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NOTE ON THE ISOMETRIC IMBEDDING OF COMPACT RIEMANNIAN MANIFOLDS IN EUCLIDEAN SPACES

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In this note, we shall state some remarks on the isometric imbedding of compact Riemannian manifolds in Euclidean spaces in connection with the works of Shiin-Shen Chern and N. H. Kuiper [1]¹⁾ and the author [2]. Theorem 1 in [2] as follows is fundamental in our considerations.

Theorem A. *Let M be a compact Riemannian manifold of dimension n and with the property that at every point there is a q -dimensional linear subspace in the tangent space along whose plane elements the sectional curvatures are non positive. Then M can not be isometrically imbedded in an Euclidean space of dimension $n + q - 1$.*

§ 1. Let M be a Riemannian manifold of dimension n whose line element is given by

$$(1) \quad ds^2 = \sum g_{ij}(x) dx^i dx^j$$

in local coordinates x^1, x^2, \dots, x^n . Let us put

$$(2) \quad \sum g_{ij}(x) dx^i dx^j = \sum \omega_i(x, dx) \omega_i(x, dx),$$

$$(3) \quad \begin{cases} d\omega_i = \sum \omega_j \wedge \omega_{ji}, \\ d\omega_{ij} = \sum \omega_{ik} \wedge \omega_{kj} + \varrho_{ij} \end{cases}$$

where ϱ_{ij} are the curvature forms of M as is well known. Let $k(p)$, $p \in M$, be the minimum number of linear differential forms in terms of which the curvature forms of M at p can be expressed. According to [1], $n - k(p)$ is called the *index of nullity* at p . Let us put

$$k(M) = \max_{p \in M} k(p)$$

A generalized Tompkins' theorem as follows was proved by means of algebraic methods in [1], [3].

1) Numbers in brackets refer to the list of references at the end of the paper.

Theorem B. *A compact Riemannian manifold M of dimension n can not be isometrically imbedded in an Euclidean space of dimension $2n - k(M) - 1$.*

We shall give more detailed results than the above theorem.

§ 2. As a preliminary we establish the following lemmas. Let R_{ijkh} be real number such that

$$(4) \quad R_{ijkh} = -R_{jikh} = -R_{ihjk} = R_{khij}$$

and let us consider the form

$$(5) \quad R(x, y) = R_{ijkh} x^i y^j x^k y^h$$

of real $2n$ variable x^i, y^i . Let m be the maximum of dimensions of linear subspaces L of the n -dimensional real vector space such that for any $x, y \in L$

$$R(x, y) < 0.$$

Let k be the rank of the system of linear equations

$$(6) \quad R_{ijkh} x^h = 0, \quad i, j, k = 1, 2, \dots, n,$$

in x^i and L_0 be the linear space of the solutions of (6). We have $\dim L_0 = n - k$ and

$$(7) \quad R(x, y) = 0, \quad x \in L_m, \quad y \in L_0.$$

Lemma 1. *If a linear subspace L of L_n has the property that for any two $x, y \in L$, $R(x, y) < 0$, then $L \cup L_0$ ²⁾ has the same property.*

Proof. For any $x, y \in L \cup L_0$, we may put

$$\begin{aligned} x &= a_1 x_1 + b_1 y_1, & y &= a_2 x_2 + b_2 y_2, \\ x_1, x_2 &\in L, & y_1, y_2 &\in L_0. \end{aligned}$$

By means of (7), we get easily

$$R(x, y) = (a_1 a_2)^2 R(x_1, x_2) < 0.$$

Since any 1-dimensional subspace has the above property, we get

1) The summation convention of tensor analysis is used in the following.
 2) We denote by $L \cup L_0$ the linear space spanned by the elements of L and L_0 .

easily the following lemma.

Lemma 2. *If $k > 0$, then $n - k + 1 \leq m$.*

Now, let us assume that (4) is of the form

$$(8) \quad \begin{aligned} R_{ijkh} &= H_{ik}H_{jh} - H_{ih}H_{jk}, \\ H_{ij} &= H_{ji}. \end{aligned}$$

Accordingly we have

$$(9) \quad R(x, y) = \psi(x, x)\psi(y, y) - (\psi(x, y))^2,$$

where $\psi(x, y) = H_{ij}x^i y^j$. From (9) we may consider that $-R(x, y)$ is a generalized discriminant of the quadratic equation $H_{ij}x^i x^j = 0$ in n variables x^i . We can easily see that

$$(10) \quad k = \text{rank}(H_{ij}).$$

Lemma 3. *If R_{ijkh} is of the form (8), then $m = n - \frac{k + \rho}{2} + 1$ where $\rho = |\text{signature of } \psi(x, x)|$.*

Proof. As stated above, the system of linear equations

$$R_{ijkh}x^h = 0, \quad i, j, k = 1, 2, \dots, n$$

is clearly equivalent to the system of linear equations

$$H_{ij}x^j = 0, \quad i = 1, 2, \dots, n.$$

Let us suppose that $k = n$. If $\psi(x, x) = H_{ij}x^i x^j$ is definite, then for any linearly independent vectors x, y , we have $R(x, y) = \psi(x, x)\psi(y, y) - (\psi(x, y))^2 > 0$ as is well known. Hence $m = 1$. Since $\rho = n$, the above stated relation holds good.

