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Associated Riemannian manifolds and motions

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ASSOCIATED RIEMANNIAN MANIFOLDS AND MOTIONS

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In this paper, we shall investigate the associated Rimannian manifold V_N $(N = n(n+1)/2)$ with a Riemannian manifold V_n that a Riemannian metric which is naturally derived from the one of V_n is given on the bundle space of the associated principal fibre bundle of V_n and the motions on V_n in connection with V_N.

In §§1—3, we shall calculate the parameters of the connection and the curvature forms of V_N with respect to the canonical frames. In §4, we shall investigate the conditions in order that V_N becomes an Einstein space. In §§5, 6, we shall give the equations of geodesics in V_N and an elementary exposition of the relations between the Levi-Civita connection of V_n and the associated Riemannian manifold V_N with V_n. In §§7, 8, we shall show that a mapping derived from the differential mapping of a motion of V_n becomes a motion of V_N and investigate the properties of motions of V_n by means of such motions of V_N. In §9, we shall investigate sequences of motions of V_n and prove that under a suitable condition, we can derive a differentiable tangent vector field of motion from a sequence of motions which is a geometrical treatment of such sequences of motions of V_n investigated in [6, 7]. Lastly in §10, we shall give an elementary exposition of holonomy groups of V_n.

§ 1. Definitions

Let V_n be an n-dimensional Riemannian manifold and let $\mathfrak{B} = \{B, p, V_n, O_n, O_n\}$ be the associated principal fibre bundle of the tangent fibre bundle of V_n as a differentiable manifold with a Riemannian metric, that is

i) $p : B \longrightarrow V_n$ be the projection,

ii) for any point $x \in V_n$, the fibre $p^{-1}(x) = O_n(x)$ is homeomorphic to the n-dimensional orthogonal group O_n,

and

iii) $O_n(x) \ni b = \{x, e_1, \cdots, e_n\}$ is an othonormal frame at x such that e_i, $i = 1, 2, \cdots, n$, are unit tangent vectors to V_n at x and mutually orthogonal.

Let be given the line element of V_n by

$$ds^2 = \sum \omega_i \omega_i \tag{1, 1}$$

and the equations of structure of V_n with respect to b are

$$\begin{cases} d\omega_i = \sum \omega_k \wedge \omega_{ki}, \\ d\omega_{ij} = \sum \omega_{ik} \wedge \omega_{kj} + \Omega_{ij}, \\ \omega_{ij} = -\omega_{ji} \end{cases} \tag{1, 2}$$

$$\Omega_{ij} = \frac{1}{2} \sum R_{ijkh} \omega_k \wedge \omega_h, \tag{1, 3}$$

where R_{ijkh} are the components of curvature tensor of V_n.

Now, define a Riemannian manifold V_N of dimension $N = n(n+1)/2$ whose underlying manifold is B and whose line element is given by

$$ds_N^2 = \sum \omega_i \omega_i + \rho^2 \sum_{i<j} \omega_{ij} \omega_{ij}, \tag{1, 4}$$

$$\rho = \text{constant} \neq 0.$$

We shall represent (1, 4) by local coordinates in V_n. Let $x^1, x^2, \cdots, x^n$ be local coordinates of a neighborhood in V_n on which the line element of V_n is written as

$$ds^2 = \sum g_{ij}(x) dx^i dx^j. \tag{1, 1'}$$

Let X_i, $i = 1, 2, \cdots, n$, be tangent vectors such as

$$X_i = \partial / \partial x^i = y_i^k e_k,$$

$$g_{ij} = X_i \cdot X_j = \sum y_i^k y_j^k.$$

In matrix notations, putting

$$G = (g_{ij}), \qquad Y = (y_i^k), \tag{1, 5}$$

then we have

$$G = YY',$$

where Y' denotes the transposed matrix of Y. If we put

$$\pi = (\pi_i^j), \quad \pi_i^j = \sum \left\{ {j \atop ik} \right\} dx^k,$$

where $\left\{ {j \atop ik} \right\}$'s are the Christoffel symbols made by g_{ij}, then we have, as is well known,

$$dG = \pi G + G \pi',$$

and

$$\omega = Y^{-1} \pi Y - Y^{-1} dY, \qquad \omega = (\omega_{ij}).$$

Hence we have

$$\sum_{i<j}\omega_{ij}\omega_{ij} = -\frac{1}{2}\,\text{Trace}\,\omega\omega$$
$$= -\frac{1}{2}\,\text{Trace}\,(Y^{-1}\pi\pi Y - Y^{-1}\pi dY - Y^{-1}dYY^{-1}\pi Y + Y^{-1}dYY^{-1}dY)$$
$$= -\frac{1}{2}\,\text{Trace}\,(\pi\pi - 2dYY^{-1}\pi + dYY^{-1}dYY^{-1}).$$

Accordingly, (4) can be written as

$$(1, 4') \qquad ds_N^2 = \sum g_{ij}(x)dx^i dx^j - \frac{1}{2}\rho^2\,\text{Trace}\,(\pi\pi - 2dYY^{-1}\pi + dYY^{-1}dYY^{-1}),$$

where

$$Y = (y_i^k), \qquad \pi = (\sum\{{}^{k}_{ih}\}dx^h), \qquad G = (g_{ij}), \qquad G = YY'.$$

§ 2. **Parameters of the connection of Levi-Civita of V_N.** In this section, we shall determine the parameters of the connection of Levi-Civita of V_N with respect to orthonormal frames of V_N. According to the ordinary method, let us put

$$(2, 1) \qquad \begin{cases} d\omega_i = \sum\omega_k \wedge \theta_{ki} + \rho\sum_{k<h}\omega_{kh} \wedge \theta_{kh;i} \\ \quad (= \sum\omega_k \wedge \omega_{ki}), \\ d\omega_{ij} = \frac{1}{\rho}\sum\omega_k \wedge \theta_{k;ij} + \sum_{k<h}\omega_{kh} \wedge \theta_{kh;ij} \\ \quad (= \sum\omega_{ik} \wedge \omega_{kj} + \frac{1}{2}\sum R_{ijkh}\omega_k \wedge \omega_h), \end{cases}$$

$$(2, 2) \qquad \theta_{ij} = -\theta_{ji}, \quad \theta_{kh;i} = -\theta_{i;kh}, \quad \theta_{ij;kh} = -\theta_{kh;ij}.$$

As is well known, we can solve (2, 1), (2, 2) with respect to θ_{ij}, $\theta_{kh;i}$, $\theta_{ij;kh}$ and shall obtain an unique solution. We get from (2, 1), (2, 2) the equations[1]

$$(2, 3) \qquad \begin{cases} \omega_k \wedge (\theta_{ki} - \omega_{ki}) + \omega_{kh} \wedge \frac{1}{2}\rho\theta_{kh;i} = 0, \\ \omega_k \wedge (\frac{1}{\rho}\theta_{k;ij} - \frac{1}{2}R_{ijkh}\omega_h) \\ \qquad + \omega_{kh} \wedge (\frac{1}{2}\theta_{kh;ij} + \delta_{ih}\omega_{kj}) = 0. \end{cases}$$

From the first of the above equations, we may put

$$(2, 4) \qquad \begin{aligned} \theta_{ki} &= \omega_{ki} + A_{kih}\omega_h + \frac{1}{2}B_{kihj}\omega_{hj}, \\ \rho\theta_{kh;i} &= B_{jikh}\omega_j + \frac{1}{2}C_{khijl}\omega_{jl}, \end{aligned}$$

1) In the following we shall use the summation convention.

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where A's, B's and C's satisfy the following relations:

$$\begin{cases} A_{kih} = A_{hik} = -A_{ikh}, \\ B_{kihj} = -B_{ikhj} = -B_{kijh}, \\ C_{khijl} = -C_{hkijl} = -C_{khilj} = C_{jlikh}. \end{cases} \tag{2, 5}$$

The first of (2, 5) yields

$$A_{kih} = 0. \tag{2, 6}$$

Substituting (2, 4) into the second of (2, 3), we get

$$\begin{aligned} -\omega_k \wedge \left\{\frac{1}{\rho^2} B_{hkij}\omega_h + \frac{1}{2\rho^2} C_{ijklm}\omega_{lm} + \frac{1}{2} R_{ijkh}\omega_h\right\} \\ + \frac{1}{4}\omega_{kh} \wedge (2\theta_{kh;ij} + \delta_{ih}\omega_{kj} - \delta_{ik}\omega_{hj} - \delta_{jh}\omega_{ki} \\ + \delta_{jk}\omega_{hi}) = 0, \end{aligned}$$

from which we may put

$$\begin{aligned} \frac{1}{\rho^2} B_{hkij}\omega_h + \frac{1}{2\rho^2} C_{ijklm}\omega_{lm} + \frac{1}{2} R_{ijkh}\omega_h \\ = D_{ijkh}\omega_h + \frac{1}{2} E_{ijkhl}\omega_{hl}, \end{aligned} \tag{2, 7}$$

$$\begin{aligned} \theta_{kh;ij} + \frac{1}{2}(\delta_{ih}\omega_{kj} - \delta_{ik}\omega_{hi} - \delta_{ih}\omega_{ki} + \delta_{jk}\omega_{hi}) \\ = -E_{ijlkh}\omega_l - \frac{1}{2} F_{khijlm}\omega_{lm}, \end{aligned} \tag{2, 8}$$

where $D's$, $E's$ and $F's$ satisfy the following relations:

$$\begin{cases} D_{ijkh} = D_{ijhk}, \\ E_{ijkhl} = -E_{ijklh} = -E_{jikhl}, \\ F_{khijlm} = -F_{hkijlm} = -F_{khjilm} = -F_{khijml} = F_{lmijkh}. \end{cases} \tag{2, 9}$$

Since ω_i, ω_{jk} are linearly independent, we get from (2, 7) the relations

$$D_{ijkh} = \frac{1}{\rho^2} B_{hkij} + \frac{1}{2} R_{ijkh}, \tag{2, 10}$$

$$E_{ijkhl} = \frac{1}{\rho^2} C_{ijkhl}. \tag{2, 11}$$

By (2, 5) and (2, 9), we get easily

$$D_{ijkh} = 0, \tag{2, 12}$$

accordingly

$$B_{hkij} = -\frac{1}{2}\rho^2 R_{ijkh} = \frac{1}{2}\rho^2 R_{hkij}. \tag{2, 13}$$

On the other hand, from (2, 2) we get

(2, 14) $$E_{ijlkh} = - E_{khlij},$$
(2, 15) $$F_{khljm} = - F_{ijkhlm}.$$

By (2, 11) and (2, 5) we get

$$E_{ijlkh} = E_{khlij}.$$

This equation and (2, 14) follow the relation

(2, 16) $$E_{ijlkh} = C_{ijlkh} = 0.$$

Analogously, we get easily from (2, 9), (2, 15)

(2, 17) $$F_{ijkhlm} = 0.$$

Thus, we obtain the solution of (2, 1) under the condition (2, 2) as follows:

(2, 18) $$\begin{cases} \theta_{ij} = \omega_{ij} + \frac{1}{4}\rho^2 R_{ijkh}\omega_{kh}, \\ \theta_{ij;k} = -\frac{1}{2}\rho R_{ijkh}\omega_h, \\ \theta_{ij;kh} = \frac{1}{2}(\delta_{ik}\omega_{jh} + \delta_{jh}\omega_{ik} - \delta_{jk}\omega_{ih} - \delta_{ih}\omega_{jk}). \end{cases}$$

§3. **Curvature forms in V_N.** Making use of (2, 1), (2, 18), we shall calculate the curvature forms of V_N.

