Mathematical Journal of Okayama University

Volume 30, *Issue* 1 1988 *Article* 16

JANUARY 1988

Operations associated with the G-equivariant unitary cobordism theory

Michikazu Fujii*

*Okayama University

Math. J. Okayama Univ. 30 (1988), 161−175

OPERATIONS ASSOCIATED WITH THE G-EQUIVARIANT UNITARY COBORDISM THEORY

MICHIKAZU FUJII

Introduction. Let G be a compact abelian Lie group. In the previous paper [4] we have introduced a G-equivariant cohomology theory which is concerned with the G-equivariant unitary cobordism theory. In the equivariant cohomology theory there is the splitting principle and Chern classes are defined for complex G-vector bundles.

In this paper we shall study on cohomology operations in the equivariant cohomology theory. In § 1 we consider Landweber-Novikov operations in our equivariant cohomology theory. And, in § 2 we observe mod p Steenrod operations in the G-equivariant unitary cobordism theory and research on them in connection with the Landweber-Novikov operations introduced in § 1.

1. Landweber-Novikov operations. Let G be a compact abelian Lie group. Let $U_G^*(-)$ and $K_G^*(-)$ be the G-equivariant unitary cobordism theory and the G-equivariant complex K-theory, respectively. By making use of Thom classes in K_G^*-theory, we can get a natural multiplicative transformation

$$\mu_G : U_G^*(-) \to K_G^*(-)$$

of the cohomologies (cf. [3], [4]). We take up a multiplicative set T_K consisting of all one dimensional representations in the representation ring $R(G) \cong K_G(\mathrm{pt})$ and we consider a multiplicative system $T = \mu_G^{-1}(T_K)$ in U_G^0. Then our multiplicative G-equivariant cohomology theory $h_G^*(-)$ is defined by

$$h_G^*(-) = T^{-1}U_G^*(-),$$

where $T^{-1}U_G^*(-)$ means a ring localized by the multiplicative system T.

Using the local triviality of complex G-vector bundles [5] and Theorem 4.5 in [4] we obtain the following splitting principle:

Proposition 1.1. *Let ξ be an n-dimensional complex G-vector bundle over a comapct G-space X. Then there exist a compact G-space $F(\xi)$, a G-map $\pi : F(\xi) \to X$ and n complex G-line bundles $\xi_1, \dots, \xi_n$ over $F(\xi)$ satisfying the following conditions :*

1) *$\pi^*: h_G^*(X) \to h_G^*(F(\xi))$ is a monomorphism.*
2) *$\pi^*\xi$ is isomorphic to the sum $\xi_1 \oplus \cdots \oplus \xi_n$.*

Proposition 1.2. *Let ξ and η be n and m-dimensional complex G-vector bundles over a compact G-space X, respectively. Then there exist a compact G-space F and a G-map $\pi: F \to X$ satisfying the following conditions:*

1) *$\pi^*: h_G^*(X) \to h_G^*(F)$ is a monomorphism.*
2) *$\pi^*(\xi)$ and $\pi^*(\eta)$ are isomorphic to the sums of n and m G-line bundles over F, respectively.*

Furthermore we have G-equivariant Chern classes $c_i^G(\xi) \in h_G^{2i}(X)$, $0 \leq i \leq n$ ($c_0^G(\xi) = 1$), of an n-dimensional complex G-vector bundle ξ over a compact G-space X [4].

We now define Landweber-Novikov operations [1, 9] in the cohomology theory $h_G^*(-)$ and call up their basic properties. Let $\boldsymbol{t} = (t_1, t_2, \ldots)$ be a sequence of indeterminates. Assigning $\deg t_i = -2i$ for each $i \geqq 1$, $U_G^*(-)[[\boldsymbol{t}]]$ and $h_G^*(-)[[\boldsymbol{t}]]$ become multiplicative G-equivariant cohomology theories.

Let ξ be an n-dimensional complex G-vector bundle over a compact G-space X and let $\pi: F(\xi) \to X$ and $\xi_1, \ldots, \xi_n$ be ones of Proposition 1.1. Consider the following

$$\prod_{i=1}^{n} (1 + e(\xi_i)t_1 + \cdots + e(\xi_i)^k t_k + \cdots) \in U_G^*(F(\xi))[[\boldsymbol{t}]]$$

and

$$(1) \quad \prod_{i=1}^{n} (1 + c_1^G(\xi_i)t_1 + \cdots + c_1^G(\xi_i)^k t_k + \cdots) \in h_G^*(F(\xi))[[\boldsymbol{t}]],$$

where $e(\xi_i) \in U_G^2(F(\xi))$ is the Euler class of ξ_i and $c_1^G(\xi_i) = \dfrac{e(\xi_i)}{1} \in h_G^2(F(\xi))$ is the first Chern class of ξ_i.