In the next place, let us assume that $\psi(x, x)$ is not definite. Taking a suitable base of L_n , we may put

$$\begin{aligned} H_{\alpha\alpha} &= 1, \quad \alpha = 1, 2, \dots, r, \\ H_{\lambda\lambda} &= -1, \quad \lambda = r + 1, \dots, n, \quad 0 < r < n, \\ H_{ij} &= 0, \quad i \neq j, \quad i, j = 1, 2, \dots, n, \end{aligned}$$

Let L_r, L_{n-r} be the subspaces of L_n given by $x^\lambda = 0, \lambda = r + 1, \dots, n; x^\alpha = 0, \alpha = 1, 2, \dots, r$, respectively, for which we have $L_r \cup L_{n-r} = L_n$ and $L_r \cap L_{n-r} = 0$.

Let L be a linear subspace such that $R(x, y) \leq 0, x, y \in L$ and

$\dim L \geq 2$. Since the quadratic form $\mathcal{F}(x, x)$ is definite on L_r and L_{n-r} , it follows that

$$(11) \quad \dim L \cap L_r < 1, \quad \dim L \cap L_{n-r} < 1.$$

Let $\pi_r : L_n \rightarrow L_r$ and $\pi_{n-r} : L_n \rightarrow L_{n-r}$ be the projections, then it must be

$$(12) \quad \dim \pi_r(L), \dim \pi_{n-r}(L) \geq \dim L - 1,$$

for otherwise we get easily $\dim L \cap L_{n-r} \geq 2$ or $\dim L \cap L_r \geq 2$ which contradicts to (11). Accordingly we get from (12)

$$\dim L \leq n - r + 1, \quad r + 1.$$

Now, we may assume that $n - r < r$. Let us take $\xi_A = (\xi_A^\alpha) \in L_r$ and $\gamma_A = (\gamma_A^\lambda) \in L_{n-r}$, $A = 1, 2, \dots, n - r$ such that

$$\sum_\alpha \xi_A^\alpha \xi_B^\alpha = \sum_\lambda \gamma_A^\lambda \gamma_B^\lambda = \delta_{AB} \quad (1)$$

Then we can define linearly independent vectors $\zeta_1, \dots, \zeta_{n-r+1}$ of L_n by

$$\begin{aligned} \zeta_A &= (\xi_A^\alpha, \gamma_A^\lambda), \quad A = 1, 2, \dots, n - r - 1, \\ \zeta_{n-r} &= (\xi_{n-r}^\alpha, 0), \quad \zeta_{n-r+1} = (0, \gamma_{n-r}^\lambda). \end{aligned}$$

Hence we have

$$(13) \quad \begin{aligned} \mathcal{F}(\zeta_A, \zeta_B) &= \mathcal{F}(\zeta_A, \zeta_{n-r}) = \mathcal{F}(\zeta_A, \zeta_{n-r+1}) = \mathcal{F}(\zeta_{n-r}, \zeta_{n-r+1}) = 0, \\ \mathcal{F}(\zeta_{n-r}, \zeta_{n-r}) &= -\mathcal{F}(\zeta_{n-r+1}, \zeta_{n-r+1}) = 1 \\ A, B &= 1, 2, \dots, n - r - 1. \end{aligned}$$

Let L be the space spanned by $\zeta_1, \dots, \zeta_{n-r+1}$. For any two vectors x, y of L

$$x = \sum_{A=1}^{n-r+1} u_A \zeta_A, \quad y = \sum_{A=1}^{n-r+1} v_A \zeta_A,$$

we get from (13)

$$\begin{aligned} \mathcal{F}(x, x) &= u_{n-r}^2 - u_{n-r+1}^2, \\ \mathcal{F}(y, y) &= v_{n-r}^2 - v_{n-r+1}^2, \\ \mathcal{F}(x, y) &= u_{n-r} v_{n-r} - u_{n-r+1} v_{n-r+1}, \end{aligned}$$

hence

1) $\delta_{AB} = 1$ if $A = B$ and $\delta_{AB} = 0$ if $A \neq B$.

$$R(x, y) = - (u_{n-r}v_{n-r-1} - u_{n-r-1}v_{n-r})^2 \leq 0.$$

Thus we see that $m = n - r + 1$ in this case. Since $\rho = r - (n - r)$ by definition, we get

$$(14) \quad m = \frac{n - \rho}{2} + 1.$$

In general case, we have by virtue of Lemma 1 the relation

$$m = (n - k) + \left(\frac{k - \rho}{2} + 1\right) = n - \frac{k + \rho}{2} + 1.$$

§ 3. Now we shall proceed geometrical considerations. For an n -dimensional Riemannian manifold M , $k(M) = 0$ is equivalent to that M is locally Euclidean. Accordingly we get from Lemma 2 and Theorem A a more detailed theorem than Theorem B in § 2.