We have

$$\begin{aligned} \Pi_{ij} &= d\theta_{ij} - \theta_{ik}\wedge\theta_{kj} - \frac{1}{2}\theta_{i;hk}\wedge\theta_{kh;j} \\ &= d\omega_{ij} + \frac{1}{4}\rho^2 d(R_{ijkh}\omega_{kh}) \\ &\quad -(\omega_{ik} + \frac{1}{4}\rho^2 R_{iklm}\omega_{lm})\wedge(\omega_{kj} + \frac{1}{4}\rho^2 R_{kjst}\omega_{st}) \\ &\quad + \frac{1}{8}\rho^2 R_{khil}\omega_l \wedge R_{khjm}\omega_m \\ &= \Omega_{ij} + \frac{1}{4}\rho^2(R_{ijkh,l}\omega_l + R_{ljkh}\omega_{il} + R_{ilkh}\omega_{jl} \\ &\qquad\qquad + R_{ijlh}\omega_{kl} + R_{ijkl}\omega_{hl})\wedge\omega_{kh} \\ &\quad + \frac{1}{4}\rho^2 R_{ijkh}(\Omega_{kh} + \omega_{kl}\wedge\omega_{lh}) \\ &\quad - \frac{1}{4}\rho^2(R_{iklm}\omega_{lm}\wedge\omega_{kj} + R_{kjst}\omega_{ik}\wedge\omega_{st}) \\ &\quad - \frac{1}{16}\rho^2 R_{iklm}R_{kjst}\omega_{lm}\wedge\omega_{st} \\ &\quad + \frac{1}{8}\rho^2 R_{khil}R_{khjm}\omega_l\wedge\omega_m, \end{aligned}$$

that is

$$
\begin{aligned}
\Pi_{ij} = \Omega_{ij} &+ \frac{1}{4}\rho^2 R_{ijkh}\Omega_{kh} + \frac{1}{8}\rho^2 R_{khil}R_{khjm}\omega_l \wedge \omega_m \\
(3, 1) \qquad &+ \frac{1}{4}\rho^2 R_{ijkh,l}\omega_l \wedge \omega_{kh} \\
&- \frac{1}{4}\rho^2 (R_{ijkh}\omega_{kl} \wedge \omega_{lh} + \frac{1}{4}\rho^2 R_{iklm}R_{kjst}\omega_{lm} \wedge \omega_{st}),
\end{aligned}
$$

where a comma denotes the covariant differentiation of V_n. We have

$$
\begin{aligned}
\Pi_{i;jk} &= d\theta_{i;jk} - \theta_{ih} \wedge \theta_{h;jk} - \frac{1}{2}\theta_{i;hl} \wedge \theta_{hl;jk} \\
&= \frac{1}{2}\rho d(R_{jkih}\omega_h) \\
&\quad - (\omega_{ih} + \frac{1}{4}\rho^2 R_{ihlm}\omega_{lm}) \wedge \left(\frac{1}{2}\rho R_{jkhs}\omega_s\right) \\
&\quad - \frac{1}{8}\rho R_{hlis}\omega_s \wedge (\delta_{hj}\omega_{lk} + \delta_{lk}\omega_{hj} - \delta_{hk}\omega_{lj} - \delta_{lj}\omega_{hk}) \\
&= \frac{1}{2}\rho(R_{jkih,l}\omega_l + R_{lkih}\omega_{jl} + R_{jlih}\omega_{kl} + R_{jklh}\omega_{il} + R_{jkil}\omega_{hl}) \wedge \omega_h \\
&\quad + \frac{1}{2}\rho R_{jkih}\omega_l \wedge \omega_{lh} + \frac{1}{2}\rho R_{jkhs}\omega_s \wedge \omega_{ih} + \frac{1}{8}\rho^3 R_{ihlm}R_{jkhs}\omega_s \wedge \omega_{lm} \\
&\quad - \frac{1}{8}\rho(R_{jlis}\omega_s \wedge \omega_{lk} + R_{hkis}\omega_s \wedge \omega_{hj} - R_{klis}\omega_s \wedge \omega_{lj} \\
&\qquad\qquad - R_{hjis}\omega_s \wedge \omega_{hk}) \\
&= \frac{1}{2}\rho R_{jkih,l}\omega_l \wedge \omega_h \\
&\quad + \frac{1}{4}\rho(R_{jlih}\omega_h \wedge \omega_{lk} + R_{hkil}\omega_l \wedge \omega_{hj}) \\
&\quad + \frac{1}{8}\rho^3 R_{ihlm}R_{jkhs}\omega_s \wedge \omega_{lm}.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
R_{jkih,l}\omega_l \wedge \omega_h &= -(R_{jkhl,i}\omega_l \wedge \omega_h + R_{jkli,h}\omega_l \wedge \omega_h) \\
&= R_{jkhl,i}\omega_h \wedge \omega_l + R_{jkil,h}\omega_l \wedge \omega_h,
\end{aligned}
$$

that is,

$$
2R_{jkih,l}\omega_l \wedge \omega_h = R_{jkhl,i}\omega_h \wedge \omega_l.
$$

Thus we obtain

$$
\begin{aligned}
\Pi_{i;jk} = &\frac{1}{4}\rho R_{jkhl,i}\omega_h \wedge \omega_l \\
(3, 2) \qquad &+ \frac{1}{4}\rho(R_{ihjl}\omega_h \wedge \omega_{lk} - R_{ihkl}\omega_h \wedge \omega_{lj}) \\
&+ \frac{1}{8}\rho^3 R_{ihlm}R_{jkhs}\omega_s \wedge \omega_{lm}.
\end{aligned}
$$

Finally, we have

$$\begin{aligned}\Pi_{ij;kh} &= d\theta_{ij;kh} - \theta_{ij;l} \wedge \theta_{l;kh} - \frac{1}{2}\theta_{ij;lm} \wedge \theta_{lm;kh} \\ &= \frac{1}{2}\{\delta_{ik}(\Omega_{jh} + \omega_{jl} \wedge \omega_{lh}) + \delta_{jh}(\Omega_{ik} + \omega_{il} \wedge \omega_{lk}) \\ &\quad - \delta_{ih}(\Omega_{jk} + \omega_{jl} \wedge \omega_{lk}) - \delta_{jk}(\Omega_{ih} + \omega_{il} \wedge \omega_{lh}) \\ &\quad + \frac{1}{4}\rho^2 R_{ijlm} R_{khls}\omega_m \wedge \omega_s \\ &\quad - \frac{1}{8}(\delta_{il}\omega_{jm} + \delta_{jm}\omega_{il} - \delta_{im}\omega_{jl} - \delta_{jl}\omega_{im}) \\ &\quad \times (\delta_{lk}\omega_{mh} + \delta_{mh}\omega_{lk} - \delta_{lh}\omega_{mk} - \delta_{mk}\omega_{lh}),\end{aligned}$$

that is

$$\begin{aligned}\Pi_{ij;kh} &= \frac{1}{2}(\delta_{ik}\Omega_{jh} + \delta_{jh}\Omega_{ik} - \delta_{ih}\Omega_{jk} - \delta_{jk}\Omega_{ih}) \\ &\quad + \frac{1}{4}\rho^2 R_{ijlm} R_{khls}\omega_m \wedge \omega_s \qquad (3, 3) \\ &\quad + \frac{1}{4}(\delta_{ik}\omega_{jl} \wedge \omega_{lh} + \delta_{jh}\omega_{il} \wedge \omega_{lk} - \delta_{ih}\omega_{jl} \wedge \omega_{lk} \\ &\qquad - \delta_{jk}\omega_{il} \wedge \omega_{lh}).\end{aligned}$$

Now, let us put

$$\Pi_{AB} = \frac{1}{2} P_{ABCD}\hat{\omega}_C \wedge \hat{\omega}_D, \tag{3, 4}$$

where

$$A, B, \cdots = i, j, [ij], \cdots,$$
$$\hat{\omega}_i = \omega_i, \quad \hat{\omega}_{ij} = \rho\omega_{ij},$$

then P_{ABCD} are components of the Riemann-Cristoffel tensor of V_N with respect to the orthonormal frames which are derived from the ones of V_n.