Given an n-tuple $\iota = (i_1, \ldots, i_n)$ of non-negative integers, denote by

$$\sum x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$$

the least symmetric polynomial in variables $x_1, \ldots, x_n$ which contains the term $x_1^{i_1} \cdots x_n^{i_n}$. The symmetric polynomial can be written as a polynomial $P_\iota(\sigma_1, \ldots, \sigma_n)$ in the elementary symmetric functions $\sigma_1, \ldots, \sigma_n$ of the variables $x_1, \ldots, x_n$:

$$P_\iota(\sigma_1, \ldots, \sigma_n) = \sum x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}.$$

As for the coefficient of $t_{i_1} t_{i_2} \cdots t_{i_n}$ in the sequence (1), by making use of the splitting principle, we get the equality

$$\begin{aligned}\sum c_1^G(\xi_1)^{i_1} c_1^G(\xi_2)^{i_2} \cdots c_1^G(\xi_n)^{i_n} &= P_\iota(\sigma_1, \ldots, \sigma_n) \\ &= \pi^* P_\iota(c_1^G(\xi), \ldots, c_n^G(\xi)),\end{aligned}$$

where $\sigma_k = \sigma_k(c_1^G(\xi_1), \ldots, c_1^G(\xi_n))$ is the k-th elementary symmetric function of the classes $c_1^G(\xi_1), \ldots, c_1^G(\xi_n)$.

Let us define the *total Chern class* $c_t(\xi)$ of ξ in the theory $h_G^*(-)[[t]]$ by

$$c_t(\xi) = \sum_\iota P_\iota(c_1^G(\xi), \ldots, c_n^G(\xi))\ t_\iota \in h_G^*(X)[[t]]$$

where $t_\iota = t_{i_1} t_{i_2} \cdots t_{i_n}$. Then, in vertue of the naturality of Euler classes of G-line bundles, the splitting principle and the external product we obtain

Proposition 1.3. *The total Chern classes satisfy the following properties:*

(1) (*naturality*) $c_t(f^*(\xi)) = f^*(c_t(\xi))$.

(2) (*multiplicativity*) $c_t(\xi \times \eta) = c_t(\xi) \times c_t(\eta)$.

(3) (*normality*) $c_t(\varepsilon) = 1$,

where $\varepsilon : \mathrm{pt} \times C \to \mathrm{pt}$ *is the trivial G-line bundle over a point.*

(4) *If* ξ *is a complex G-line bundle, then*

$$c_t(\xi) = 1 + c_1^G(\xi) t_1 + \cdots + c_1^G(\xi)^k t_k + \cdots.$$

Let $T(\xi)$ be the Thom space of an n-dimensional complex G-vector bundle ξ over a compact G-space X. Then, by making use of the Thom isomorphism

$$\phi(\xi) : h_G^*(X) \to \tilde{h}_G^{*+2n}(T(\xi)),$$

we obtain the Thom isomorphism

$$\phi_t(\xi) : h_G^*(X)[[t]] \to \tilde{h}_G^{*+2n}(T(\xi))[[t]],$$

which is defined by

$$\phi_t(\xi)(\sum a_{i_1, \ldots, i_k} t_1^{i_1} \cdots t_k^{i_k}) = \sum \phi(\xi)(a_{i_1, \ldots, i_k}) t_1^{i_1} \cdots t_k^{i_k}.$$

Put

$$s_t(\xi) = \phi_t(\xi)(c_t(\xi)) \in \tilde{h}_G^*(T(\xi))[[t]].$$

Then we have

Proposition 1.4. *The classes $s_t(\ )$ satisfy the following properties:*

(1) (*naturality*) $s_t(f^*(\xi)) = f^*(s_t(\xi))$.

(2) (*multiplicativity*) $s_t(\xi \times \eta) = s_t(\xi) \times s_t(\eta)$.

(3) (*normality*) $s_t(\varepsilon) = t_h(\varepsilon) \in \tilde{h}^2_G(S^2)$.

(4) *If ξ is a complex G-line bundle, then*

$$s_t(\xi) = t_h(\xi) + t_h(\xi)^2 t_1 + \cdots + t_h(\xi)^k t_{k-1} + \cdots$$

where $t_h(\xi)$ is the Thom class of ξ in the theory $h^*_G(-)$.

Let γ^n_G be the universal complex G-vector bundle and denote by $M_n(G)$ the Thom space of γ^n_G. Let W be a complex G-module and $G_n(W)$ the Grassmann manifold of complex n-planes. Then γ^n_G and $M_n(G)$ are the limit of the canonical n-dimensional G-vector bundle

$$\gamma^n_G(W) = (E_n(W),\ \pi,\ G_n(W))$$

and the Thom space $M_n(W) = T(\gamma^n_G(W))$, respectively.

Let $x \in U^{2n}_G(X)$ be represented by $f:\ V^C \wedge X^+ \to M_{\|V\|+n}(W) \subset M_{\|V\|+n}(G)$, where $X^+ = X \cup \{\infty\}$ (disjoint union), V^C means the one point compactification of a complex G-module V and $\|V\| = \dim_C V$. Defining

$$s_t:\ U^*_G(X) \to h^*_G(X)[[t]]$$

by

$$s_t(x) = \phi_t(V)^{-1} f^*(s_t(\gamma^{\|V\|+n}_G(W))),$$

we obtain a natural transformation

$$s_t:\ U^*_G(-) \to h^*_G(-)[[t]]$$

of G-equivariant cohomology theories.

Proposition 1.5. *The natural transformation s_t has the following properties:*

(1) (*naturality*) $s_t(g^*(x)) = g^*(s_t(x))$.

(2) (*multiplicativity*) $s_t(xy) = s_t(x)s_t(y)$.

