

# *Mathematical Journal of Okayama University*

---

*Volume 5, Issue 2*

1955

*Article 5*

MARCH 1956

---

## On fibres of fibre spaces whose total space is contractible

Masahiro Sugawara\*

\*Okayama University

Copyright ©1955 by the authors. *Mathematical Journal of Okayama University* is produced by  
The Berkeley Electronic Press (bepress). <http://escholarship.lib.okayama-u.ac.jp/mjou>

## ON FIBRES OF FIBRE SPACES WHOSE TOTAL SPACE IS CONTRACTIBLE

MASAHIRO SUGAWARA

1. In this note, we consider a fibre space  $(E, F, B, p)$ , where  $E$  is the total space,  $B$  the base space, the map  $p: E \rightarrow B$  the projection, and  $F = p^{-1}(b_0)$  the fibre over a point  $b_0 \in B$ , in the sense of Serre, (i. e., being assumed the truth of the covering homotopy theorem for maps of finite polyhedra). The purpose of this note is to prove the following theorem.

**Theorem 1.** *In a fibre space  $(E, F, B, p)$  such that the total space  $E$  is a CW-complex and the fibre  $F$  is a locally finite CW-complex<sup>1)</sup>, we assume that  $E$  is contractible to a vertex  $x_0 \in F$  in itself (with  $x_0$  stationary throughout the contraction). Then  $F$  is a homotopy-associative H-space having a (two-sided) inversion<sup>2)</sup>.*

2. As the first step of proofs of Theorem 1, we prove the following theorem, which is an analogy of the result of E. H. Spanier and J. H. C. Whitehead [4, Theorem (1.1)].

**Theorem 2.** *If the fibre space  $(E, F, B, p)$  satisfies the hypotheses of Theorem 1, then there exists a (continuous) map  $\bar{\mu}$  of  $E \times F$  into  $E$  having the following properties:*

(1)  $p \circ \bar{\mu}(u, x) = p(u)$  for every  $u \in E$  and  $x \in F$ .

(2)  $\bar{\mu}|_{F \times F} = \mu$  is an H-structure of  $F$  having  $x_0$  as an unit element.

(3) This H-structure  $\mu$  has a left inversion<sup>2)</sup>.

*Proof of (1) and (2).* We notice that  $E \times F$  and

1) For definitions, Cf. [6], § 5, p. 223.

2) A space  $X$  is an H-space (has an H-structure) if there is a multiplication  $\mu$  in  $X$ , i. e. a map  $\mu: X \times X \rightarrow X$ , such that  $\mu(x_0, x) = \mu(x, x_0) = x$  for some  $x_0$  (called an unit) and every  $x \in X$ . If two maps  $(x, y, z) \rightarrow \mu(x, \mu(y, z))$  and  $(x, y, z) \rightarrow \mu(\mu(x, y), z)$  of  $X \times X \times X$  into  $X$  are homotopic each other, we say  $\mu$  is homotopy-associative.  $\mu$  has a (two-sided) inversion, if there exists a map  $\sigma: X \rightarrow X$  such that the maps  $x \rightarrow \mu(\sigma(x), x)$  and  $x \rightarrow \mu(x, \sigma(x))$  of  $X$  into  $X$  are homotopic to the constant map  $x \rightarrow x_0$  respectively. If only one of these maps has this property, we say  $\sigma$  is an one-sided (left or right) inversion of  $\mu$ .

$$E \vee F = (E \times x_0) \cup (x_0 \times F) \text{ and } F \vee F = (F \times x_0) \cup (x_0 \times F)$$

are CW-complexes, because  $E$  and  $F$  are CW-complexes and the latter is locally finite<sup>3)</sup>.

