}_i) \quad (i = 1, 2, \cdots, 2p-1).$$

Then we have

$$
\begin{aligned}
a_{2p-1}(X) \leq a_{2p-1}(x_1, x_2, \cdots, x_{2p-1}) &= \frac{1}{2p-1} \sum_{i=1}^{2p-1} dist(x_0, x_i) \\
&\leq \frac{1}{2p-1} \sum_{i=1}^{2p-1} dist(\tilde{x}_0, \tilde{x}_i) \leq a_{2p-1}(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_{2p-1}) \\
&= a_{2p-1}(S^n) = a_{2p-1}(X).
\end{aligned}
\tag{3.6}
$$

It follows that

$$a_{2p-1}(X) = a_{2p-1}(S^n) = \frac{1}{2p-1} \sum_{i=1}^{2p-1} dist(\tilde{x}_0, \tilde{x}_i), \tag{3.7}$$

and we obtain for any $i$

$$dist(x_0, x_i) = dist(\tilde{x}_0, \tilde{x}_i) \quad (i = 1, 2, \cdots, 2p-1). \tag{3.8}$$

Then from Theorem 1.1 $\tilde{x}_0$ is the antipodal point of some $\tilde{x}_i$, namely, we have $dist(\tilde{x}_0, \tilde{x}_i) = \pi$, and hence $dist(x_0, x_i) = \pi$ for some $x_i (1 \leq i \leq 2p-1)$. □

The following lemma is given in the previous paper([So]).

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**Lemma 3.2.** *Suppose $X = \Sigma_1 Y$, where $Y$ is an $(n-1)$-dimensional Alexandrov space with curvature $\geq 1$ and $diamY < \pi$ and $n \geq 2$. Let $x_1,x_2 \in X$ be the pole points of the spherical suspension $X = \Sigma_1 Y$. Then there is no pair of points whose distance is $\pi$ except for $x_1,x_2$.*

Next we show that $X$ is isometric to $\Sigma_2 Z$ if $\dim X \geq 2$ as in the case of $k = 3$. By Lemma 3.2 we have the following lemma.

**Lemma 3.3.** *Let $X$ be an $n$-dimensional Alexandrov space with curvature $\geq 1$ and $n \geq 2$. Suppose $a_{2p-1}(X) = a_{2p-1}(S^n) = \frac{2p^2-2p+1}{(2p-1)^2}\pi$. Then $X$ is isometric to $\Sigma_2 Z$, where $Z$ is an $(n-2)$-dimensional Alexandrov space with curvature $\geq 1$.*

*Proof.* By Lemma 3.1 we may write $X = \Sigma_1 Y$. Suppose $diamY < \pi$. In the proof of Lemma 3.1 we may take a point $p$ as an arbitrary regular point of $X$, and the set of regular points is dense in $X$. If the base point $p \in X$ is shifted, the points $x_1, x_2, \cdots, x_{2p-1}$ that realize $a_{2p-1}(X)$ can be moved. Then there exists another pair of points $x_0, x_i (i = 1, 2, \cdots, 2p-1)$ whose distance is equal to $\pi$. This contradicts Lemma 3.2. Therefore, we have $diamY = \pi$ and $X = \Sigma_2 Z$. □

We prepare one more lemma.

**Lemma 3.4.** *Let $X$ be an $n$-dimensional Alexandrov space with curvature $\geq 1$. Suppose $a_{2p-1}(X) = a_{2p-1}(S^n) = \frac{2p^2-2p+1}{(2p-1)^2}\pi$. If $X$ is isometric to $S^{k-1} * Y^{n-k} (1 \leq k \leq n-1)$, where $Y^{n-k}$ is an $(n-k)$-dimensional Alexandrov space with curvature $\geq 1$, then $diamY^{n-k} = \pi$.*

*Proof.* Suppose $diamY^{n-k} < \pi$. Take any points $(x, y, t_1), (x', y', t_2) \in X (x, x' \in S^{k-1}, y, y' \in Y^{n-k}, 0 \leq t_1, t_2 \leq \pi/2)$, and set

$$l = dist((x, y, t_1), (x', y', t_2)).$$

Then we will show that $l = \pi$ holds exactly when $t_1 = t_2 = 0$ and $dist(x, x') = \pi$, namely, $(x, y, t_1), (x', y', t_2)$ are antipodal pair of $S^{k-1}$. Indeed, by the distance formula (3.5) we have

$$\begin{aligned} -1 &= \cos l \\ &= \cos t_1 \cos t_2 \cos dist(x, x') + \sin t_1 \sin t_2 \cos dist(y, y') \qquad (3.9) \\ &\geq \cos(\pi + t_1 - t_2) \geq -1. \end{aligned}$$

Then we have $t_1 = t_2 = 0$ because of $\cos(dist(y, y')) > -1$, and also $dist(x, x') = \pi$ holds.

Now in the proof of Lemma 3.1 we may choose a point $p$ arbitrarily as long as $p$ is regular. Since $radX = a_1(X) \geq a_{2p-1}(X) = (2p^2 - 2p + 1)\pi/(2p-1)^2 > \pi/2$, $X$ is homeomorphic to $S^n$ by the radius sphere theorem, and the set $S$ of non regular points is a closed set of dimension $\leq n-2$([G-P1]). Also we may choose identification between $S_p$ and $S_{\tilde{p}} = S^{n-1}$ up to isometries of $S^{n-1}$. If the point $p \in X$ is shifted or $v_1, v_2, \cdots, v_{2p-1}$ are rotated around $p$ in $S_p = S^{n-1}$, the points $x_1, x_2, \cdots, x_{2p-1}$ that realize $a_{2p-1}(X)$ can be moved outside of $S^{k-1}$. Then there exists another pair of points $x_0, x_i (i = 1, 2, \cdots, 2p-1)$ with $dist(x_0, x_i) = \pi$ such that they are not antipodal pair in $S^{k-1}$. This contradicts $diamY^{n-k} < \pi$. It follows that $diamY^{n-k} = \pi$ holds. □

Now we show that $X$ is isometric to $S^n$.

*Proof of Theorem* 1.5. By Lemma 3.1 and Lemma 3.3 we have $X = S^0 * Y^{n-1} = S^1 * Y^{n-2}$. Next we assume that $X = S^{k-1} * Y^{n-k}$ holds for $k(1 \leq k \leq n-1)$. By Lemma 3.4 we have $diamY^{n-k} = \pi$. It follows that $X = S^k * Y^{n-k-1}$. By induction on $k$ we see that $X$ is isometric to $S^{n-2} * Y^1$. Since $radX = rad(\Sigma_1 Y^{n-1}) = radY^{n-1}$, we have $radY^{n-1} > \pi/2$(see [G-P1]). It follows that $radY^1 > \pi/2$ and $Y^1$ is homeomorphic to the circle $S^1$. By Lemma 3.4 $diamY^1 = \pi$ and therefore $Y^1$ is isometric to $S^1$. It follows that $X$ is isometric to $S^n$. This completes the proof of Theorem 1.5. □

**Remark 3.1.** *If* $a_{2p-1}(X)$ *is close to* $a_{2p-1}(S^n) = (2p^2-2p+1)\pi/(2p-1)^2 > \pi/2$, *then* $X$ *is homeomorphic to* $S^n$ *since* $radX > \pi/2$.

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