(3) (*normality*) i) $s_t(t(\xi)) = s_t(\xi)$ *for the Thom class* $t(\xi) \in \tilde{U}^{2n}_G(T(\xi))$ *of an n-dimensional complex G-vector bundle ξ*, ii) $s_t(1) = 1$, *and* iii) $s_t(V) = t_h(V)$.

Let $\omega = (\omega_1, \omega_2, \ldots)$ be a sequence of non-negative integers with $\omega_i = 0$ except for a finite number of terms, $|\omega| = \sum \omega_i$ and $\underline{t}^\omega = t_1^{\omega_1} t_2^{\omega_2} \cdots$. Put

$$s_t(x) = \sum_\omega s_\omega(x) \underline{t}^\omega$$

for $x \in U_G^*(X)$. Then, from the properties of s_t it follows

Theorem 1.6. *For each sequence* $\omega = (\omega_1, \omega_2, \ldots)$ *there exists an operation*

$$s_\omega : U_G^*(-) \to h_G^{*+2|\omega|}(-)$$

with the following properties:

(1) (*natural*) $s_\omega(g^*(x)) = g^*(s_\omega(x))$.

(2) (*multiplicative*) $s_\omega(xy) = \sum_{\alpha+\beta=\omega} s_\alpha(x)\, s_\beta(y)$

where $\alpha+\beta = (\alpha_1+\beta_1, \alpha_2+\beta_2, \ldots)$.

(3) $s_0(x) = \dfrac{x}{1}$ *for* $0 = (0, 0, \ldots)$.

(4) (*stable*) $s_\omega \sigma(V) = \sigma_h(V)\, s_\omega$, *where* $\sigma(V)$ *and* $\sigma_h(V)$ *are suspension isomorphisms in the theories* $U_G^*(-)$ *and* $h_G^*(-)$.

(5) *If* ξ *is* 1-*dimensional, then*

$$s_\alpha(t(\xi)) = \begin{cases} t_h(\xi)^{\alpha_i+1} & \text{for } \alpha = (0, \ldots, 0, \alpha_i, 0, 0, \ldots) \\ 0 & \text{otherwise.} \end{cases}$$

2. Steenrod operations. In this section we observe the mod p Steenrod operations in the theory $U_G^*(-)$ and reseach on them in connection with the Landweber-Novikov operations which are introduced in the previous section.

Let G be a compact Lie group and Z_p a cyclic group of order p with a generator ρ. By a (G, Z_p)-space X we mean a Hausdorff space X having both actions of G and Z_p which commute. Let V be a complex G-module. Throughout this section we only treat finite dimensional complex G-modules. We consider the G-module V a (G, Z_p)-space with a Z_p action defined by $\rho^k v = \exp \dfrac{2\pi\sqrt{-1}\, k}{p} v$ $(v \in V)$. Then $S(V)^+ = S(V) \cup \{\infty\}$ is a pointed (G, Z_p)-space with a fixed base point ∞, where $S(V)$ is the unit sphere in V.

Example 1. For a pointed G-space X, the p-fold reduced join $\overset{p}{\wedge} X = X \wedge \cdots \wedge X$ is a pointed (G, Z_p)-space with a Z_p-action defined by $\rho(x_1 \wedge \cdots$

$\wedge x_p) = x_2 \wedge \cdots \wedge x_p \wedge x_1$. We consider the p-fold product $\overset{p}{\times} X = X \times \cdots \times X$ a (G, Z_p)-space for a G-space X, too.

Example 2. Let $\xi : E \to X$ be a complex G-vector bundle and denote by $\overset{p}{\times} \xi$ the p-fold product bundle of ξ. Then the total space $E(\overset{p}{\times} \xi) = E \times \cdots \times E$ of $\overset{p}{\times} \xi$ is a (G, Z_p)-space with a Z_p-action defined by $\rho(v_1, \ldots, v_p) = (v_2, \ldots, v_p, v_1)$.

Let us define a G-space and a pointed G-space as follows:

$$E_V(X) = (S(V) \times X)/Z_p \qquad \text{for } (G, Z_p)\text{-space } X,$$

and

$$\tilde{E}_V(X) = (S(V)^+ \wedge X)/Z_p \qquad \text{for pointed } (G, Z_p)\text{-space } X.$$

Then we have

Proposition 2.1. *For a (G, Z_p)-space X, there holds*

$$\tilde{E}_V(X^+) = E_V(X)^+.$$

Proposition 2.2. *For a complex G-vector bundle ξ over a compact G-space X*

$$E_V(\overset{p}{\times} \xi) : E_V(E(\overset{p}{\times} \xi)) \to E_V(\overset{p}{\times} X)$$

is a complex G-vector bundle.

Let $\mathscr{C}(G, Z_p)$ be the category of pointed (G, Z_p)-spaces and $\mathscr{C}(G)$ the category of pointed G-spaces. Then $\tilde{E}_V : \mathscr{C}(G, Z_p) \to \mathscr{C}(G)$ is a covariant functor.

