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ON REGULAR RINGS WHOSE CYCLIC FAITHFUL MODULES ARE GENERATOR

Dedicated to Professor Hisao Tominaga on his 60th birthday

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In [4], S. Page has proved that a regular ring R is right FPF (= every finitery generated faithful right R-module is a generator in the category of right R-modules) if and only if R is isomorphic to a finite direct product of full matrix rings over abelian regular self-injective rings.

On the other hand, in [3], we have characterized a right semihereditary right FPF-ring R as (1) R is right bounded and right non-singular, (2) For all positive integer n, nR_R has the extending property of modules for $L_r(nR)$ (= the lattice of right R-submodules of nR_R), (3) For any finitely generated idempotent right ideal I of R, there exists a central idempotent e of R such that $RI = eR$.

If R is a regular ring, by using our characterization, we can easily seen that R is right FPF if and only if every faithful right R-module, which is generated at most two elements, is a generator in the category of right R-modules.

Therefore we are interested in regular rings whose every cyclic faithful right R-module is a generator in the category of right R-modules.

In this paper, we are concerned with rings whose cyclic faithful modules are generator.

In section 1, we shall determine the structure of regular rings whose cyclic faithful modules are generator in the case every non-zero two-sided ideal contains a non-zere central idempotent.

In section 2, we are concerned with rings whose faithful modules, which is generated at most two elements, are generator. We shall give a simillar characterization to non-singular FPF-rings.

0. Preliminaries. Throughout this paper, we assume that R is a ring with identity and all modules are unitary.

Let M be any R-module. Then we use $r_R(M)$ (resp. $l_R(M)$) to denote the right (resp. left) annihilator ideal of M, i.e. $r_R(M) = \{r \in R \mid Mr = 0\}$.

Similary for $l_R(M)$, and we use $\mathrm{Tr}_R(M)$ to denote the trace ideal of M. Further the notation $M_n(R)$ means the ring of $n\times n$-matrices over R.

Let e be an idempotent of a regular ring R. Then e is said to be faithful and abelian provided that the right R-module eR is faithful and the ring eRe is abelian regular, where R is called abelian regular if every idempotent is central.

If R is a right self-injective regular ring and contains a faithful abelian idempotent, then R is said to be Type I.

We say that R is biregular if for any idempotent e of R, the two-sided ideal ReR is generated by a central idempotent, and that R is right bounded if every essential right ideal contains a non-zero two-sided ideal which is essential as a right ideal.

1. Regular rings whose cyclic faithful modules are generator. In this section, we determine the structure of regular rings whose cyclic faithful modules are generator.

First we prepare some lemmas.

Lemma 1. *Let R be a ring. Then the following conditions are equivalent.*

(1) *R is right FPF.*

(2) *For any positive integer n, every cyclic faithful right $M_n(R)$-module is a generator in the category of right $M_n(R)$-modules.*

Proof. (1) ⇒ (2). It is easily seen since FPF-rings are Morita equivalent.

(2) ⇒ (1). Let M be a finitely generated faithful right R-module. Then there exists a positive integer n and submodule K of R^n such that $R^n/K \cong M$. Now we set $T = M_n(R)$ and

$$I = \left\{ (K, K, \dots, K) = \begin{pmatrix} a_{11} & & a_{n1} \\ \vdots & \dots, & \vdots \\ a_{1n} & & a_{nn} \end{pmatrix} \in M_n(K) \,\middle|\, \begin{pmatrix} a_{i1} \\ \vdots \\ a_{in} \end{pmatrix} \in K,\ i = 1, \dots, n \right\}.$$

Then I is a right ideal of $M_n(R)$ and I contains no non-zero two-sided ideal of $M_n(R)$, since R^n/K is a faithful right R-module. Therefore T/I is a cyclic faithful right T-module, so T/I is a generator in the category of right T-modules. We note that $l_T(I)T = T$. On the other hand, for any element $(a_{ij})_{i,j=1}^n \in l_T(I)$, we can define a map $(a_{i1}, \dots, a_{in}) : R^n/K \to R$.