In the first place, we get from (3, 1) the equations as follow.

$$\begin{aligned}P_{ijkh} &= R_{ijkh} + \frac{1}{4}\rho^2 R_{ijlm} R_{lmkh} \\ &\quad + \frac{1}{8}\rho^2(R_{lmik}R_{lmjh} - R_{lmih}R_{lmjk}),\end{aligned} \tag{3, 5}$$

$$P_{ij[kh]} = \frac{1}{2}\rho R_{ijkh,l}. \tag{3, 6}$$

Since

$$-\frac{1}{4}\rho^2 R_{ijkh}\omega_{kl} \wedge \omega_{lh} - \frac{1}{16}\rho^4 R_{iklm} R_{kjst}\omega_{lm} \wedge \omega_{st}$$

$$= \frac{1}{8}\rho^2(R_{ijml}\delta_{kh} - R_{ijkh}\delta_{ml})\omega_{kl} \wedge \omega_{mh}$$
$$+ \frac{1}{32}\rho^4(R_{iskl}R_{jsmh} - R_{ismh}R_{jskl})\omega_{kl} \wedge \omega_{mh},$$

we have

$$P_{i j[kl][mh]} = \frac{1}{2}(R_{ijkm}\delta_{lh} - R_{ijlm}\delta_{kh} - R_{ijkh}\delta_{lm} + R_{ijlh}\delta_{km}) \tag{3, 7}$$
$$+ \frac{1}{4}\rho^2(R_{iskl}R_{jsmh} - R_{ismh}R_{jskl}).$$

Since we have

$$\frac{1}{4}\rho(R_{ihjl}\omega_h \wedge \omega_{lk} - R_{ihkl}\omega_h \wedge \omega_{lj})$$
$$+ \frac{1}{8}\rho^3 R_{ihlm}R_{jkhs}\omega_s \wedge \omega_{lm}$$
$$= \frac{1}{4}\rho(R_{ihjl}\delta_{km} - R_{ihkl}\delta_{jm})\omega_h \wedge \omega_{lm}$$
$$+ \frac{1}{8}\rho^3 R_{islm}R_{jksh}\omega_h \wedge \omega_{lm},$$

we get from (3, 1)

$$P_{i[jk]h[lm]} = \frac{1}{4}(R_{ihjl}\delta_{km} - R_{ihkl}\delta_{jm} - R_{ihjm}\delta_{kl} + R_{ihkm}\delta_{jl})$$
$$- \frac{1}{4}\rho^2 R_{silm}R_{shjk}, \tag{3, 8}$$

and

$$P_{i[jk][lm][st]} = 0. \tag{3, 9}$$

Lastly, since

$$\frac{1}{4}\{\delta_{ik}\omega_{jl} \wedge \omega_{lh} + \delta_{jh}\omega_{il} \wedge \omega_{lk} - \delta_{ih}\omega_{jl} \wedge \omega_{lk} - \delta_{jk}\omega_{il} \wedge \omega_{lh}\}$$
$$= \frac{1}{16}\{\delta_{ik}\delta_{jl}\delta^{st}_{mh} - \delta_{ik}\delta_{jm}\delta^{st}_{lh} + \delta_{jh}\delta_{il}\delta^{st}_{mk} - \delta_{jh}\delta_{im}\delta^{st}_{lk}$$
$$- \delta_{ih}\delta_{jl}\delta^{st}_{mk} + \delta_{ih}\delta_{jm}\delta^{st}_{lk} - \delta_{jk}\delta_{il}\delta^{st}_{mh} + \delta_{jk}\delta_{im}\delta^{st}_{lh}\}\omega_{lm} \wedge \omega_{st}$$
$$= \frac{1}{16}\{\delta^{kl}_{ij}\delta^{st}_{mh} - \delta^{km}_{ij}\delta^{st}_{lh} + \delta^{lh}_{ij}\delta^{st}_{mk} - \delta^{mh}_{ij}\delta^{st}_{lk}\}\omega_{lm} \wedge \omega_{st}$$
$$= \frac{1}{32}\{\delta^{kl}_{ij}\delta^{st}_{mh} - \delta^{km}_{ij}\delta^{st}_{lh} + \delta^{lh}_{ij}\delta^{st}_{mk} - \delta^{mh}_{ij}\delta^{st}_{lk}$$
$$- \delta^{ks}_{ij}\delta^{lm}_{th} + \delta^{kt}_{ij}\delta^{lm}_{sh} - \delta^{sh}_{ij}\delta^{lm}_{tk} + \delta^{th}_{ij}\delta^{lm}_{sk}\}\omega_{lm} \wedge \omega_{st},$$ [1)]

we have

1) Where δ^{kh}_{ij} denote the generalized Kronecker's δ.

$$
\begin{aligned}
(3,10)\qquad & P_{[ij][kh][lm][st]} \\
&= \frac{1}{4}\{\delta_{ij}^{ks}\delta_{lm}^{ht} - \delta_{ij}^{kt}\delta_{lm}^{hs} - \delta_{ij}^{hs}\delta_{lm}^{kt} + \delta_{ij}^{ht}\delta_{lm}^{ks} \\
&\qquad - \delta_{ij}^{kl}\delta_{st}^{hm} + \delta_{ij}^{km}\delta_{st}^{hl} + \delta_{ij}^{hl}\delta_{st}^{km} - \delta_{ij}^{hm}\delta_{st}^{kl}\}
\end{aligned}
$$

(3, 10) shows that V_N *can not always become flat.*

Now, we shall calculate components of the Ricci curvature tensor with respect to the canonical orthonormal frames derived from those of V_n from (3, 5)—(3, 10).

Let us put

$$
(3,11)\qquad P_{AB} = \sum_C P_{ACBC}.
$$

We have

$$
\begin{aligned}
P_{ij} &= \sum_k P_{ikjk} + \frac{1}{2}\sum_{h,k} P_{i[hk]j[hk]} \\
&= R_{ikjk} + \frac{1}{4}\rho^2 R_{iklm}R_{lmjk} \\
&\qquad + \frac{1}{8}\rho^2(R_{lmij}R_{lmkk} - R_{lmik}R_{lmkj}) \\
&\qquad + \frac{1}{8}(R_{ijhh}\delta_{kk} - R_{ijkh}\delta_{hk} - R_{ijhk}\delta_{kh} - R_{ijkk}\delta_{hh}) \\
&\qquad\qquad - \frac{1}{8}\rho^2 R_{sihk}R_{sjhk},
\end{aligned}
$$

that is

$$
(3,12)\qquad P_{ij} = R_{ij} + \frac{1}{4}\rho^2 R_{iklm}R_{lmjk}.
$$

Nextly, we have

$$
\begin{aligned}
P_{i[jk]} &= P_{ih[jk]h} + \frac{1}{2}P_{i[lm][jk][lm]} \\
&= P_{ih[jk]h} = -\frac{1}{2}\rho R_{ihjk,h} \\
&= \frac{1}{2}\rho(R_{ihkh,j} + R_{ihhj,k}),
\end{aligned}
$$

that is

$$
(3,13)\qquad P_{i[jk]} = -\frac{1}{2}\rho(R_{ij,k} - R_{ik,j}).
$$

Then, we have

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$$
\begin{aligned}
P_{[ij][kh]} &= P_{[ij]l[kh]l} + \frac{1}{2} P_{[ij][st][kh][st]} \\
&= -\frac{1}{4}\rho^2 R_{ijlm} R_{lmkh} \\
&\quad + \frac{1}{8}\{\delta^{ss}_{ij}\delta^{tt}_{kh} - \delta^{st}_{ij}\delta^{ts}_{kh} - \delta^{ts}_{ij}\delta^{st}_{kh} + \delta^{tt}_{ij}\delta^{ss}_{kh} \\
&\quad - \delta^{sk}_{ij}\delta^{th}_{st} + \delta^{sh}_{ij}\delta^{tk}_{st} + \delta^{tk}_{ij}\delta^{sh}_{st} - \delta^{th}_{ij}\delta^{sk}_{st}\} \\
&= -\frac{1}{4}\rho^2 R_{ijlm} R_{lmkh} - \frac{1}{2}(n-2)\,\delta^{kh}_{ij},
\end{aligned}
$$

hence

$$
P_{[ij][kh]} = -\frac{1}{4}\rho^2 R_{ijlm} R_{lmkh} - \frac{1}{2}(n-2)\delta^{kh}_{ij}. \tag{3, 14}
$$

Lastly we get from (3, 12), (3, 14) the scalar curvature of V_N as follows.

$$
\begin{aligned}
P &= P_{ii} + \frac{1}{2} P_{[ij][ij]} \\
&= R_{ii} + \frac{1}{4}\rho^2 R_{iklm} R_{iklm} - \frac{1}{8}\rho^2 R_{ijlm} R_{ijlm} - \frac{n(n-1)(n-2)}{4},
\end{aligned}
$$

that is

$$
P = -\frac{1}{4} n(n-1)(n-2) + R + \frac{1}{8}\rho^2 R_{ijlm} R_{ijlm}. \tag{3, 15}
$$

§ 4. **Some special cases.** In this section, we shall consider the spaces V_n whose associated Riemannian spaces V_N are Einstein spaces, that is

$$
P_{AB} = \frac{P}{N}\delta_{AB}. \tag{4, 1}
$$

These equations are written by (3, 12)−(3, 14) as

$$
R_{ij,k} - R_{ik,j} = 0, \tag{4, 2}
$$

$$
R_{ij} + \frac{1}{4}\rho^2 R_{iklm} R_{jklm} = \frac{P}{N}\delta_{ij}, \tag{4, 3}
$$

$$
-\frac{(n-2)}{2}\delta^{kh}_{ij} - \frac{1}{4}\rho^2 R^{i}{}_{jlm} R_{lmkh} = \frac{P}{N}\delta^{kh}_{ij}. \tag{4, 4}
$$

By contraction, we get from (4, 3), (4, 4), (3, 15)

$$R+\frac{1}{4}\rho^2 R_{ijlm}R_{ijlm}=\frac{P}{N}n$$
$$=\frac{2}{n+1}\left\{-\frac{n(n-1)(n-2)}{4}+R+\frac{1}{8}\rho^2 R_{ijlm}R_{ijlm}\right\}$$

and

$$-\frac{1}{2}n(n-1)(n-2)-\frac{1}{4}\rho^2 R_{ijlm}R_{ijlm}=\frac{P}{N}n(n-1)$$
$$=\frac{2(n-1)}{n+1}\left\{-\frac{n(n-1)(n-2)}{4}+R+\frac{1}{8}\rho^2 R_{ijlm}R_{ijlm}\right\},$$

that is

$$(n-1)R+\frac{1}{4}n\rho^2 R_{ijlm}R_{ijlm}=-\frac{1}{2}n(n-1)(n-2), \tag{4, 5}$$

which shows that (4, 3) and (4, 4) are linearly dependent.

Now, we get by (3, 15) and (4, 5) the equation

$$P=\frac{(n+1)}{2n}R-\frac{(n+1)(n-1)(n-2)}{4} \tag{4, 6}$$

Substituting (4, 6) into (4, 3) and (4, 4), we have

$$R_{ij}+\frac{1}{4}\rho^2 R_{iklm}R_{jklm}=\left(\frac{1}{n^2}R-\frac{(n-1)(n-2)}{2n}\right)\delta_{ij}, \tag{4, 3'}$$

$$-\frac{1}{4}\rho^2 R_{ijlm}R_{khlm}=\left(\frac{1}{n^2}R+\frac{n-2}{2n}\right)\delta^{kh}_{ij}. \tag{4, 7}$$

By contraction, we get from (4, 7)

$$-\frac{1}{4}\rho^2 R_{iklm}R_{jklm}=\left(\frac{n-1}{n^2}R+\frac{(n-1)(n-2)}{2n}\right)\delta_{ij}, \tag{4, 8}$$

hence this and (4, 3′) follow the equation

$$R_{ij}=\frac{1}{n}R\delta_{ij}. \tag{4, 9}$$

(4, 9) shows that if V_N is an Einstein space, then V_n is also an Einstein space.