Forthermore we have

Proposition 2.3. *If ξ is a complex (G, Z_p)-vector bundle over a compact (G, Z_p)-space X, then*

$$E_V(\xi) : E_V(E(\xi)) \to E_V(X)$$

is a complex G-vector bundle. And, as for the Thom spaces of them, it follows that

$$T(E_V(\xi)) = \tilde{E}_V(T(\xi)).$$

Proposition 2.4. *For a pair (X, A) of a (G, Z_p)-space and its subspace, there exist G-homeomorphisms*

$$\tilde{E}_V(X/A) \approx E_V(X)/E_V(A) \approx \tilde{E}_V(X^+)/\tilde{E}_V(A^+).$$

Proposition 2.5. *For a pointed G-space X with the trivial Z_p-action and a pointed (G, Z_p)-space Y, there exists a G-homeomorphism*

$$\tilde{E}_V(Y \wedge X) \approx \tilde{E}_V(Y) \wedge X.$$

Proposition 2.6. *For a pointed G-space X with the trivial Z_p-action and a G-module W, there exists a G-homeomorphism*

$$\tilde{E}_V((\overset{p}{\bigwedge} W^c) \wedge X) \approx T(E_V(\overset{p}{\times} W) \times X)/T(E_V(\overset{p}{\times} W) \times *_X),$$

where W^c means the one point compactification of W.

Proof. We have the following G-homeomorphisms

$$\begin{aligned}
\tilde{E}_V((\overset{p}{\bigwedge} W^c) \wedge X) &\approx \tilde{E}_V(\overset{p}{\bigwedge} W^c) \wedge X && \text{(by 2.5)}\\
&\approx \tilde{E}_V((\overset{p}{\times} W)^+) \wedge X \\
&\approx E_V(\overset{p}{\times} W)^+ \wedge X && \text{(by 2.1)}\\
&\approx T(E_V(\overset{p}{\times} W)) \wedge X && \text{(by 2.2)}\\
&= T(E_V(\overset{p}{\times} W)) \wedge (X^+/*_X^+)\\
&= T(E_V(\overset{p}{\times} W)) \wedge X^+/T(E_V(\overset{p}{\times} W)) \wedge *_X^+\\
&= T(E_V(\overset{p}{\times} W) \times X)/T(E_V(\overset{p}{\times} W) \times *_X).
\end{aligned}$$

q.e.d.

By the same way as in the non-equivariant case we have the following Thom isomorphism theorem of a pair (cf. [2]):

Theorem 2.7. *For an n-dimensional complex G-vector bundle ξ over a compact G-space X and a closed G-subspace A of X, the Thom homomorphism*

$$\phi: U_G^*(X, A) \to U_G^{*+2n}(T(\xi), T(\xi | A))$$

is an isomorphism.

In virtue of Proposition 2.3, for a G-module W and a pointed G-space

X with the trivial Z_p-action,

$$\overset{p}{\times} W : E_V((\overset{p}{\times} W) \times X) \to E_V((*, \ldots, *) \times X)$$

is a G-vector bundle. Therefore, by making use of Theorem 2.7 and Propositions 2.4 and 2.6, we obtain a Thom isomorphism

$$\phi : \tilde{U}_G^*(\tilde{E}_V(X)) \to \tilde{U}_G^*(\tilde{E}_V((\overset{p}{\wedge} W^C) \wedge X)).$$

We now would like to define the external mod p Steenrod operation

$$P_V^{2k} : \tilde{U}_G^{2k}(X) \to \tilde{U}_G^{2pk}(E_V(X))$$

for each G-module V and a pointed G-space X.

Let $x \in U_G^{2k}(X)$ be represented by $f : W^C \wedge X \to M_{\|W\|+k}(U) \subset M_{\|W\|+k}(G)$. Consider the composition of G-maps

$$\begin{aligned}\tilde{E}_V(\overset{p}{\wedge} f) : \tilde{E}_V(\overset{p}{\wedge}(W^C \wedge X)) &\to \tilde{E}_V(\overset{p}{\wedge} M_{\|W\|+k}(U)) \\ &= \tilde{E}_V(T(\overset{p}{\times} \gamma_G^{\|W\|+k}(U))) \\ &= T(E_V(\overset{p}{\times} \gamma_G^{\|W\|+k}(U))) \quad (\text{by } 2.3) \\ &\xrightarrow{\mu_p} T(\gamma_G^{p(\|W\|+k)}) = M_{p(\|W\|+k)}(G),\end{aligned}$$

where μ_p is the map of Thom spaces induced by the classifying map of the complex G-vector bundle $E_V(\overset{p}{\times} \gamma_G^{\|W\|+k}(U))$. The map μ_p represents the Thom class

$$[\mu_p] = t(E_V(\overset{p}{\times} \gamma_G^{\|W\|+k}(U))) \in \tilde{U}_G^{2p(\|W\|+k)}(T(E_V(\overset{p}{\times} \gamma_G^{\|W\|+k}(U))).$$

Define a map $\tilde{d} : (\overset{p}{\wedge} W^C) \wedge X \to \overset{p}{\wedge}(W^C \wedge X)$ by $\tilde{d}((w_1 \wedge \cdots \wedge w_p) \wedge x) = (w_1 \wedge x) \wedge \cdots \wedge (w_p \wedge x)$. Then we get a G-map

$$\tilde{E}_V(\tilde{d}) : \tilde{E}_V((\overset{p}{\wedge} W^C) \wedge X) \to \tilde{E}_V(\overset{p}{\wedge}(W^C \wedge X)).$$

Now we define $P_V^{2k}(x)$ by

$$P_V^{2k}(x) = \phi^{-1}\tilde{E}_V(\tilde{d})^*\tilde{E}_V(\overset{p}{\wedge} f)^*(t(E_V(\overset{p}{\times} \gamma_G^{\|W\|+k}(U)))).$$

And we have the following properties:

Proposition 2.8. *For a G-module V there exists an operator*

$$P_V : U_G^*(-) \to U_G^*(E_V(-))$$

with the following properties :

(1) (*naturality*) $P_V^{2k} h^*(x) = E_V(h)^* P_V^{2k}(x)$.