Conversely, for any map f from R^n/K to R, we can construct an element $\begin{pmatrix} f \\ 0 \end{pmatrix}$ of $1_T(I)$. Now we have an one to one corespondence $\mathrm{Hom}_R(R^n/K, R)$ with $1_T(I)$. Thus

$$\mathrm{Tr}_R(R^n/K) = \sum_{f \in \mathrm{Hom}_R(R^n/K, R)} f(R^n/K) = \sum_{i,j=1}^{t,n} (a_{ij}) \begin{pmatrix} R \\ \vdots \\ R \end{pmatrix} = R,$$

where $(a_{ij}) \in 1_T(I)$. Therefore $M \cong R^n/K$ is a generator in the category of right R-modules, so R is right FPF-ring.

The proof of following is essentially due to Professor Y. Hirano. He simplyfied our original proof.

Lemma 2. *Let R be a regular ring. Then the following conditions are equivalent.*

(1) *$R \cong \prod_{i=1}^{t} M_{n_i}(S_i)$, where each S_i is an abelian regular ring.*

(2) *R is biregular and has a faithful abelian idempotent.*

Proof. (1) ⇒ (2). It is clear that R is biregular and contains a faithful abelian idempotent.

(2) ⇒ (3). Let g be a faithful abelian idempotent of R. Then since R is biregular and gR is a faithful right R-module, $RgR = R$, so gR is a generator. Thus R and gRg are Morita equivalent. Let P be a progenerator right gRg-module such that $R \cong \mathrm{End}_{gRg}(P)$. Then by [1 ; Prop 2.6], $P \cong K_1 \oplus \dots \oplus K_t$, where each K_i is a principal right ideal of gRg. While since gRg is an abelian regular ring, each K_i is a direct summand of gRg as a ring. Now we refine K_i more, we obtain that $gRg = K_1 \oplus \dots \oplus K_m$ as a ring. Hence there exist positive integer $n_1, \dots, n_m$ such that $P \cong K_1^{(n_1)} \oplus \dots \oplus K_m^{(n_m)}$ as a gRg-module. In this case, $R \cong \prod M_{n_i}(S_i)$, where each S_i is an abelian regular ring.

Now we can prove the main theorem of this section.

Theorem 1. *Let R be a regular ring. Then R has the conditions, "every cyclic faithful right R-module is a generator in the category of right R-modules, and every non-zero two-sided ideal contains a non-zero central idempotent", if and only if $R \cong R_1 \times \prod M_{n_i}(T_i)$, where R_1 is an abelian regular ring and each T_i is an abelian regular self-injective regular ring* *and* $n(i) \geqslant 2$.