If $n>2$ and V_n is an Einstein space, then, as is well known, (4, 2) is automatically satisfied. If $n=2$, since $R_{ij}=\frac{1}{2}R\,\delta_{ij}$, (4, 2) becomes

$$R_{,1}=R_{,2}=0,$$

that is $R=$ constant. But, if V_N for V_2 is an Einstein space, then, by

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means of (4, 6), R is constant because $N > 2$. Then we obtain the theorem.

Theorem 1. *Let V_n be an n-dimensional Riemannian space and V_N be the Riemannian space of dimension $N = \frac{1}{2}n(n+1)$ associated with V_n. In order that V_N be an Einstein space, it is necessary and sufficient that V_n be an Einstein space and satisfy the equation*

$$-\frac{1}{4}\rho^2 R_{ij}{}^{lm} R_{lm}{}^{kh} = \frac{1}{n}\left(\frac{R}{n} + \frac{n-2}{2}\right)\delta_{ij}^{kh}.$$

Proof. The necessity of the conditions is evident by the arguments above. We shall prove the sufficiency.

Since V_n is an Einstein space, (4, 2) is clearly satisfied for $n > 2$. When $n = 2$, (4, 2) is equivalent to $R =$ constant but it can be derived from (4, 7).

We get from (3, 15), (4, 7)

$$\frac{P}{N} = \frac{2}{n(n+1)}\left\{-\frac{n(n-1)(n-2)}{4} + R - \frac{1}{2}\left(\frac{n-1}{n}R + \frac{(n-1)(n-2)}{2}\right)\right\}$$

$$= -\frac{(n-1)(n-2)}{2n} + \frac{R}{n^2}.$$

On the other hand, we get from (4, 7)

$$-\frac{n-2}{2}\delta_{ij}^{kh} - \frac{1}{4}\rho^2 R_{ijlm} R_{lmkh}$$

$$= -\frac{n-2}{2}\delta_{ij}^{kh} + \left(\frac{1}{n^2}R + \frac{n-2}{2n}\right)\delta_{ij}^{kh}$$

$$= \left(-\frac{(n-1)(n-2)}{2n} + \frac{R}{n^2}\right)\delta_{ij}^{kh} = \frac{P}{N}\delta_{ij}^{kh}.$$

Analogously we have

$$R_{ij} + \frac{1}{4}\rho^2 R_{iklm} R_{jklm}$$

$$= \frac{R}{n}\delta_{ij} - \left(\frac{n-1}{n^2}R + \frac{(n-1)(n-2)}{2n}\right)\delta_{ij} = \frac{P}{N}\delta_{ij}.$$

Thus we see that the system of equations (4, 2), (4, 8) and (4, 4) is equivalent to the one of (4, 7) and (4, 9).

Now, let V_n be a space of constant curvature, that is whose curvature tensor satisfies the equations

$$R_{ijkh} = -K(\delta_{ik}\delta_{jh} - \delta_{ih}\delta_{jk}),$$
$$K = \text{constant}$$

with respect to orthonormal frames. Since

$$R_{ij} = -(n-1)K_{ij}, \quad R = -n(n-1)K,$$
$$\frac{1}{n^2}R + \frac{n-2}{2n} = -\frac{n-1}{n}K + \frac{n-2}{2n}$$

and

$$-\frac{1}{4}\rho^2 R_{ij}{}^{lm} R_{lm}{}^{kh} = -\frac{1}{2}\rho^2 K^2 \delta_{ij}^{kh}$$

it follows that if ρ is a constant such that

(4, 10) $$\rho^2 K^2 = \frac{2(n-1)}{n}K - \frac{n-2}{n},$$

then V_N becomes an Einstein space. Thus we have a corollary.

Corollary. *An n-dimensional Riemann space of constant curvature K has an Einstein space as its associated Riemann space if and only if* $K > \frac{n-2}{2(n-1)}$ (or $-R > \frac{n(n-2)}{2}$).

Lastly, we shall consider the special case $n = 3$. Putting

(4, 11) $$\begin{aligned} R_{2323} &= -K_{11}, & R_{3112} &= -K_{23} = -K_{32}, \\ R_{3131} &= -K_{22}, & R_{1223} &= -K_{31} = -K_{13}, \\ R_{1212} &= -K_{33}, & R_{2331} &= -K_{12} = -K_{21}, \end{aligned}$$

we have

(4, 12) $$\begin{cases} R_{11} = -K_{22} - K_{33}, & R_{23} = K_{32} \\ R_{22} = -K_{33} - K_{11}, & R_{31} = K_{13}, \\ R_{33} = -K_{11} - K_{22}, & R_{12} = K_{21} \end{cases}$$

and

(4, 13) $$R = -2\sum_{i=1}^{3} K_{ii}.$$

Then, (4, 9) becomes

(4, 14) $$K_{11} = K_{22} = K_{33} = \kappa,$$
$$K_{23} = K_{31} = K_{12} = 0.$$

Since

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$$R_{23lm}R_{lm23} = 2\sum K_{1i}K_{i1} = 2\kappa^2,$$
$$R_{23lm}R_{lm31} = 2\sum K_{1j}K_{j2} = O,$$
etc.,

(4, 7) becomes

$$\frac{1}{2}\rho^2\kappa^2 = \frac{2}{3}\kappa - \frac{1}{6} \tag{4, 15}$$

Since $R = -6\kappa$, (4, 15) yields the following corollary.

Corolloary. *A 3-dimensional Einstein space has an Einstein space as its associated Riemann space if and only if* $-R > \frac{3}{2}$.

Remark. (4, 7) follows that $R + \frac{1}{2}n(n-2) < 0$.

§ **5. Geodesics in V_N.** The differential equations of geodesics in V_N are

$$\frac{d\omega_i + \theta_{ki}\omega_k + \frac{1}{2}\rho\gamma_{kh;i}\omega_{kh}}{\omega_i} = \frac{\rho d\omega_{ij} + \theta_{k;ij}\omega_k + \frac{1}{2}\rho\gamma_{kh;ij}\omega_{kh}}{\rho\omega_{ij}}.$$

Since we have by means of (2, 18)

$$\begin{aligned} &d\omega_i + \theta_{ki}\omega_k + \frac{1}{2}\rho\gamma_{kh;i}\omega_{kh} \\ &= d\omega_i + (\omega_{ki} + \frac{1}{4}\rho^2 R_{kilh}\omega_{lh})\omega_k - \frac{1}{4}\rho^2 R_{khij}\omega_j\omega_{kh} \\ &= d\omega_i + \omega_{ki}\omega_k - \frac{1}{2}\rho^2 R_{ijkh}\omega_j\omega_{kh}, \end{aligned}$$

and

$$\begin{aligned} &\rho d\omega_{ij} + \theta_{k;ij}\,\omega_k + \frac{1}{2}\rho\gamma_{kh;ij}\omega_{kh} \\ &= \rho d\omega_{ij} + \frac{1}{2}\rho R_{ijkh}\omega_k\omega_h + \frac{1}{4}(\delta_{ki}\omega_{hj} + \delta_{hj}\omega_{ki} - \delta_{kj}\omega_{hi} - \delta_{hi}\omega_{kj})\omega_{kh} \\ &= \rho d\omega_{ij}, \end{aligned}$$

the equations of geodesics in V_N become

$$(5,1)\qquad \frac{d\omega_i+\omega_{ki}\omega_k-\frac{1}{2}\rho^2R_{ijkh}\omega_i\omega_{kh}}{\omega_i}=\frac{d\omega_{ij}}{\omega_{ij}}$$

Let $\bar{C}$ be a geodesic in V_N and C be its image in V_n by the projection $p: V_N\to V_n$. Let τ, s be the arclengths of $\bar{C}$, C respectively. Then, (5, 1) is written as

$$\begin{cases}\dfrac{d\omega_i}{d\tau^2}+\dfrac{\omega_{ki}}{d\tau}\dfrac{\omega_k}{d\tau}-\dfrac{1}{2}\rho^2R_{ijkh}\dfrac{\omega_j}{d\tau}\dfrac{\omega_{kh}}{d\tau}=0,\\ \dfrac{d\omega_{ij}}{d\tau^2}=0.\end{cases}$$

Hence, we have

$$(5,2)\qquad \begin{cases}\dfrac{d\omega_i}{d\tau^2}+\dfrac{\omega_{ki}}{d\tau}\dfrac{\omega_k}{d\tau}=\dfrac{1}{2}\rho^2R_{jikh}\dfrac{\omega_j}{d\tau}c_{kh},\\ \omega_{ij}=c_{ij}d\tau,\end{cases}$$

where $c_{ij}=-c_{ji}$ are constants. In a local coordinate neighborhood (x^λ), if we put

$$(5,3)\qquad dx^\lambda=y_i^\lambda\omega_i,\qquad g^{\lambda\mu}=y_i^\lambda y_i^\mu,$$

then since we have

$$\omega_{ik}y_k^\lambda=y_i^\mu\{{}^{\lambda}_{\mu\nu}\}dx^\nu+dy_i^\lambda,$$

where $\{{}^{\lambda}_{\mu\nu}\}$'s are the Christoffel symbols made by $g_{\lambda\mu}$, (5, 2) is written as

$$(5,2')\qquad \begin{cases}\dfrac{D}{d\tau}\dfrac{dx^\lambda}{d\tau}=\dfrac{1}{2}\rho^2R^\lambda{}_{\mu\nu\omega}\dfrac{dx^\mu}{d\tau}y_i^\nu y_j^\omega c_{ij},\\ \dfrac{Dy_i^\lambda}{d\tau}=c_{ij}y_j^\lambda,\end{cases}$$

where $\{{}^{\lambda}_{\mu\nu}\}$'s are the Christoffel symbols made by $g_{\lambda\mu}$ and D denotes the covariant differential in V_n. From (5, 2), we see that

$$\frac{ds}{d\tau}=k=\text{constant}\qquad 0\leqslant k\leqslant 1.$$

This equation shows that a geodesic in V_N has a constant angle with the field Γ of n-dimensional horizontal tangent subspaces $\Gamma_b\subset T_b(B)$, $b\in B$, which will be defined in §6. By means of (1, 4), we must have

(5, 3) $$1=k^2+\frac{1}{2}\rho^2 c_{ij}c_{ij}.$$

In the case $k=0$, *C is clearly a point curve, hence* $\bar{C}$ *is the image of a one parameter subgroup of* O_n *by an admissible mapping of the fibre bundle* $\mathfrak{B}$. *In the case* $k=1$, *we have* $c_{ij}=0$ *by the above equation, hence C is a geodesic in* V_n *and the points of* $\bar{C}$ *are the parallel displaced orthonormal frames of* V_N *along C.* For an example, if V is a 2-sphere, then C is a circle on the sphere.