Theorem 1. *A compact locally non Euclidean Riemannian manifold M of dimension n can not be isometrically imbedded in an Euclidean space of dimension $2n - k(M)$.*

Now, let M be an n -dimensional Riemannian manifold whose curvature forms

$$(15) \quad \Omega_{ij} = \frac{1}{2} \sum R_{ijkl} \omega_k \wedge \omega_l$$

are of the form (8). If $k(p) \geq 3$, $p \in M$, then H_{ij} is determined uniquely save for signs as is well known. Accordingly the absolute value of signature of the quadratic form $H_{ij}x^i x^j$ is an invariant of M at p . We denote this by $\rho(p)$ and put $\rho(M) = \max_{p \in M} \rho(p)$. Furthermore if $k(p) \leq 4$, then the field of H_{ij} satisfies the Codazzi's equation

$$(16) \quad H_{ij,k} - H_{ik,j} = 0.$$

Since (8) is the Gauss equations, M can be locally isometrically imbedded in an Euclidean space of dimension $n + 1$. By means of Lemma 3 and Theorem A, we obtain the following theorem.

Theorem 2. *Let M be a compact Riemannian manifold of dimension n on which there exists a symmetric tensor field H_{ij} such that*

$$R_{ijkl} = H_{ik}H_{jl} - H_{il}H_{jk}.$$

Then M can not be isometrically imbedded in an Euclidean space of dimension $2n - \frac{1}{2}(k(M) + \rho(M))$.¹⁾

This theorem shows that even though M is of imbedding class 1, that is it can be locally isometrically imbedded in an Euclidean space of dimension $n + 1$, it does not so in the large and shows negatively an order of imbeddability of M into Euclidean spaces.

Lastly we consider the case $k(M) = 2$. Then there exists a skew-symmetric tensor S_{ij} such that

$$(17) \quad R_{ijkl} = \sigma S_{ij}S_{kl},$$

that is M is a space of separated curvature. Thus we get generally the following theorem.

Theorem 3. *A compact Riemannian manifold of dimension n with separated curvature and with non-positive scalar curvature can not be isometrically imbedded in an Euclidean space of dimension $2n - 1$.*

§ 4. In this section, we shall investigate especially compact Riemannian manifolds of dimension 3. By means of orthonormal frames, we put

$$\begin{aligned} R_{2323} &= K_{11}, & R_{3112} &= K_{23} = K_{32}, \\ R_{3131} &= K_{22}, & R_{1223} &= K_{31} = K_{13}, \\ R_{1212} &= K_{33}, & R_{2331} &= K_{12} = K_{21} \end{aligned}$$

and

$$x^2y^3 - x^3y^2 = v^1, \quad x^3y^1 - x^1y^3 = v^2, \quad x^1y^2 - x^2y^1 = v^3.$$

Then we have easily the equations

$$(18) \quad R_{ij} = R_{i^kjk} = \frac{1}{2} R \delta_{ij} - K_{ij}, \quad R = R_i^i$$

and

$$(19) \quad R(x, y) = \frac{1}{2} R \Sigma v^i v^i - R_{ij} v^i v^j.$$

1) We put $\rho(M) = \max_{p \in M} \rho(p)$, $\rho(p) = |\text{signature of } (H_{ij}(p))|$.

Making use of frames such that $R_{ij} = 0$, $i \neq j$, we get

$$(20) \quad 2R(x, y) = (-R_{11} + R_{22} + R_{33})v^1v^1 + (R_{11} - R_{22} + R_{33})v^2v^2 \\ + (R_{11} + R_{22} - R_{33})v^3v^3.$$

In order that $m=3$, it is necessary and sufficient that $R_{11}, R_{22}, R_{33} \leq 0$ and $|R_{11}|, |R_{22}|, |R_{33}|$ are the lengths of the sides of a triangle including the case in which the triangle degenerates. In order that $m=1$, it is necessary and sufficient that $R_{11}, R_{22}, R_{33} > 0$ and R_{11}, R_{22}, R_{33} are the lengths of the sides of a triangle. Thus we obtain the following theorems.

Theorem 4. *A compact Riemannian manifold of dimension 3 with the property that at every point its Ricci tensor R_{ij} is negative semi-definite and the absolute values of its eigen values are the lengths of the three sides of a triangle (including the case in which the triangle degenerates), can not be isometrically imbedded in an Euclidean space of dimension 5.*

Corollary. *A compact Einstein space of dimension 3 with non-positive scalar curvature at every point can not be isometrically imbedded in an Euclidean space of dimension 5.*

Theorem 5. *A compact Riemannian manifold of dimension 3 with the property that there exists no point at which its Ricci tensor R_{ij} is positive definite and its eigen values are the lengths of the three sides of a triangle, can not be isometrically imbedded in an Euclidean space of dimension 4.*

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