(2) (*multiplicativity*) *For* $x \in U_G^{2k}(X)$ *and* $y \in U_G^{2l}(Y)$

$$P_V^{2k+2l}(x \times y) = P_V^{2k}(x) \times P_V^{2l}(y).$$

(3) *For the Thom class* $t(\xi) \in \tilde{U}_G^{2k}(T(\xi))$ *of a k-dimensional G-vector bundle* ξ,

$$P_V^{2k}(t(\xi)) = \tilde{E}_V(\tilde{d})^*(t(E_V(\overset{p}{\times} \xi))) = t(E_V(\xi)).$$

Let L be the canonical 1-dimensional Z_p-module and consider it a trivial G-module. Put

$$\Delta = L \oplus L^2 \oplus \cdots \oplus L^{p-1}.$$

Then we obtain

Proposition 2.9. *Let* ξ *be a complex G-vector bundle over a compact G-space X. Consider the p-fold sum* $\xi \oplus \cdots \oplus \xi$ *a* (G, Z_p)*-bundle over X with a* Z_p*-action defined by* $\rho(v_1, \ldots, v_p) = (v_2, \ldots, v_p, v_1)$ *for* $(v_1, \ldots, v_p) \in E(\xi \oplus \cdots \oplus \xi)$. *Then it follows that*

(1) *the vector bundles* $\xi \oplus \cdots \oplus \xi$ *and* $\xi \otimes (C \oplus \Delta)$ *are* (G, Z_p)*-isomorphic, and*

(2) *the diagram*

$$\begin{array}{ccc} \xi & \xrightarrow{\hat{d}} & \xi \oplus \cdots \oplus \xi \\ & {\scriptstyle i} \searrow & \downarrow \cong \\ & & \xi \otimes (C \oplus \Delta) \end{array}$$

is G-homotopy commutative, where $\hat{d}$ *is the diagonal map and* i *is the natural inclusion defined by* $i(v) = v \otimes 1 \in \xi \otimes C$.

Proof. (1) Let us consider a (p, p)-matrix A and a unitary matrix $U = (u_{ij})$ such that

$$A = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 \\ & 1 \ddots & \ddots & \vdots \\ 0 & & 1 & 0 \end{pmatrix}, \quad U^{-1}AU = \begin{pmatrix} 1 & & & 0 \\ & \rho & & \\ & & \ddots & \\ 0 & & & \rho^{p-1} \end{pmatrix}.$$

Then a (G, Z_p)-bundle isomorphism

$$h : \xi \oplus \cdots \oplus \xi \to \xi \otimes (C \oplus \Delta) = \xi \otimes C \oplus \xi \otimes L \oplus \cdots \oplus \xi \otimes L^{p-1}$$

and its inverse h^{-1} are given by

$$h(v_1, \ldots, v_p) = (\sum_{j=1}^{p} u_{j1} v_j \otimes 1, \sum_{j=1}^{p} u_{j2} v_j \otimes 1, \ldots, \sum_{j=1}^{p} u_{jp} v_j \otimes 1)$$

and

$$h^{-1}(v_1 \otimes z_1, \ldots, v_p \otimes z_p) = (\sum_{j=1}^{p} u'_{j1} z_j v_j, \ldots, \sum_{j=1}^{p} u'_{jp} z_j v_j)$$

for $(v_1, \ldots, v_p) \in \xi \oplus \cdots \oplus \xi$ and $(v_1 \otimes z_1, \ldots, v_p \otimes z_p) \in \xi \otimes (C \oplus \Delta)$, where $U^{-1} = (u'_{ij})$.

(2) Since there holds $h\rho = \rho h$ for the generator $\rho \in Z_p$, it follows that

$$\begin{aligned} h\hat{d}(v) &= h\rho(v, \ldots, v) = (\sum_{j=1}^{p} u_{j1} v \otimes 1, \ldots, \sum_{j=1}^{p} u_{jp} v \otimes 1) \\ &= \rho(\sum_{j=1}^{p} u_{j1} v \otimes 1, \ldots, \sum_{j=1}^{p} u_{jp} v \otimes 1) \\ &= (\sum_{j=1}^{p} u_{j1} v \otimes 1, \sum_{j=1}^{p} u_{j2} v \otimes \rho \cdot 1, \ldots, \sum_{j=1}^{p} u_{jp} v \otimes \rho^{p-1} \cdot 1). \end{aligned}$$

Hence we have

$$(\sum_{j=1}^{p} u_{jk})(v \otimes 1) = \rho^{k-1} (\sum_{j=1}^{p} u_{jk})(v \otimes 1) \qquad \text{in } \xi \otimes L^{k-1}.$$

This implies

$$\sum_{j=1}^{p} u_{jk} = 0 \ (k = 2, \ldots, p), \text{ that is,}$$

$$h\hat{d}(v) = (\sum_{j=1}^{p} u_{j1} v \otimes 1, 0, \ldots, 0).$$

Since (u_{j1}) is an eigenvector for 1 of A, we have $u_{11} = \cdots = u_{p1}$ and $|u_{11}| = \dfrac{1}{\sqrt{p}}$. Hence a G-homotopy connecting $h\hat{d}$ and i is given easily. q.e.d.