§ 6. The Levi-Civita's connection of V_n and its explanation in V_N.

According to § 1, let $\mathfrak{B}=\{B, p, V_n, O_n, O_n\}$ be the associated principal fibre bundle of V_n. Any point $b \in B$ is represented as

(6, 1) $$b=(x(b),\ e_i(b))$$

where $x(b)=p(b)$ and $e_i(b)$, $i=1, 2, \cdots n$, are unit tangent vectors to V_n at $x(b)$ and orthonal each others.

Let $\mathfrak{v}_i(b)$, $\mathfrak{v}_{ij}(b)$, $i<j$, $i, j=1, 2, \cdots, n$, be tangent vectors to B dual to $\omega_i(b)$, $\omega_{ij}(b)$, $i<j$, $i, j=1, 2, \cdots, n$.

In the following, for a differentiable mapping f of a differentiable manifold X into another differentiable manifold Y, we shall denote the differential mapping of f by $f_*: T(X)\rightarrow T(Y)$ and the dual mapping of f_* by $f^*: T^*(Y)\rightarrow T^*(X)$, where $T(X)$, $T(Y)$ ($T^*(X)$, $T^*(Y)$) are the spaces of tangent (cotangent) vectors to X, Y respectively.

Since $\omega_i(b)=p^*\omega_i(b)$, where $\omega_i(b)$ in the right-hand side is regarded as a cotangent vector to V_n at $x(b)$ such that $\delta_{ij}=\langle \omega_i(b),\ e_j(b)\rangle$, we have $\langle \omega_i(b), p_*\mathfrak{v}_j(b)\rangle=\langle \omega_i(b), \mathfrak{v}_j(b)\rangle=\delta_{ij}$, hence

(6. 2) $$p_*\mathfrak{v}_i(b)=e_i(b).$$

Analogously we have

(6, 3) $$p_*\mathfrak{v}_{ij}(b)=0.$$

For any $\alpha=((a_i^j(\alpha)))\in O_n$, we denoted the right translation corresponding to α by $r(\alpha)$ which is defined by

(6, 4) $$r(\alpha)(b)=(x(b),\ a_i^j(\alpha)e_j(b)),$$

where $((a_i^j(\alpha)))$ is an n-dimensional orthogonal matrix. Since we have

$$a_i^j(\alpha_2\alpha_1)=a_k^j(\alpha_2)a_i^k(\alpha_1),\quad \alpha_1,\ \alpha_2\in O_n,$$

it follows that

(6, 5) $$r(\alpha_1)\circ r(\alpha_2)=r(\alpha_2\alpha_1).$$

Now, we shall consider $(r(\alpha))^*$. Let $b=f(x)=(x, e_i(x))$ be a differentiable local cross-section of $\mathfrak{B}$ defined on a neighborhood U in V_n. Let us put

(6, 6) $$e_i(b)=y_i^k(b)\,\mathrm{e}_k(x),\quad x=p(b),$$

then we can consider $x(b)$, $y_i^j(b)$ as local coordinates of the point b. Let us put

(6, 7) $$\theta_k(x)=f^*\omega_k(b),\quad \theta_{kh}(x)=f^*\omega_{kh}(b),$$

then we have the equations

(6, 8) $$\begin{cases}\omega_i(b)=z_k^i(b)\theta_k(x),\\ \omega_{ij}(b)=y_i^k(b)z_h^j(b)\theta_{kh}(x)+z_k^j(b)dy_i^k(b)\end{cases}$$

in the coordinates $x(b)$, $y_i^j(b)$, where $y_i^k(b)\,z_k^j(b)=\delta_i^j$.

Since we have from (6, 4) (6, 6)

$$r(\alpha)(b)=(x, a_i^j(\alpha)y_j^k(b)e_k(x)),\quad x=p(b),$$

we get

(6, 9) $$\begin{aligned}&r(\alpha)^*\omega_i(r(\alpha)(b))=a_j^i(\alpha^{-1})\omega_j(b)=a_i^j(\alpha)\omega_j(b),\\ &r(\alpha)^*\omega_{ij}(r(\alpha)(b))=a_i^k(\alpha)a_j^h(\alpha)\omega_{kh}(b).\end{aligned}$$

Accordingly, we get

(6, 10) $$\begin{cases}r(\alpha)_*\mathfrak{v}_j(b)=a_i^j(\alpha)\mathfrak{v}_i(r(\alpha)(b)),\\ r(\alpha)_*\mathfrak{v}_{kh}(b)=a_i^k(\alpha)a_j^h(\alpha)\mathfrak{v}_{ij}(r(\alpha)(b)).\end{cases}$$

Now, let be Γ_b the tangent subspace to B at b spaned by $\mathfrak{v}_1(b),\cdots,\mathfrak{v}_n(b)$ which define a differentiable field Γ on V_N. By (6, 2), (6, 10) it follows that

(6, 11) $$\begin{cases}r(\alpha)_*\Gamma_b=\Gamma_{r(\alpha)(b)},\quad \alpha\in O_n,\\ p_*\Gamma_b=T_{p(b)}(V_n).\end{cases}$$

Let $\mu_b: T_b(B)\rightarrow T_b(O_n(x))$, $p(b)=x$, be the projection defined by

(6, 12) $$\mu_b(\sum v_i \mathfrak{v}_i(b) + \sum_{i<k} v_{ij}\mathfrak{v}_{ij}(b)) = \sum_{i<j} v_{ij}\mathfrak{v}_{ij}(b)$$

and denote also by b the admissible mapping $O_n \to O_n(x)$ defined by $b(\alpha) = r(\alpha)(b)$, $\alpha \in O_n$. Putting $\pi_b = (b_*)^{-1}\mu_b : T_b(B) \to T_e(O_n)$, $e =$ the identity of O_n, we obtain a $L(O_n)$-valued differential from π defined on B by $\pi \mid T_b(B) = \pi_b$. Then it follows from (6, 11) that $r(\alpha)^*\pi = ad(\alpha^{-1})\cdot\pi$.

Let $\iota_x : O_n(x) \to B$ be the imbedding mapping then we get from (6, 8)

(6, 13) $$b^*\iota_x^*\omega_{ij}(b(\alpha)) = a_k^j(\alpha^{-1})da_i^k(\alpha) = a_j^k(\alpha)da_i^k(\alpha) = \hat{\omega}_{ij}(\omega),$$

which are left invariant differential forms on O_n.

Let $(\hat{\mathfrak{v}}_{ij}(\alpha))$ be the tangent vector fields on O_n dual to $(\hat{\omega}_{ij}(\alpha))$. Since

$$\begin{aligned}
&< \omega_{ij}(b), (\iota_x b)_*\hat{\mathfrak{v}}_{kh}(e) > = < \hat{\omega}_{ij}(e), \hat{\mathfrak{v}}_{kh}(e) >, \\
&< \omega_i(b), (\iota_x b)_*\hat{\mathfrak{v}}_{kh}(e) > = < \iota_x^*\omega_i(b), b_*\hat{\mathfrak{v}}_{kh}(e) > \\
&= < 0, b^*\hat{\mathfrak{v}}_{kh}(e) > = 0,
\end{aligned}$$

we have

(6, 14) $$(\iota_x b)_*\hat{\mathfrak{v}}_{ij}(e) = \mathfrak{v}_{ij}(b).$$

Let $\iota : p^{-1}(U) \to B$ be the imbedding mapping and define a mappings $\rho : U \to U \times O_n$, $\phi : U \times O_n \to B$ by

$$\begin{aligned}
&\rho(x) = x \times e, \\
&\phi(x, \alpha) = (x, a_i^j(\alpha)e_j(f(x))).
\end{aligned}$$

Then we have a $L(O_n)$-valued differential from $\theta = (\iota\phi\rho)^*$ on U. Since $(\iota\phi\rho)(x) = (x, e_i(f(x)))$, we get from (6, 8)

$$\begin{aligned}
&(\iota\phi\rho)^*\omega_i(f(x)) = \theta_i(x), \\
&(\iota\phi\rho)^*\omega_{ij}(f(x) = \theta_{ij}(x) = \Gamma_{ik}^j(x)\vartheta_k(x)
\end{aligned}$$

hence

$$(\iota\phi\rho)_*e_k(f(x)) = \mathfrak{v}_k(f(x)) + \frac{1}{2}\Gamma_{ik}^j\mathfrak{v}_{ij}(f(x)).$$

Accordingly, we have by (6, 14)

$$\begin{aligned}
&< \hat{\theta}, e_k(x) > = < \pi, (\iota\phi\rho)_*e_k(x) > \\
&= < \pi, \mathfrak{v}_k(f(x)) + \frac{1}{2}\Gamma_{ik}^j(x)\mathfrak{v}_{ij}(f(x)) > \\
&= \frac{1}{2} < \pi, \Gamma_{ik}^j(x)\mathfrak{v}_{ij}(f(x)) > = \frac{1}{2}\Gamma_{ik}^j(x)\hat{\mathfrak{v}}_{ij}(e).
\end{aligned}$$

On the other hand, we can define canonically a $L(O_n)$-valued differential form θ on U from θ_{ij} by

$$\theta(v) = \langle \theta_i^j(x),\ v \rangle\ \hat{\mathfrak{v}}_{ij}(e),\ \ v \in T_x(V_n)$$

this shows that

$$\theta = \hat{\theta}.$$

That is, *the parameters* $\hat{\theta}$ *on* $U \subset V_n$ *derived from the connection in the sense of C. Ehresmann* [3] *defined by the field of tangent subspaces* Γ_b *by the local cross section* $f: U \to B$ *are the parameters* θ *on* U *of the Levi-Civita connection of* V_n *with respect to the field of orthonormal frames defined by* f.