Let  $k_t: (E, x_0) \rightarrow (E, x_0)$  ( $0 \leq t \leq 1$ ) be the homotopy between  $k_1 =$  the identity map and  $k_0: E \rightarrow x_0$  the constant map, which gives a contraction of  $E$ ; and we define the map  $g_0: E \times F \rightarrow E$  by  $g_0(u, x) = x$ . and the homotopy  $g'_t: E \vee F \rightarrow E$  ( $0 \leq t \leq 1$ ) by

$$\begin{cases} g'_t(u, x_0) = k_t(u), \\ g'_t(x_0, x) = x, \end{cases} \quad \text{for } u \in E \text{ and } x \in F.$$

Then, there is a homotopy  $g_t: E \times F \rightarrow E$  ( $0 \leq t \leq 1$ ) of  $g_0$  which is an extension of  $g'_t$ ,<sup>4)</sup> and it holds relations  $g_1(u, x_0) = u$  and  $g_1(x_0, x) = x$ . Therefore the map  $p \circ g_1: E \times F \rightarrow B$  satisfies  $p \circ g_1(u, x) = p(u)$  for  $(u, x) \in E \vee F$ .

Let  $h': E \times E \times I \rightarrow B$  ( $I = [1, 2]$ ) be defined by

$$h'(u, x, t) = \begin{cases} p \circ g_{3-2t}(u, x) & \text{for } 1 \leq t \leq 3/2, \\ p \circ k_{2t-3}(u) & \text{for } 3/2 \leq t \leq 2. \end{cases}$$

Then  $h'(u, x, 1) = p \circ g_1(u, x)$ ,  $h'(x_0 \times F \times I) = b_0$  and  $h'(u, x, 2) = p(u)$ . Also  $h'|_{(E \vee F) \times I}$  is homotopic, relative  $(E \times x_0 \times 1) \cup (E \times x_0 \times 2) \cup (x_0 \times F \times I)$ , to the map  $h: (E \vee F) \times I \rightarrow B$  such that  $h(u, x, t) = p(u)$ . This homotopy can be extended, first to a homotopy of  $(E \times F \times 1) \cup (E \times F \times 2) \cup (E \vee F) \times I$  as stationary on  $(E \times F \times 1) \cup (E \times F \times 2)$ , and then to a homotopy of  $E \times F \times I$  into  $B$  (by applying the homotopy extension theorem for CW-complex). The last map  $h: E \times F \times I \rightarrow B$  defines the homotopy  $h_t: E \times F \rightarrow B$  ( $1 \leq t \leq 2$ ) which satisfies  $h_1 = p \circ g_1$  and

$$h_t(u, x) = p(u), \quad \text{for } \begin{cases} t = 2 \text{ and } (u, x) \in E \times F, \\ 1 \leq t \leq 2 \text{ and } (u, x) \in E \vee F. \end{cases}$$

Because  $(E, F, B, p)$  is a fibre space in the sense of Serre and  $E \times F$  is a CW-complex, we can apply the covering homotopy theorem due to I. M. James and J. H. C. Whitehead [2, Theorem 5.1], and hence there

3) Cf. [6], § 5, property (H), p. 227.

4) This is a consequence of the homotopy extension theorem for CW-complexes, cf. [6], § 5, property (J), p. 228.

exists a homotopy  $g_t: E \times F \rightarrow E$  ( $1 \leq t \leq 2$ ) such that  $p \circ g_t = h_t$  and stationary with  $h_t$ . It follows immediately that the map  $\bar{\mu} = g_2$  satisfies the properties (1) and (2).

3. *Proof of (3).* As  $(E, F, B, p)$  is a fibre space, it follows evidently that  $(E \times F, F \times F, B, p')$  is also a fibre space, where  $p': E \times F \rightarrow B$  is defined by  $p'(u, x) = p(u)$  and  $F \times F = p'^{-1}(b_0)$  is the fibre over  $b_0 \in B$ .

Making use of  $\bar{\mu}$ , let  $\bar{\varphi}: E \times F \rightarrow E \times F$  be defined by

$$\bar{\varphi}(u, x) = (\bar{\mu}(u, x), x) \quad \text{for } u \in E \text{ and } x \in F,$$

and set  $\varphi = \bar{\varphi}|_{F \times F}$ . Then, by property (1),  $p' \circ \bar{\varphi}(u, x) = p'(u, x)$  for every  $u \in E$  and  $x \in F$ . Therefore  $\bar{\varphi}$  induces an endomorphism of the homotopy sequence of the fibre space  $(E \times F, F \times F, B, p')$ :