Proposition 2.10. *For complex G-vector bundles ξ and η over a compact G-space X, let*

$$i_\xi : T(\xi) \to T(\xi \oplus \eta)$$

be an inclusion given by $i_\xi(v) = (v, 0)$ *for* $v \in T(\xi)$, *and*

$$\phi_\xi : U_G^*(X) \to \tilde{U}_G^*(T(\xi))$$

the Thom isomorphism. Then there holds

$$i_\xi{}^*(t(\xi \oplus \eta)) = \phi_\xi(e(\eta))$$

for the Thom class $t(\xi \oplus \eta) \in \tilde{U}_G^*(T(\xi \oplus \eta))$ *and the Euler class* $e(\eta) \in U_G^*(X)$.

Proof. Consider the following commutative diagram

$$\begin{array}{ccccc}
\tilde{U}_G^*(T(\xi)) \otimes \tilde{U}_G^*(T(\eta)) & \xrightarrow{\times} & \tilde{U}_G^*(T(\xi) \wedge T(\eta)) & \xrightarrow{\bar{d}^*} & \tilde{U}_G^*(T(\xi \oplus \eta)) \\
\downarrow 1 \otimes s^* & & \downarrow (1 \wedge s)^* & & \downarrow i_\xi{}^* \\
\tilde{U}_G^*(T(\xi)) \otimes \tilde{U}_G^*(X^+) & \xrightarrow{\times} & \tilde{U}_G^*(T(\xi) \wedge X^+) & \xrightarrow{\bar{d}^*} & \tilde{U}_G^*(T(\xi)),
\end{array}$$

where $s : X^+ \to T(\eta)$ is the 0-section and $\bar{d} : T(\xi \oplus \eta) \to T(\xi \times \eta) = T(\xi) \wedge T(\eta)$ is the map induced by the diagonal map. Then it follows that

$$\begin{aligned}
i_\xi{}^*(t(\xi \oplus \eta)) &= i_\xi{}^*\bar{d}^*(t(\xi) \times t(\eta)) \\
&= \bar{d}^*(t(\xi) \times s^*(t(\eta)) \\
&= \bar{d}^*(t(\xi) \times e(\eta)) \\
&= \phi_\xi(e(\eta)).
\end{aligned}$$

q.e.d.

Proposition 2.11. *For an n-dimensional complex G-vector bundle* ξ *over a compact G-space X, there holds*

$$P_V^{2n}(t(\xi)) = \phi_{E_V\xi}(e(E_V(\xi \otimes \Delta))).$$

Proof. By Proposition 2.8 we have

$$P_V^{2n}(t(\xi)) = \tilde{E}_V(\tilde{d})^*(t(E_V(\overset{p}{\times} \xi))).$$

Since the commutative diagram

$$\begin{array}{ccc}
T(\xi) & \xrightarrow{\hat{d}} & T(\xi \oplus \cdots \oplus \xi) \\
& \searrow^{\tilde{d}} & \downarrow \bar{d} \\
& & T(\xi \times \cdots \times \xi)
\end{array}$$

induces $\tilde{E}_V(\tilde{d})^* = \tilde{E}_V(\hat{d})^* \tilde{E}_V(\bar{d})^*$, we get

$$\begin{aligned}\tilde{E}_V(\tilde{d})^*(t(E_V(\overset{p}{\times}\xi))) &= \tilde{E}_V(\hat{d})^*(t(E_V(\xi\oplus\cdots\oplus\xi)))\\ &= \tilde{E}_V(i_\xi)^*(t(E_V(\xi\oplus\xi\otimes\varDelta)))\quad\text{(by 2.9 (2))}\\ &= i_{E_V(\xi)}{}^*(t(E_V(\xi)\oplus E_V(\xi\otimes\varDelta)))\\ &= \phi_{E_V(\xi)}(e(E_V(\xi\otimes\varDelta)))\quad\text{(by 2.10)}\qquad \text{q.e.d.}\end{aligned}$$

Let us consider a connection of the operations P_V with the Landweber-Novikov operations introduced in § 1. Therefore, let us assume the compact Lie group G abelian hereafter.

Let $V = L_1\oplus\cdots\oplus L_m$ and $W = L'_1\oplus\cdots\oplus L'_n$ be complex G-modules, where L_i and L'_j are 1-dimensional complex G-modules. Let $P(V)$ be the complex projective space for the G-module V and $\eta(V;\ C)$ the canonical complex G-line bundle over $P(V)$. Then, according to [4, Theorems 4.2 and 4.5] we see that

$$h_G^*(P(V)\times P(W)) = h_G^*(\mathrm{pt})[x_V, y_W]/(\theta_V(x_V),\ \theta_W(y_W))$$