§ 7. **Motions of V_N derived from motions of V_n.** Let f be a motion of V_n, that is a homeomorphism onto itself such that $(f(x_1), f(x_2)) = (x_1, x_2)$, $x_1, x_2 \in V_n$, where (x_1, x_2) denotes the Riemannian distance in V_n between x_1 and x_2. As is well known, f is differentiable. Furthermore we have

$$(7,1)\qquad (f_* X_1)\cdot(f_* X_2) = X_1 \cdot X_2,\ \ X_1, X_2 \in T_x(V_n),$$

where $X_1 \cdot X_2$ denotes the inner product of X_1 and X_2. Accordingly, we can define a differentiable homeomorphism $\bar{f} = \chi(f)$ on B by

$$(7,2)\qquad \bar{f}(b) = (f(x(b)),\ f_* e_i(b)),\ \ b \in B.$$

Since $p\bar{f} = fp$, we have $p_* \bar{f}_* \mathfrak{v}_i(b) = f_* p_* \mathfrak{v}_i(b) = f_* e_i(b) = e_i(\bar{f}(b))$ by (6, 2), (7, 2), hence $\bar{f}_* \mathfrak{v}_i(b) = \mathfrak{v}_i(\bar{f}(b)) +$ a linear combinations of $\mathfrak{v}_{ij}(\bar{f}(b))$ and $\bar{f}^* \omega_i(\bar{f}(b)) = \omega_i(B) +$ a linear combination of $\omega_{ij}(b)$. On the other hand, since we can consider $\omega_i(b)$ as differential forms in V_n, we obtain

$$(7,3)\qquad \bar{f}^* \omega_i(\bar{f}(b)) = \omega_i(b).$$

Furthermore, we get from (1, 2), (7, 3)

$$d\omega_i = \sum \omega_k \wedge \bar{f}^* \omega_{ki},$$
$$\bar{f}^* \omega_{ij} = -\bar{f}^* \omega_{ij},$$

hence we have

$$(7,4)\qquad \bar{f}^* \omega_{ij}(\bar{f}(b)) = \omega_{ij}(b).$$

Thus we obtain the following theorem.

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Theorem 2. *If f is a motion of V_n, then the transformation $\bar{f} = \chi(f)$ derived from f by* (7, 2) *is also a motion of V_N and*

$$\chi(f_1 \circ f_2) = \chi(f_1) \circ \chi(f_2). \tag{7, 5}$$

Now, denoting the group of motions of V_n by $M(V_n)$, we get easily from (6, 9) the following theorem.

Theorem 3. *Any right translation of B is a motion of V_N and commutes with $\chi(f)$, $f \in M(V_n)$.*

It is sufficient to prove the second part of the theorem.

For $\alpha \in O_n$, $f \in M(V_n)$, $b \in B$, we have

$$\begin{aligned} r(\alpha)(\chi(f)(b)) &= r(\alpha)((f(x(b)), f_* e_i(b))) \\ &= (f(x(b)), a_i^j(\alpha) f_* e_j(b)) \\ &= (f(x(b)), f_*(a_i^j(\alpha) e_j(b))) \\ &= \chi(f)((x)(b), a_i^j(\alpha) e_j(b))) \\ &= \chi(f)(r(\alpha)(b)). \end{aligned}$$

Hence we have the relation

$$r(\alpha) \circ \chi(f) = \chi(f) \circ r(\alpha). \tag{7, 6}$$

We see also easily that

$$r(O_n) \cap \chi(M(V_n)) = 1, \tag{7, 7}$$

where 1 denotes the identity transformation.

§ 8. **Some mappings on V_N.** Now, let S^{n-1} be the $(n-1)$-dimensional unit sphere: $\sum w^i w^i = 1$ in an n-dimensional Euclidean space R^n. For any complete Riemannian manifold V_n, we define a mapping Φ: $B \times S^{n-1} \times R \to B$ as follows:

For $b \in B$, $\mathfrak{w} = (w^1, \cdots, w^n) \in S^{n-1}$, $s \in R$, let $\gamma(b, \mathfrak{w}, s)$ be the geodesic arc in V_n starting at $p(b) = x(b)$ whose tangent unit vector at $x(b)$ is $w^i e_i(b)$ and whose length is s. Let $F(b, \mathfrak{w}, s)$ be the end point of $\gamma(b, \mathfrak{w}, s)$. By parallel displacing $e_i(b)$ along this geodesic, we get a curve $\bar{\gamma}(b, \mathfrak{w}, s)$ in B whose points are these frames, hence $p(\bar{\gamma}(b, \mathfrak{w}, s)) = \gamma(b, \mathfrak{w}, s)$. Let $\Phi(b, \mathfrak{w}, s)$ be the end point of $\bar{\gamma}(b, \mathfrak{w}, s)$.

The mapping Φ is clearly differentibale and have the following properties:

$$p(\Phi(b, \mathfrak{w}, s)) = F(b, \mathfrak{w}, s). \tag{8, 1}$$

(8, 2) $\qquad r(\alpha)\circ\Phi(b, \mathfrak{w}, s) = \Phi(r(\alpha)b, \ \alpha^{-1}\mathfrak{w}, s), \ \alpha\in O_n,$

(8, 3) $\qquad F(b, \mathfrak{w}, s) = F(r(\alpha)b, \ \alpha^{-1}\mathfrak{w}, s),$

(8, 4) $\qquad \Phi(b, \mathfrak{w}, 0) = b.$

Furthermore, since any motion of V_n preserves geodesics and parallel displaced vector fields, it follows that

(8, 5)
$$\bar{f}(\Phi(b, \mathfrak{w}, s)) = \Phi(\bar{f}(b), \mathfrak{w}, s),$$
$$\bar{f} = \chi(f), \quad f\in M(V_n),$$

and by $p\circ\bar{f} = f\circ p$ and (8, 2),

(8, 6)
$$f\circ F(b, \mathfrak{w}, s) = F(\bar{f}(b), \mathfrak{w}, s).$$

Let $\{f_m\}$, $m = 1, 2, \cdots,$ be a sequence of motions on V_n. For a fixed point $b_0\in B$, we suppose that $\bar{f}_m(b_0)$, $\bar{f}_m = \chi(f_m)$, converge to a point b'_0. For any point $b\in B$, we can take an $\alpha\in O_n$, a $\mathfrak{w}\in S^{n-1}$ and a real number s such that $b = r(\alpha)((\Phi(b_0, \mathfrak{w}, s))$. Hence, by (7,'5), (8, 5), we get

$$\begin{aligned}\lim \bar{f}_m(b) &= \lim \bar{f}_m(r(\alpha)\ (\Phi(b_0, \mathfrak{w}, s)))\\ &= \lim r(\alpha)\ (\bar{f}_m(\Phi(b_0, \mathfrak{w}, s)))\\ &= r(\alpha)\ (\lim \Phi(\bar{f}_m(b_0), \mathfrak{w}, s)),\end{aligned}$$

that is

(8, 7)
$$\lim \bar{f}_m(b) = r(\alpha)\ (\Phi(b'_0, \mathfrak{w}, s)).$$

Thus we can define a limiting map $\hat{f}: B\to B$ by

(8, 8)
$$\hat{f}(b) = \lim \chi(f_m)\ (b), \quad b\in B,$$

which is clearly a motion of V_N. By the above equation, we get easily

(8, 9)
$$\hat{f}\circ r(\alpha_1) = r(\alpha_1)\circ\hat{f}, \quad \alpha_1\in O_n.$$

Furthermore, since we have $p(\hat{f}(b)) = \lim f_m(p(b))$, we get a limiting map $f: V_n\to V_n$, by

$$f(x) = \lim f_m(x), \quad x\in V_n,$$

such that $f\in M(V_n)$ and

$$f\circ p = p\circ\hat{f}.$$

Now, we have by (8, 1), (8, 7) the relation

$$f(x) = F(b'_0, \mathfrak{w}, s),$$
$$x = p(b), \quad b = r(\alpha)(\Phi(b_0, \mathfrak{w}, s)).$$

On the other hand, we get from (8, 5) the equation

$$p(\bar{f}(\Phi(b_0, \mathfrak{w}, s))) = f(p(\Phi(b_0, \mathfrak{w}, s))) = f(x)$$
$$= p(\Phi(\bar{f}(b_0), \mathfrak{w}, s)) = F(\bar{f}(b_0), \mathfrak{w}, s),$$
$$\bar{f} = \chi(f),$$

hence

$$F(\dot{f}(b_0), \mathfrak{w}, s) = F(\bar{f}(b_0), \mathfrak{w}, s)$$
$$w \in S^{n-1}, \quad s \in R.$$

It follows that $\dot{f}(b_0) = \bar{f}(b_0)$ and by (8, 5), (7, 5)

$$\dot{f}(b) = r(a)(\Phi(\bar{f}(b_0), \mathfrak{w}, s))$$
$$= r(a)(\bar{f}(\Phi(b_0, \mathfrak{w}, s)) = \bar{f}(r(a)(\Phi(b_0, \mathfrak{w}, s))$$
$$= \bar{f}(b),$$

that is

(8, 10) $$\lim \chi(f_m)(b) = \chi(\lim f_m)(b), \quad b \in B$$

For any V_n which is not complete, we can carry the same argument by means of a finite number of points of B such as b_0, Thus, we obtain.