$$\begin{array}{ccccccc} \cdots & \rightarrow & \pi_{n+1}(B) & \rightarrow & \pi_n(F \times F) & \rightarrow & \pi_n(E \times F) \rightarrow \pi_n(B) \rightarrow \cdots \\ & & \downarrow i_* & & \downarrow \varphi_* & & \downarrow \bar{\varphi}_* & & \downarrow i_* \\ \cdots & \rightarrow & \pi_{n+1}(B) & \rightarrow & \pi_n(F \times F) & \rightarrow & \pi_n(E \times F) \rightarrow \pi_n(B) \rightarrow \cdots \end{array}$$

where  $i_*$  is the induced isomorphism of the identity map  $i: B \rightarrow B$ . Because  $E$  is contractible to  $x_0$ , the map  $q: E \times F \rightarrow x_0 \times F$  such that  $q(u, x) = (x_0, x)$  induces an isomorphism of  $\pi_n(E \times F)$  onto  $\pi_n(x_0 \times F)$ , and hence  $\bar{\varphi}_*$  is an isomorphism onto, by  $q = q \circ \bar{\varphi}$ . Therefore, by five lemma, it is immediately seen that  $\varphi_*$  is an isomorphism onto; and this shows that the map  $\varphi: F \times F \rightarrow F \times F$  is a homotopy equivalence, as  $F \times F$  is a CW-complex<sup>5)</sup>. We denote by  $\theta: F \times F \rightarrow F \times F$  a homotopy inverse of  $\varphi$ , i. e., a map such that  $\theta \circ \varphi$  and  $\varphi \circ \theta$  are homotopic to the identity map, respectively.

We now define maps  $\sigma$  and  $\tau$  of  $F$  into  $F$  by

$$\begin{cases} \sigma(x) = q_1 \circ \theta(x_0, x), \\ \tau(x) = q_2 \circ \theta(x_0, x), \end{cases} \quad \text{for } x \in F,$$

where  $q_1$  is the projection of  $F \times F$  onto  $F$  of the first factor, and  $q_2$  onto  $F$  of the second factor. As  $\varphi$  can be written by  $\varphi(w) = (\mu(q_1(w), q_2(w)), q_2(w))$  for  $w \in F \times F$ , we have  $\varphi \circ \theta \circ j(x) = (\mu(\sigma(x), \tau(x)), \tau(x))$  where  $j: F \rightarrow x_0 \times F$  is  $j(x) = (x_0, x)$ . Because  $\varphi \circ \theta$  is homotopic to the identity map, let  $i_t: F \times F \rightarrow F \times F$  be a homotopy between  $i_0 = \varphi \circ \theta$  and  $i_1 =$  the identity map; then  $q_1 \circ i_t \circ j: F \rightarrow F$  defines a homo-

5) This is a consequence of Theorem 1 of [6, §1, p. 215].

topy between maps  $x \rightarrow \mu(\sigma(x), \tau(x))$  and  $x \rightarrow x_0$  of  $F$  into  $F$ , and also  $q_2 \circ i_1 \circ j: F \rightarrow F$  between maps  $x \rightarrow \tau(x)$  and  $x \rightarrow x$ . Combining these properties, it is seen that the map  $x \rightarrow \mu(\sigma(x), x)$  of  $F$  into  $F$  is homotopic to the map  $x \rightarrow \mu(\sigma(x), \tau(x))$  and hence to the constant map  $x \rightarrow x_0$ . This shows that  $\sigma$  is an inversion of  $\mu$  and so (3) is valid; and hence proofs of Theorem 2 is completed.

4. As the second step, we prove the following theorem, which is an modification of results of H. Samelson [3, Theorem 1] and [5, Theorem 1].

**Theorem 3.** *If the fibre space  $(E, F, B, p)$  satisfies:*

(1) *There exists a map  $\bar{\mu}: E \times F \rightarrow E$ , having properties (1), (2) and (3) of Theorem 2.*

(2)  *$E$  is contractible to  $x_0$  (with  $x_0$  stationary).*

*Then there exists an H-homomorphism<sup>6)</sup>  $f$ , which is also a weak homotopy equivalence<sup>7)</sup>, of  $F$  into the space  $A(B)$  of loops in  $B$  with the base point  $b_0$ .*

*If  $F$  is a locally finite CW-complex in addition, the H-structure  $\mu$  of  $F$  is homotopy-associative, and also has a (two-sided) inversion.*

*Proof.* The existence of such map  $f$  can be shown by applying [3, Proposition 1] and [5, Theorem 1].