where $x_V = e_h(\eta(V;\ C)\mathbin{\hat{\otimes}} 1)$ and $y_W = e_h(1\mathbin{\hat{\otimes}}\eta(W;\ C))$ are the Euler classes of the G-line bundles, and $(\theta_V(x_V),\ \theta_W(y_W))$ is an ideal generated by polynomials $\theta_V(x_V) = (x_V - e_h(L_1))\cdots(x_V - e_h(L_m))$ and $\theta_W(y_W) = (y_W - e_h(L'_1))\cdots(y_W - e_h(L'_n))$. As usual we put

$$h_G^*(P_\infty\times P_\infty) = \lim h_G^*(P(V)\times P(W))$$

where the limmit depends on the inverse system defined by inclusion maps of G-modules. Then we get

$$h_G^*(P_\infty\times P_\infty) = h_G^*(\mathrm{pt})[[x,\ y]].$$

As usual, by commutativity and associativity of tensor products of G-vector bundles, we obtain a commutative formal group

$$F(x,\ y) = \sum a_{ij}x^iy^j \in h_G^2(P_\infty\times P_\infty)$$

such that $F(x,\ y)\,|\,P(V)\times P(W) = e_h(\eta(V;\ C)\mathbin{\hat{\otimes}}\eta(W;\ C))$ and $a_{10} = a_{01} = 1$. And, for G-line bundles ξ and η over a compact G-space X, we have

$$e_h(\xi\otimes\eta) = F(e_h(\xi),\ e_h(\eta)) = e_h(\xi)+e_h(\eta)+\textit{higher terms}.$$

Lemma 2.12. *For an n-dimensional complex G-vector bundle ξ over a compact G-space X, there holds*

$$s_0P_V^{2n}(t(\xi)) = \sum_{|\alpha|\leq n} e_h(E_V(\varDelta))^{n-|\alpha|}b_\alpha(v)s_\alpha(t(\xi))$$

where $v = e_h(E_V(L))$, $|\alpha| = \sum \alpha_i$ *for each sequence* $\alpha = (\alpha_1, \alpha_2, \dots)$ *and* $b_\alpha(v) \in h_G^*(\mathrm{pt})[[v]]$ *is a power series.*

Proof. By Proposition 2.11 we have

$$\begin{aligned} s_0 P_V^{2n}(t(\xi)) &= s_0 \phi_{E_V(\xi)}(e(E_V(\xi \otimes \Delta))) \\ &= \phi(E_V(\xi))(e_h(E_V(\xi \otimes \Delta))). \end{aligned}$$

1) When ξ is a sum of G-line bundles $\xi_1, \dots, \xi_n$, it follows that

$$\begin{aligned} e_h(E_V(\xi \otimes \Delta)) &= e_h(E_V(\xi_1 \otimes \Delta \oplus \cdots \oplus \xi_n \otimes \Delta)) \\ &= e_h(\xi_1 \hat{\otimes} E_V(\Delta) \oplus \cdots \oplus \xi_n \hat{\otimes} E_V(\Delta)) \\ &= e_h(\xi_1 \hat{\otimes} E_V(\Delta)) \cdot \dots \cdot e_h(\xi_n \hat{\otimes} E_V(\Delta)). \end{aligned}$$

For each k we have

$$\begin{aligned} e_h(\xi_k \hat{\otimes} E_V(\Delta)) &= e_h(\xi_k \hat{\otimes} E_V(L) \oplus \cdots \oplus \xi_k \hat{\otimes} E_V(L^{p-1})) \\ &= e_h(\xi_k \hat{\otimes} E_V(L)) \cdot \dots \cdot e_h(\xi_k \hat{\otimes} E_V(L^{p-1})) \\ &= F(e_h(\xi_k), e_h(E_V(L))) \cdot \dots \cdot F(e_h(\xi_k), e_h(E_V(L^{p-1}))) \\ &= \prod_{l=1}^{p-1} (e_h(E_V(L^l)) + \sum_{j \geq 1} a_j(v) e_h(\xi_k)^j) \\ &= e_h(E_V(\Delta)) + \sum_{j \geq 1} b_j(v) e_h(\xi_k)^j, \end{aligned}$$

where $a_j(v)$ and $b_j(v)$ are formal power series of v. Hence we have

$$\begin{aligned} e_h(E_V(\xi \otimes \Delta)) &= \prod_{k=1}^{n} (e_h(E_V(\Delta)) + \sum_{j \geq 1} b_j(v) e_h(\xi_k)^j) \\ &= \sum_{|\alpha| \leq n} e_h(E_V(\Delta))^{n-|\alpha|} b_\alpha(v) c_\alpha(\xi) \end{aligned}$$

where $c_\alpha(\xi) = \sum e_h(\xi_1)^{\alpha_1} \cdot \dots \cdot e_h(\xi_n)^{\alpha_n}$ and $b_\alpha(v)$ is a formal power series of v. Therefore we have

$$\begin{aligned} s_0 P_V^{2n}(t(\xi)) &= \phi(E_V(\xi))(\sum_{|\alpha| \leq n} e_h(E_V(\Delta))^{n-|\alpha|} b_\alpha(v) c_\alpha(\xi)) \\ &= \sum_{|\alpha| \leq n} e_h(E_V(\Delta))^{n-|\alpha|} b_\alpha(v) s_\alpha(\xi). \end{aligned}$$

2) General case is shown by making use of the splitting principle in the theory $h_G^*(-)$. q.e.d.

Now we obtain an h_G^*-theoretic version of [15, Proposition 3.17].