Theorem 4. *Let V_n be a Riemannian manifold and let $\{f_m\}$, $m =$ 1, 2, ⋯, be a sequence of motions of V_n. Then the sequence $\{\chi(f_m)\}$ is simultaneously convergent or do not convergent at every point of B. In the first case, we have*

$$\lim \chi(f_m)(b) = \chi(\lim f_m)(b), \quad b \in B.$$

In the next place, we suppose that for a sequence $\{f_m\}$ of motions of V_n, $\lim f_m(x_0) = x'_0$. For a subsequence $\{f_{m_\lambda}\}$ of $\{f_m\}$, we may suppose that $\lim \bar{f}_{m_\lambda}(b_0) = b'_0$, where b_0 is a fixed element in $p^{-1}(x_0)$. Then, by means of the above theorem there exists a $f \in M(V_n)$ such that $f(x) = \lim_{\lambda\to\infty} f_{m_\lambda}(x)$ and $\chi(f)(b) = \lim_{\lambda\to\infty} \chi(f_{m_\lambda})(b)$, $x \in V_n$, $b \in B$. Accordingly, we see that if $\lim f_m(x) = f(x)$, then $\lim \chi(f_m)(b) = \chi(f)(b)$. Thus, we obtain

Theorem 5. *For any V_n, $\chi: M(V_n) \to M(V_N)$ is continuous in the sense of weakly convergence, that is, if $\lim f_m(x) = f(x)$, then $\lim \chi(f_m)(b) = \chi(f)(b)$, $x \in V_n$, $b \in V_N$.*

§ 9. **Tangent vector fields over V_n derived from sequences of motions of V_n.** For the sake of simplicity, let V_n be a complete Riemannian manifold. For any $f \in M(V_n)$, $x \in V_n$, since we have by Theorem 3

$$(b, \chi(f)(b)) = (r(\alpha)(b), r(\alpha)(\chi(f)(b))) \\ = (r(\alpha)(b), \chi(f)(r(\alpha)(b))), \\ b \in p^{-1}(x), \ \alpha \in O_n,$$

we define a function $u_f : V_n \to R$ by

(9, 1) $$u_f(x) = (b, \chi(f)(b)), \ b \in p^{-1}(x),$$

which is differentiable. If $f \neq 1$, then everywhere $u_f(x) \neq 0$ by (8, 5).

Now, let be given a sequence $\{f_m\}$, $m = 1, 2, \cdots$, of motions of V_n which are mutually distinct and weakly converge to the identity transformation. For simplicity, we put

$$u_m(x) = u_{f_m}(x), \ x \in V_n.$$

By Theorem 5, we have

$$\lim_{m \to \infty} \chi(f_m)(b) = b, \ b \in B.$$

Then we define a tangent vector field γ over V_n by

(9, 2) $$\gamma(x)(h) = \lim_{m \to \infty} \frac{(f_m{}^*h)(x) - h(x)}{u_m(x)},$$

where $x \in V_n$ and h is any differentiable function defined on an open neighborhood of x. We shall show that $\gamma(x)$ *can be defined by the right hand side of* (9, 2) *and* γ *is a differentiable tangent vector field over* V_n.

Now, we define a differentiable function by

(9, 3) $$w(b, b', \mathfrak{w}, s) = (\Phi(b, \mathfrak{w}, s), \Phi(b', \mathfrak{w}, s)), \\ b, b' \in B, \ \mathfrak{w} \in S^{n-1}, \ s \in R.$$

By Theorem 3, (8, 2), we get

(9, 4) $$w(b, b', \mathfrak{w}, s) = w(r(\alpha)(b), r(\alpha)(b'), \alpha^{-1}\mathfrak{w}, s) \ \ \alpha \in O_n.$$

For any point x_0, we can take a spherical neighborhood U_{x_0} such that for a fixed $b_0 \in p^{-1}(x_0)$, $F(b_0, \mathfrak{w}, s)$ gives a geodesic polar coordinate system on U_{x_0}. Then we can define a function u by

(9, 5) $$u(b_0, b_1, x) = w(b_0, b_1, \mathfrak{w}, s),$$

where $x \in U_{p(b_0)}$, $x = F(b_0, \mathfrak{w}, s)$, $b_1 \in B$. By (8, 3), (8, 4), (9, 4), we get a relation as

$$u(b_0, b_1, x) = u(r(\alpha)(b_0), r(\alpha)(b_1), x). \tag{9, 6}$$

For a motion f on V_n, we have

$$\begin{aligned} u_f(F(b_0, \mathfrak{w}, s)) &= (\Phi(b_0, \mathfrak{w}, s), \chi(f)(\Phi(b_0, \mathfrak{w}, s))) \\ &= (\Phi(b_0, \mathfrak{w}, s), \Phi(\chi(f)(b_0), \mathfrak{w}, s)) \\ &= w(b_0, \chi(f)(b_0), \mathfrak{w}, s), \end{aligned}$$

hence

$$u_f(x) = u(b_0, \chi(f)(b_0), x), \ x \in U_{p(b_0)}, \ b_0 \in B. \tag{9, 7}$$

For a fixed $\mathfrak{w} \in S^{n-1}$, a fixed $s \in R$, the differentiable mapping $\Phi_{\mathfrak{w},s} : B \to B$ by

$$\Phi_{\mathfrak{w},s}(b) = \Phi(b, \mathfrak{w}, s), \quad b \in B \tag{9, 8}$$

is a differentiable homeorphism on B and it is evident from the definition of Φ that

$$\begin{cases} \Phi_{-\mathfrak{w},s} = \Phi_{\mathfrak{w},-s}, \\ \Phi_{\mathfrak{w},s}\Phi_{-\mathfrak{w},s} = 1. \end{cases} \tag{9, 9}$$

Accordingly, if for a point $b_0 \in B$, the tangent unit vectors to elementary geodesic arcs $\gamma(b_0, \bar{f}_m(b_0))$, $\bar{f}_m = \chi(f_m)$, from b_0 to $\bar{f}_m(b_0)$ at b_0 converge to a tangent vector to B at b_0, then for any point $b \in B$, the same is true. Furthermore, since for the function $u(b_0, b_1, x)$ which is differentiable with respect to b_0, $b_1 \in B$, $x \in U_{p(b_0)}$, we have

$$u(b_0, b_1, x_0) = (b_0, b_1), \ x_0 = p(b_0),$$

we can take an open neighborhood of x_0, $U'_{x_0} \subset \bar{U}'_{x_0} \subset U_{x_0}$ such that

$$\lim_{m\to\infty} \frac{u(b_0, b_m, x)}{(b_0, b_m)} = \lim_{m\to\infty} \frac{U_m(x)}{(b_0, b_m)} \neq 0, \tag{9, 10}$$
$$b_m = \bar{f}_m(b_0), \quad x \in U'_{x_0}.$$

In U_{x_0}, we get from the equation $x = F(b_0, w, s)$ the inverse mapping

$$\mathfrak{w} = \mathfrak{w}(x), \quad s = s(x), \quad x \neq x_0,$$

which are differentiable. Then, we have by (8, 6)

(9, 11) $$f_m(x) = F(b_m, \mathfrak{w}(x), s(x)).$$

Therefore, we have by (9, 10), (9, 11)

$$\lim_{m\to\infty} \frac{(f_m{}^*h)(x) - h(x)}{u_m(x)}$$

$$= \lim_{m\to\infty} \frac{\dfrac{h(F(b_m, \mathfrak{w}(x), s(x)) - h(x)}{(b_0, b_m)}}{\dfrac{u(b_0, b_m, x)}{(b_0, b_m)}}$$

This equation shows that $\eta(x)$ is defined and differentiable on $U'_{x_0} - x_0$. We have proved that we can define a vector field η over V_n by (9, 2) and it is differentiable on it.

On the other hand, from the above consideration, we can define a differentiable scalar field σ_{x_0} over V_n for a point $x_0 \in V_n$ by

(9, 12) $$\sigma_{x_0}(x) = \lim_{m\to\infty} \frac{u_m(x)}{(b_0, X(f_m)(b_0))}, \quad x \in V_n, \quad b_0 \in p^{-1}(x_0),$$

which is everywhere positive. Then, we can define a differentiable covariant vector field τ over V_n as follows: in local coordinates $x^1, \cdots, x^n$ on $U \subset V_n$

(9, 13) $$\tau_i(x) = \frac{\partial}{\partial x^i}\log \sigma_{x_0}(x) \left(= \frac{\partial}{\partial x^i}\log \sigma_{x_1}(x)\right)$$
$$x_0, x_1 \in V_n, \quad x \in U,$$

which does not depend on the point x_0.

Now, in the coordinate neighborhood U'_{x_0}, for sufficiently large m, we must have

(9, 14) $$g_{ij}(x) = g_{kh}(f_m(x)) \frac{\partial f_m{}^*x^k}{\partial x^i}(x) \frac{\partial f_m{}^*x^h}{\partial x^j}(x)$$

where $g_{ij}(x)$ are the components of the fundamental tensor of V_n with respect to the coordinates $x^1, \cdots, x^n$. Taking a suitable neighborhood W of b_0 in V_N, we can consider differentiable functions $H_{ij}(b, x)$ defined on $W \times U'_{x_0}$ by

$$H_{ij}(b, x) = - g_{ij}(x)$$

(9, 15) $$+ g_{kh}(F(b, \mathfrak{w}(x), s(x)) \frac{\partial}{\partial x^i} x^k(F(b, \mathfrak{w}(x), s(x))) \times$$
$$\times \frac{\partial}{\partial x^j} x^h(F(b, \mathfrak{w}(x), s(x))), b \in W, \quad x \in U'_{x_0},$$

where we use $\frac{\partial}{\partial x^i}$ conventionally but there will be no confusion.
By (9, 11), (9, 14), we have for sufficiently large m the equation

$$H_{ij}(b_m, x) = 0.$$

Then we get easily the equation

$$0 = \lim_{m\to\infty} \frac{H_{ij}(b_m, x)}{u_m(x)}$$
$$= \gamma_{i,j}(x) + \gamma_{j,i}(x) + \gamma_i(x)\tau_j(x) + \gamma_j(x)\tau_i(x),$$

that is

(9, 16) $$\gamma_{i,j}(x) + \gamma_{j,i}(x) + \gamma_i(x)\tau_j(x) + \gamma_j(x)\tau_i(x) = 0,$$

where $\gamma_i(x) = g_{ij}(x)\gamma^j(x)$ and a comma denotes the covariant differentiation of V_n. This relations are clearly true on any coordinate neighborhood since the fields γ, τ are defined on V_n and do not depend on the point x_0.

If we define a differentiable contravariant vector field ξ by

$$\xi = \sigma_{x_0}\gamma,$$

that is

(9, 17) $$\xi(x)(h) = \lim_{m\to\infty} \frac{(f_m{}^*h)(x) - h(x)}{(b_0, \chi(f_m)(b_0))},$$

where $x \in V_n$, h is any differentiable function defined on an open neighborhood of x and b_0 is a fixed point of B.
Then we get by (9, 13), (9, 16)

$$\xi_{i,j}(x) + \xi_{j,i}(x) = 0$$

in any local coordinate neighborhood. This is the equation of Killing.

Since we can omit, in the above consideration, *the condition that V_n is complete as in* §8 *by means of a finite number of points in B such as b_0, we obtain the* classical theorem [6].