Proof. First we assume that every cyclic faithful right R-modules is a generator in the category of right R-modules, and every non-zero two-sided ideal contains a non-zero central idempotent. We shall determine the structure of the maximal right quotient ring Q of R. Let M be a cyclic faithful right Q-module. We may assume that $M = Q/I$ for some right ideal I of Q. Set $N = R/(R \cap I)$ and $H = r_R(R/(R \cap I))$. If H is not zero, then there exists a non-zero central idempotent f of R such that $fR \subseteq I \cap R$. Hence $fQ \subseteq I$. But in this case, f must be zero since Q/I is faithful, which is a contradiction. Thus N is faithful, so N generates R_R. Hence $N^n \overset{\phi}{\to} R \to 0$ is exact for some positive integer n. Since maps of N to R lift to maps of M to Q, we have a R-homomorphism $\bar{\phi}$ from M^n to Q, while it is easy to see that $\bar{\phi}$ is also a Q-homomorphism. Therefore $\bar{\phi}$ is a Q-epimorphism, so M is a generator in the category of right Q-modules. This shows that Q has also the condition, every cyclic faithful right Q-module is a generator in the category of right Q-modules. Next we show that Q is biregular. Let e be any idempotent element of Q. Then $r_Q(eQ) = (1-h)Q$ for some central idempotent $1-h$ of Q. Thus eQ is a faithful right hQ-module. Note that hQ has also the condition, every cyclic faithful right hQ-module is a generator. Hence eQ is a generator in the category of right hQ-modules, so that $\mathrm{Tr}_{hQ}(eQ) = hQ$. It is easily seen that $QeQ = \mathrm{Tr}_Q(eQ) = \mathrm{Tr}_{hQ}(eQ) = hQ$. Therefore Q is biregular. . In this case, by the same proof of Proposition 1 of [2], Q is right bounded. Thus [2, Corollary of Theorem 2] shows that Q is isomorphic to a finite direct product of full matrix rings over abelian regular self-injective rings. Next we show that R is isomorphic to a finite direct product of full matrix rings over abelian regular rings. To show this assertion, by Lemma 2, it suffices to show that R is biregular and contains a faithful abelian idempotent element. Since Q is Type I, Q contains a faithful abelian idempotent e. Set $M = R/((1-e)Q \cap R)$. We claim that M is faithful. Assume not, then there exists a non-zero central idempotent g of R such that $gR \subseteq (1-e)Q \cap R$. Hence $(1-e)g = g$, so $eg = 0$. But this is impossible since eQ is faithful. Thus M is faithful, as claimed. Now M generates R_R. Note that $l_R((1-e)Q \cap R)R = R$. Therefore there exist $a_i \in l_R((1-e)Q \cap R)$ and $r_i \in R$ such that $\sum_{i=1}^n a_i r_i = 1$. We can write that $\sum_{i=1}^n Ra_i = Re'$ for some idempotent e' of R. Then Re' is faithful and $Re' \subseteq l_R((1-e)Q \cap R)$. Since $l_R((1-e)Q \cap R) \subseteq Qe$, $Re' \subseteq Qe$, hence $e' \cdot R \cdot e' \lesssim eQe$. This implies that e' is a faithful abelian idempotent of R. Further since $e'R$ is a generator in the category of right R-modules, R is a Morita equivalent to the abelian regular ring $e'\cdot R\cdot e'$. Since $e'\cdot R\cdot e'$ is biregular, we conclude that R is also biregular. Therefore R is isomorphic to a finite direct product of full matrix rings over abelian regular rings, i.e. $R\cong\prod_{i=1}^{t}M_{n(i)}(S_i)$, where each S_i is an abelian regular ring. Finally, we assert that each S_i is self-injective if $n(i)\geqslant 2$. To prove this assertion, there is no loss in assuming that $R\cong M_n(S)$ for some abelian regular ring S and for some positive integer $n\geqslant 2$. Then the maximal right quotient ring Q of R is isomorphic to $M_n(Q(S))$, where $Q(S)$ is the maximal right quotient ring of S. Let w be any element of $Q(S)$ and let $e=(e_{ij})\in Q$ such that $e_{11}=1$ and $e_{21}=w$ and $e_{ii}=1$ if $i\geqslant 3$, and $e_{ij}=0$ otherwise. e is a faithful idempotent of Q. We set $M=R/(eQ\cap R)$. We show that M is faithful. If M is not faithful, then there exists a non-zero central idempotent f of R such that $f\in eQ\cap R$, so $fe=f$. While since eQ is faithful, $f=1$. But this is impossible since $eQ\neq Q$. Therefore M is a generator in the category of right R-modules. On the other hand, it is easy to see that $eQ\cap R=\{(x_{ij})\in R\,|\,x_{1j}\in J$ for all $j=1,\dots,n$, and $x_{2j}=wx_{1j}$ for all $j=1,\dots,n$, and $x_{ij}\in S$ for all $i=3,4,\dots,n$ and $j=1,\dots,n\}$, where $J=\{r\in S\,|\,ws\in S\}$. Let $y=(y_{ij})$ be any element of $l_R(eQ\cap R)$. Then for any $a\in J$, $y_{i1}a+y_{i2}wa=0$ for all $i=1,\dots,n$, and $y_{ij}=0$ for all $i=1,\dots,n$ and $j=3,4,\dots,n$. Thus $y_{i1}+y_{i2}w=0$ for all $i=1,\dots,n$ since J is an essential right ideal of S. This shows that y_{i2} is in $J'=\{r\in S\,|\,rw\in S\}$. Note that J' is an essential left ideal of S. On the other hand, since $l_R(eQ\cap R)R=R$, $\sum_{i=1}^{m}(-r_iw)S+r_iS=S$ for some $r_i\in S$. We note that each r_iw is in J'. Hence this implies that $J'=S$ since J' is a two-sided ideal of S. So w is in S. Concequently, we obtain that $Q(S)=S$. Conversely, if $R\cong R_1\times\prod_{i=1}^{t}M_{n(i)}(S_i)$, where R_1 is an abelian regular ring and each S_i is an abelian regular self-injective ring and $n(i)\geqslant 2$. Then since each one-sided ideal of R_i is two-sided, and each idempotênt is central, R_1 has the desired condition. Further $\prod_{i=1}^{t}M_{n(i)}(S_i)$ has also the desired condition by [2, Theorem 2]. Now the proof is complete.