Because  $f$  induces isomorphisms of all homotopy groups of  $F$  and  $A(B)$ , two maps of a CW-complex into  $F$  are homotopic if and only if two composed maps of these maps and  $f$  are homotopic each other<sup>8)</sup>. Therefore, if we assume  $F$  is a locally finite CW-complex, the homotopy-associativity of  $F$ , i. e., the fact that two maps  $(x, y, z) \rightarrow \mu(x, \mu(y, z))$  and  $(x, y, z) \rightarrow \mu(\mu(x, y), z)$  of  $F \times F \times F$  into  $F$  are homotopic, is an immediate consequence of the fact that  $f$  is an H-homomorphism and the

6) For H-spaces  $X$  and  $Y$  with multiplication  $\mu$  and  $\mu'$  respectively, we say a map  $f: X \rightarrow Y$  is an H-homomorphism, if two maps  $(x_1, x_2) \rightarrow f \circ \mu(x_1, x_2)$  and  $(x_1, x_2) \rightarrow \mu'(f(x_1), f(x_2))$  of  $X \times X$  into  $Y$  are homotopic. If  $\mu$  and  $\mu'$  have inversions  $\sigma$  and  $\sigma'$  respectively, we assume in addition that two maps  $x \rightarrow f \circ \sigma(x)$  and  $x \rightarrow \sigma' \circ f(x)$  of  $X$  into  $Y$  are homotopic each other.

For the map  $f$  of this theorem, the multiplication of  $F$  is  $\mu = \bar{\mu} | F \times F$  and that of the H-space  $A(B)$  is the natural multiplication (composition of loops).

7) This means that  $f$  induces isomorphisms of all the homotopy groups of  $F$  and  $A(B)$ .

8) This property is easily proved, for example, by making use of Theorem 1.7 of [1, Chapter VII, p. 98].

H-space  $A(B)$  of loops in  $B$  with natural multiplication (composition of loops) is homotopy-associative.

As the consequence of the homotopy-associativity of  $\mu$ , it is easy to show that the left inversion  $\sigma$  of  $\mu$  is also the right inversion, as follows. As the map  $x \rightarrow \mu(\sigma(x), x)$  is homotopic to the constant map  $x \rightarrow x_0 =$  an unit, the map  $x \rightarrow \sigma \circ \sigma(x)$  of  $F$  into  $F$  to the map  $x \rightarrow \mu(\sigma \circ \sigma(x), \mu(\sigma(x), x))$ , and the latter to the map  $x \rightarrow \mu(\mu(\sigma \circ \sigma(x), \sigma(x)), x)$ , and so to the identity map  $x \rightarrow x$ . Therefore the map  $x \rightarrow \mu(x, \sigma(x))$  is homotopic to the map  $x \rightarrow \mu(\sigma \circ \sigma(x), \sigma(x))$ , and hence to the constant map  $x \rightarrow x_0$  of  $F$  into  $F$ . This proves  $\sigma$  is also a right inversion of  $\mu$ , and proofs of Theorem 3 is completed.

Combining Theorems 2 and 3, we have Theorem 1 immediately.

#### REFERENCES

- [1] P. J. HILTON, Homotopy Theory, Cambridge Univ. Press, 1953.
- [2] I. M. JAMES AND J. H. C. WHITEHEAD, Note on fibre spaces, Proc. London Math. Soc. (3), 4 (1954), 129—137.
- [3] H. SAMUELSON, Groups and spaces of loops, Comm. Math. Helv., 28 (1954), 278—287.
- [4] E. H. SPANIER AND J. H. C. WHITEHEAD, On fibre spaces in which the fibre is contractible, Comm. Math. Helv., 29 (1955), 1—8.
- [5] M. SUGAWARA, H-spaces and spaces of loops, Math. J. Okayama Univ., 5 (1955), 5—11.
- [6] J. H. C. WHITEHEAD, Combinatorial homotopy. I, Bull. Amer. Math. Soc., 55 (1949), 245—277.

DEPARTMENT OF MATHEMATICS,  
OKAYAMA UNIVERSITY

(Received February 4, 1956)