Theorem 6. *Let V_n be a Riemannian manifold and let $\{f_m\}$ be a sequence of motions of V_n which are mutually distinct and weakly converge to the identity transformation. If the tangent unit vector to elementary geodesic arcs $\gamma(b_0, \chi(f_m)(b_0))$ from b_0 to $\chi(f_m)(b_0)$ at a fixed point $b_0 \in B$ converge to a tangent vector, then we can obtain a* *differentiable tangent vector field which represents an infinitesimal transformation of motion by* (9, 17).

§ 10. $\Phi(b, \xi)$ **and holonomy groups.** In this paragraph, we shall investigate the automorphisms on V_N which are generalizations of $\Phi(b, \mathfrak{w}, s)$ in § 8 and the holonomy group of V_n.

Let W be the set of piecewise differentiable arcs parameterized with arclengths in an n-dimentional Euclidean space. We shall classify the elements of W as follows: $W \ni \gamma_a: 0 \leqslant s \leqslant l_a \to R^n$, $a = 1, 2$, are $\gamma_1 \sim \gamma_1$, (1) if there exists a translation such that $\gamma_2 = \psi \circ \gamma_1$, (2) if for some $k, c \geqslant 0$ such that $0 \leqslant k - c \leqslant k + c \leqslant l_1 = l_2 + 2c$, and we have

$$\begin{aligned} \gamma_1(s) &= \gamma_2(s) && \text{for } 0 \leqslant s \leqslant k - c, \\ \gamma_1(s) &= \gamma_1(2k - s) && \text{for } k - c \leqslant s \leqslant k, \\ \gamma_1(s) &= \gamma_2(s - 2c) && \text{for } k - c \leqslant s \leqslant l_1 \end{aligned}$$

or (3) if there exists a relation between γ_1 and γ_2 exchanged γ_1 and γ_2 in (2).

Let $\mathfrak{W}$ be the set of equivalent classes of W by the above equivalent relation.

For $\gamma_1, \gamma_2 \in W$ such that the end point of γ_1 is the starting point of γ_2, $\gamma = \gamma_1\gamma_2$ is usually defined by

$$\gamma(s) = \begin{cases} \gamma_1(s) & 0 \leqslant s \leqslant l_1, \\ \gamma_2(s - l_1) & l_1 \leqslant s \leqslant l_1 + l_2. \end{cases}$$

We define multiplication in $\mathfrak{W}$ as follows:

$\xi_1, \xi_2 \in \mathfrak{W}$, we take $\gamma_1 \in \xi_1$, $\gamma_2 \in \xi_2$ such that the end point of γ_1 is the starting point of γ_2 and we denote the class containing $\gamma_1\gamma_2$ by $\xi_1 \cdot \xi_2$. Clearly $\xi_1 \cdot \xi_2$ does not depend on the choice of $\gamma_1 \in \xi_1$ and $\gamma_2 \in \xi_2$.

We can easily prove that $\mathfrak{W}$ *is a group with respect to this multiplication.* $\mathfrak{W}$ *contains the n-dimensional translation group* $\mathfrak{T}_n$ *of* R^n *as a subgroup.*

We define a homomorphism $\sigma: \mathfrak{W} \to \mathfrak{T}_n$, as follows: For any $\xi \in \mathfrak{W}$, let γ be a representative with the minimum length in ξ, and let $\sigma(\ )$ be the translation corresponding to the sensed segment from the starting point to the end point of γ. We can easily see that $\sigma(\cdot)$ does not depend on the choice of γ and σ is a homomorphism onto. Let $\mathfrak{W}_0$ be the kernel of σ. We obtain easily the relations.

(10, 1) $$\mathfrak{W}_0 \cap \mathfrak{T}_n = 1,$$

(10, 2) $$\mathfrak{W} = \mathfrak{W}_0 \cdot \mathfrak{T}_n = \mathfrak{T}_n \cdot \mathfrak{W}_0,$$

(10, 3) $$\sigma(\mathfrak{T}_n) = \mathfrak{T}_n.$$

Now, for any $\xi \in \mathfrak{W}$, we define a homeomorphism $\Psi_\xi: V_N \to V_N$ as follows: Let $\gamma \in \xi$ be a representative with its end point at the origin O of R^n. For any point $b \in V_N$, we take a curve C in V_n and a curve $\bar{C}$ in B such that

(i) $p(\bar{C}) = C$,

(ii) *the points of* $\bar{C}$ *are the parallel displaced frames along* C.

(iii) *the point* b *is the end point of* $\bar{C}$,

(iv) *by the linear mapping* I_b: $T_{p(b)}(V_n) \to R_n$, $I_b(e_i(b)) = \mathfrak{w}_i$, *the tangent unit vector* C *at* $p(b)$ *is transformed to the tangent unit vector to* γ *at* O, *where* $\mathfrak{w}_i$ *is the i-th unit vector at* O *of* R^n,

and

(v) *the developement of* C *on* R^n *so that the condition in* (iv) *is satisfied at* $p(b)$ *is* γ.

As is well known, for γ and b, C, $\bar{C}$ are uniquely determined under these conditions (i)—(v).

Let b' be the starting point of $\bar{C}$ which depends only on ξ, b and put $b' = \Psi_\xi(b)$. Ψ_ξ is clearly a homeomorphism on V_N and from the above definition it follows that

(10, 4) $$\Psi_{\xi_1} \circ \Psi_{\xi_2} = \Psi_{\xi_1 \circ \xi_2}, \quad \xi_1, \xi_2 \in \mathfrak{W}$$

(10, 5) $$r(\alpha) \circ \Psi_\xi = \Psi_{\alpha^{-1}(\xi)} \circ r(\alpha), \quad \alpha \in O_n.$$

The set $\mathfrak{K}$ of all the $\Psi_\xi, \xi \in \mathfrak{W}$, is a group of automorphism on V_N and the correspondence $\Psi: \mathfrak{W} \to \mathfrak{K}$ by $\Psi(\cdot) = \Psi_\xi$ is a homomorphism by (10, 4).

For any $\mathfrak{w} \in S^{n-1}$, $s \in R$, we get easily the relation

(10, 6) $$\Psi_{s\mathfrak{w}}(b) = \Phi(b, \mathfrak{w}, -s) = \Phi(b, -\mathfrak{w}, s).$$

By means of (10, 2), putting

$$\mathfrak{K}_0 = \Psi(W_0), \quad S_n = \Psi(T_n),$$

k_0 is an invariant subgroup of k and

(10, 7) $$\mathfrak{K} = \mathfrak{K}_0 \cdot S_n = S_n \cdot \mathfrak{K}_0.$$

Now. for a fixed point $x \in V_n$, let Ω_x be the set of piecewise differentiable closed curves in V_n starting and ending at x and parameterized with arclength. Classifying the elements of Ω_x by the equivalent relation (2) which was used when we derived $\mathfrak{W}$ from W in the beginning of this paragraph, we define a group Π_x with multiplication by the usual method in it.

For any $b \in p^{-1}(x)$, $C \in \Omega_x$, we obtain $\bar{C} \subset B$, $\gamma \subset R^m$ such that C, $\bar{C}$, γ have the above mentioned properties (i)—(v). Then, let $\psi_C : O_n(x) \to O_n(x)$ be defined by

$$(10, 8) \qquad \psi_C(b) = \Psi_\xi(b), \qquad \xi = \xi(C, b)$$

where ξ denotes the class containing γ depending on b and C. Since by a right translation $r(\alpha)$, $\alpha \in O_n$, a system $\{C, \bar{C}, \xi\}$ is transformed to $\{C, r(\alpha)(\bar{C}), \alpha^{-1}(\xi)\}$, we get

$$(10, 9) \qquad r(\alpha) \circ \psi_C = \psi_C \circ r(\alpha).$$

By definition, we get easily

$$(10, 10) \qquad \psi_{C_1} \circ \psi_{C_2} = \psi_{C_1 \cdot C_2}.$$

Since ψ_C dependends only on the element in Π_x containing C, it define a homomorphism of Π_x onto a group of automorphisms on $O_n(x)$ by means of (10, 10). For any $b \in O_n(x)$, $C \in \zeta \in \Pi_x$, let $\beta_b(\zeta)$ be defined by

$$\psi_C(b) = r(\beta_b(\zeta))b,$$

then for any $\zeta_1, \zeta_2 \in \Pi_x$, we have by (11, 9), (11, 10)

$$\begin{aligned} r(\beta_b(\zeta_1)\beta_b(\zeta_2))b &= r(\beta_b(\zeta_2)(r(\beta_b(\zeta_1)b) \\ &= r(\beta_b(\zeta_2))\psi_{C_1}(b) = \psi_{C_1}(r(\beta_b(\zeta_2)b) \\ &= \psi_{C_1}(\psi_{C_2}(b)) = \psi_{C_1 \cdot C_2}(b) \\ &= r(\beta_b(\zeta_1\zeta_2))b, \quad C_1 \in \zeta_1, \ C_2 \in \zeta_2, \end{aligned}$$

hence

$$(10, 10) \qquad \beta_b(\zeta_1)\beta_b(\zeta_2)) = \beta_b(\zeta_1 \cdot \zeta_2)$$

The transformation $\beta_b : \Pi_x \to O_n$ is a homomorphism. For $b_1 = r(\alpha_1)b$, we have

$$\begin{aligned} \psi_C(b_1) &= r(\beta_{b_1}(\zeta))b_1 = r(\beta_{b_1}(\zeta))\,(r(\alpha_1)b) \\ &= r(\alpha_1\beta_{b_1}(\zeta))b \end{aligned}$$

$$= r(\alpha_1)\psi_{rC}(b) = r(\alpha_1)\,(r(\beta_b(\zeta))b)$$
$$= r(\beta_b(\zeta)\alpha_1)b,$$

henee

(10, 11) $$\beta_{r(\alpha)b}(\zeta) = \alpha^{-1}\beta_b(\zeta)\alpha, \quad \zeta \in \Pi_x, \quad \alpha \in O_n.$$

$H_{x,b} = \beta_b(\Pi_x)$ is the holonomy group of V_n at x with respect to b.

With regards to $\xi(C, b)$, we get analogously the formulas:

(10, 12) $$\xi(C, r(\alpha)b) = \alpha^{-1}(\xi(C, b)),$$

(10, 13) $$\xi(C_1C_2, b) = \xi(C_1, \psi_{rC_2}(b)) \cdot \xi(C_2, b)$$
$$= (\beta_b(\zeta_2^{-1})\,(\xi(C_1, b)))\cdot\xi(C_2, b),$$
$$C_a \in \zeta_a \in \Pi_x, \quad a = 1, 2.$$

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