Corollary. *Let R be a regular ring whose cyclic faithful modules are generator. Then the following conditions are equivalent.*

(1) *R has the condition "every non-zero two-sided ideal contains a non-zero central idempotent of R".*

(2) *The maximal right quotient ring Q of R is right FPF-ring, i.e. Q*

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is isomorphic to a finite direct product of full matrix rings over abelian regular self-injective rings.

Proof. (1) ⇨ (2). We have already shown in the proof of Theorem 1.

(2) ⇨ (1). Clearly, Q is of Type I_f. Therefore by [2, Theorem 2], R statisfies the condition of (1).

Without the condition, every non-zero two-sided ideal contains a non-zero central idempotent, we do not know the structure of regular rings whose cyclic faithful modules are generator.

However, if R is a prime regular ring, we have the following theorem.

Theorem 2. *Let R be a prime regular ring. Then R has the condition, every cyclic faithful right R-module is a generator in the category of right R-modules, if and only if R is a simple artinian ring.*

Proof. Let x be any non-zero element of R. Then $xR = eR$ for some idempotent e of R. We consider that $R/(1-e)R$. If $R/(1-e)R$ is not faithful, then $(1-e)R$ contains a non-zero two-sided ideal I of R. While since R is a prime ring, I is essential as a right ideal, so $(1-e)R$ must equal to R. Hence $eR = 0$, which is a contradiction. Thus $R/(1-e)R \cong eR$ generates R_R, hence $ReR = R$. This shows that R is a simple ring. Next we show that R is right bounded. Let J be an essential right ideal of R. If R/J is faithful, then R/J generates R_R. But this is impossible since R is non-singular. Thus J contains a non-zero two-sided ideal, so R is right bounded. However, since R is a simple ring, R has no proper non-zero two-sided ideals. Therefore R is a simple artinian. The converse is clear. Now the proof is complete.

2. Rings whose faithful module, which is generated at most two elements, are generator. In this section, we consider a ring R which has the follwing condition,

$(*)$: every faithful right R-module, which is generated at most two elements, are generator in the category of right R-modules.

In the sequel of this paper, we call this ring R as a ring R with property $(*)$.

By Lemma 1, the condition $(*)$ is equivalent to the condition "every cyclic faithful right $M_2(R)$-module is a generator in the category of right $M_2(R)$-modules".