Theorem 2.13. *Let* $x \in \tilde{U}_G^{2n}(X)$ *be represented by a map* $f: W^c \wedge X \to M_{|W|+n}(U) \subset M_{|W|+n}(G)$. *Then there holds*

$$e_h(E_V(\Delta))^{\|W\|}s_0P_V^{2n}(x) = \sum_{|\alpha|\leq\|W\|+n} e_h(E_V(\Delta))^{n+\|W\|-|\alpha|}b_\alpha(v)s_\alpha(x),$$

where $b_\alpha(v) \in h_G^*(\mathrm{pt})[[v]]$ *is a well defined power series.*

Proof. There holds $x = \sigma_W^{-1}f^*(t(\gamma_G^m(U)))$, where $m = \|W\|+n$ and σ_W is the suspension isomorphism. Hence, by the previous lemma, the naturality of s_ω and P_V^{2m} and the stability of s_ω, we have

$$\begin{aligned} s_0P_V^{2m}(f^*(t(\gamma_G^m(U)))) &= \tilde{E}_V(f)^*(s_0P_V^{2m}(t(\gamma_G^m(U)))) \\ &= \tilde{E}_V(f)^*(\sum_{|\alpha|\leq m} e_h(E_V(\Delta))^{m-|\alpha|}b_\alpha(v)s_\alpha(t(\gamma_G^m(U)))) \\ &= \sum_{|\alpha|\leq m} e_h(E_V(\Delta))^{m-|\alpha|}b_\alpha(v)s_\alpha(f^*(t(\gamma_G^m(U)))) \\ &= \sum_{|\alpha|\leq m} e_h(E_V(\Delta))^{m-|\alpha|}b_\alpha(v)s_\alpha(\sigma_W x) \\ &= \sum_{|\alpha|\leq m} e_h(E_V(\Delta))^{m-|\alpha|}b_\alpha(v)\sigma_W s_\alpha(x). \end{aligned}$$

On the other hand we have

$$\begin{aligned} s_0P_V^{2m}(\sigma_W x) &= s_0P_V^{2m}(\sigma_W(1) \times x) \\ &= s_0P_V(\sigma_W(1)) \times s_0P_V(x). \end{aligned}$$

Since $\sigma_W(1) = t(W)$, by the previous lemma, we have

$$s_0P_V(\sigma_W(1)) = \sum_{|\alpha|\leq\|W\|} e_h(E_V(\Delta))^{\|W\|-|\alpha|}b_\alpha(v)s_\alpha(\sigma_W(1)).$$

Here

$$s_\alpha\sigma_W(1) = \sigma_W s_\alpha(1) = \begin{cases} \sigma_W(1) & \text{for } \alpha = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Thus we have

$$s_0P_V^{2m}(\sigma_W x) = e_h(E_V(\Delta))^{\|W\|}\sigma_W s_0P_V^{2n}(x).$$

This completes the proof. q.e.d.

Acknowledgement. The author wishes to express his thanks to Professor M. Kamata for his kind advices and discussions.

References

[1] J. F. Adams : P. S. Novikov's work on operations on complex cobordism, mimeographed, Univ. of Chicago, 1967.

[2] T. Bröcker und T. tom-Dieck : Kobordismentheorie, Lecture Notes in Math. 178, Springer-

Verlag, 1970.

[3] P. E. CONNER and E. E. FLOYD : The relation of cobordism to K-theories, Lecture Notes in Math. 28, Springer-Verlag, 1966.

[4] M. FUJII and M. KAMATA : On the completion of the G-equivariant unitary cobordism rings of G-spaces, Publ. R. I. M. S., Kyoto Univ. 19 (1983), 577−600.

[5] M. FUJII and S. KÔNO : A note on the local triviality of G-vector bundles, Yokohama Math. J. 30 (1982), 49−52.

[6] T. tom-DIECK : Kobordismen-Theorie und Transformations-Gruppen, Preprint Series, Aarhus Univ., (1968/69).

[7] T. tom-DIECK : Bordism of G-manifolds and integrality theorems, Topology 9 (1970), 345−358.

[8] T. tom-DIECK : Steenrod-Operationen in Kobordismen-Theorien, Math. Z. 107 (1968), 380−401.

[9] P. S. LANDWEBER : Cobordism operations and Hopf algebras, Trans. Amer. Math. Soc. 129 (1967), 94−110.

[10] M. NAKAOKA : Characteristic Classes with values in complex Cobordism, Osaka J. Math. 10 (1973), 521−543.

[11] P. S. NOVIKOV : The method of algebraic topology from viewpoint of cobordism theories (Russian), Izv. Akad. Nauk SSSR 31 (1967), 855−951.

[12] C. OKONEK : Der Conner-Floyd-Isomorphismus fur Abelsche Gruppen, Math. Z. 179 (1982), 201−212.

[13] C. OKONEK : Die äquivariante unitäre Kobordismentheorie und ihre äquivariante formale Gruppe, Arch. Math. 40 (1983), 208−220.

[14] D. QUILLEN : On the formal group laws of unoriented and complex cobordism theory, Bull. Amer. Math. Soc. 75 (1969), 1293−1298.

[15] D. QUILLEN : Elementary proofs of some results of cobordism theory using Steenrod operations, Advances in Math. 7 (1971), 29−56.

OKAYAMA UNIVERSITY

(Received February 15, 1988)