Lemma 3. *Let R be a right non-singular ring with property $(*)$, and Q be the maximal right quotient ring of R and let $B(R)$ be the set of all central idempotents of R. Then $B(R) = B(Q)$.*

Proof. Set $T = M_2(R)$ and note that $M_2(Q)$ is a maximal right quotient ring of T. Let e be any element of $B(Q)$, and let $x = \begin{pmatrix} 1 & e \\ 0 & 0 \end{pmatrix}$. Then it is easy to see that xT is a faithful right T-module, so generates T_T. Now f be any non-zero element of $\mathrm{Hom}_T(xT, T)$. Since $\mathrm{Hom}_T(xT, Y) \subseteq \mathrm{Hom}_{M_2(Q)}(xM_2(Q), M_2(Q)) = M_2(Q)x$, we can write $f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a, b and c, d are elements of $M_2(Q)$. Further since $f(xT) \subseteq T$, easy calulation of matrix shows that $a, c \in J = \{r \in R \mid er \in R\}$. On the other hand, since xT generates T_T, $J(R+eR) = R$. Finally, we conclude that $J = R$ since J is a two-sided ideal of R and e is a central idempotent of Q. Now e is in R.

In [3], we have characterized non-singular right FPF-rings. In the following Proposition, we have a similar characterization on non-singular rings with property $(*)$.

Proposition 1. *Let R be a right non-singular ring with property $(*)$ and Q be the maximal right quotient ring of R. Then R satisfied the following conditions.*

(i) *R is right bounded.*

(ii) *$Q \otimes_R Q \cong Q$ and Q is flat as a right R-module.*

(iii) *For any right ideal, which is generated at most two elements, I of R,*

$$\mathrm{Tr}_R(I) \oplus \mathrm{r}_R(I) = R \ (\textit{as ideals}).$$

Proof. (i) It is easily seen from the assumption that every cyclic faithful right R-module is a generator in the category of right R-modules. Now let J be an essential right ideal of R. If R/J is faithful, then R/J generates R_R, which is contradiction since R is right non-singular. Thus $\mathrm{r}_R(R/J) = H \neq 0$. In this case, by Lemma 3, there exists a central idempotent e of R such that $\mathrm{r}_R(J) = (1-e)R$. Therefore we can apply the proof of Proposition 1 of [2].

(ii) We can apply the proof of Corollary of [5].

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(iii) Let I be a right ideal, which is generated at most two elements. Then there exists a central idempotent e of R such that $\mathrm{r}_R(I) = (1-e)R$. Thus I is faithful right $R/(1-e)R$-module, so I generates $R/(1-e)R = eR_{eR}$ since eR also is a ring with property (*). We see that $\mathrm{Tr}_{eR}(I) = eR$. Note that $\mathrm{Tr}_R(I) = \mathrm{Tr}_{eR}(I) = eR$. Therefore $\mathrm{Tr}_R(I) \oplus \mathrm{r}_R(I) = R$.

Corollary 2. *Let R be a commutative semiprime ring with property* (*). *Then R is a FPF-ring.*

Proof. By Proposition 1, it is easy to see that every right ideal, which is generated at most two elements, are projective. Therefore, by Corollary 3 of [3], it suffices to show that for any $M \in \mathrm{BS}(R)$, R/MR is a Prüfer domain, where $\mathrm{BS}(R)$ is the collection of all maximal ideal of $\mathrm{B}(R)$. On the other hand, by [4, Lemma 10], in order to prove that a domain is Prüfer, it suffices to show that every ideal, which is generated at most two elements, are projective. Now let $\bar{I} = \bar{a}R/M + \bar{b}R/M$ be an ideal of R/M, where a and b are in R. We set $I = aR + bR$. Then there exists a central idempotent e of R such that $\mathrm{r}_R(I) = eR$. In this case, $\mathrm{Tr}_R((aR + bR) \oplus eR) = \mathrm{Tr}_R(aR + bR) \oplus eR = R$, by Proposition 1. Thus $(aR + bR) \oplus eR$ is a generator in the category of R-modules. Further since M is a maximal ideal of $\mathrm{B}(R)$, e is in M, so $\bar{I}$ is a homomorphic image of $I \oplus eR$. Now $\bar{I}$ generates $R/M_{R/M}$, hence $\bar{I}$ is projective.

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