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Bordism theory with reality and duality theorem of Poincaré type

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BORDISM THEORY WITH REALITY AND DUALITY THEOREM OF POINCARÉ TYPE

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Introduction. In the paper [10] we introduced the cobordism theory with reality. The purpose of this paper is to form a bordism theory of closed differentiable manifolds with a given real structure on the stable tangent bundle.

In this paper, by a *τ-space* (X, t) we mean a Hausdorff space X together with an involution $t : X \to X$, and by a *τ-map* we mean an equivariant map between τ-spaces (which we called a real space and a real map in the previous papers [10], [11]). By a *τ-manifold* (M, t) we mean a compact C^∞-manifold M together with a C^∞-involution $t : M \to M$.

In § 1 we consider weakly real structures on equivariant vector bundles over τ-spaces. In § 2 we introduce an *R-structure* on the τ-manifold, which is an equivalence class of the real structures on the stable tangent bundle. The τ-manifold with an R-structure is called an *R-manifold*. In § 3 we form the bordism groups of R-manifolds and in § 4 we show that there is a duality of Poincaré type between the bordism groups and the cobordism groups with reality.

1. Weakly real structures of equivariant vector bundles.

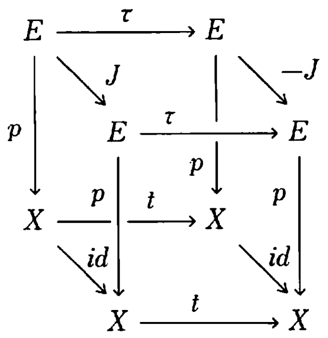
By a *τ-space* (X, t) we mean a Hausdorff space X together with an involution $t = t_X : X \to X$, and by a *τ-map* we mean an equivariant map between τ-spaces.

By an *equivariant vector bundle* $\xi = (E, p, X, \tau)$ over a τ-space (X, t) we mean a vector bundle (E, p, X) such that

i) the total space (E, τ) is a τ-space, and
ii) the projection $p : E \to X$ is a τ-map.

Furthermore, by a *real vector bundle* $\xi = (E, p, X, J, \tau)$ over a τ-space (X, t) we mean an equivariant vector bundle (E, p, X, τ) such that

iii) $J : E \to E$ is a bundle automorphism such that $J^2 = -id$, $\tau J = (-J)\tau$ and $p(-J) = p$. That is, the diagram



commutes, (E, p, X, J) is a complex vector bundle and the bundle map $\tau : E \to E$ is an isomorphism from the complex vector bundle (E, J) to the conjugate bundle $(E, -J)$. (J, τ) will be called a *real structure* of the vector bundle $\xi = (E, p, X)$.

Let $\xi_i = (E_i, p_i, X_i, J_i, \tau_i)$ be a real vector bundle over a τ-space (X_i, t_i) $(i = 1$ or $2)$. Then a *real vector bundle map* $f : \xi_1 \to \xi_2$ is a bundle map f such that $f\tau_1 = \tau_2 f$ and $fJ_1 = J_2 f$. If $(X_1, t_1) = (X_2, t_2)$ and f is a bundle isomorphism, then the real bundle map $f : \xi_1 \to \xi_2$ is said to be an *isomorphism* of real vector bundles. When $(X_1, t_1) = (X_2, t_2)$, we can form a *Whitney sum* of the real vector bundles ξ_1 and ξ_2 by defining

$$\xi_1 \oplus \xi_2 = (E_1 \oplus E_2, p, X_1, J_1 \oplus J_2, \tau_1 \oplus \tau_2).$$

Let R^n be the n-dimensional Euclidean space and let $R^{r,s} = R^r \times R^s$. Consider the *standard involution* $c_{r,s}$ on $R^{r,s}$ defined by

$$c_{r,s}(x_1, \dots, x_r, y_1, \dots, y_s) = (-x_1, \dots, -x_r, y_1, \dots, y_s)$$

for $(x_1, \dots, x_r, y_1, \dots, y_s) \in R^{r,s}$. Furthermore, consider the standard map $I_n : R^{n,n} \to R^{n,n}$ defined by

$$I_n(x_1, \dots, x_n, y_1, \dots, y_n) = (y_1, \dots, y_n, -x_1, \dots, -x_n)$$

for $(x_1, \dots, x_n, y_1, \dots, y_n) \in R^{n,n}$. Then it holds a relation

$$c_n I_n = -I_n c_n,$$

where $c_n = c_{n,n}$. That is, I_n is the *standard complex structure* on $R^{n,n}$ $(\equiv R^{2n} \equiv C^n)$, c_n is the *standard conjugation* on $R^{n,n}$ and (I_n, c_n) is the *standard real structure* on $R^{n,n}$ in the sense of Atiyah [5].

Real vector bundles $\xi = (E, p, X, J, \tau)$ and $\eta = (E', p', X, J', \tau')$ over a τ-space (X, t) are said to be *stably equivalent* if and only if there exist positive integers m, n and an isomorphism

$$\xi \oplus (X\times R^{m,m},\ p_1,\ X,\ id\times I_m,\ t\times c_m)$$
$$\cong \eta \oplus (X\times R^{n,n},\ p_1,\ X,\ id\times I_n,\ t\times c_n)$$

of real vector bundles.

Let $\xi=(E,\ p,\ X,\ \tau)$ be an equivariant vector bundle over a τ-space $(X,\ t)$. When there exist positive integers r, s and a real structure $(J,\ \tau\oplus c_{r,s})$ of the vector bundle $\xi\oplus(X\times R^{r,s},\ p_1,\ X,\ t\times c_{r,s})$, the real structure $(J,\ \tau\oplus c_{r,s})$ will be called a *weakly real structure* of the equivariant vector bundle ξ. Two weakly real structures $(J_1,\ \tau\oplus c_{p,q})$ and $(J_2,\ \tau\oplus c_{r,s})$ of the equivariant vector bundle ξ are said to be *equivalent* if the real vector bundles $(E\oplus X\times R^{p,q}, p,\ X,\ J_1,\ \tau\oplus c_{p,q})$ and $(E\oplus X\times R^{r,s},\ p,\ X,\ J_2,\ \tau\oplus c_{r,s})$ are stably equivalent.

Let $\xi=(E,\ p,\ X,\ \tau)$ and $\eta=(E',\ p',\ X,\ \tau')$ be equivariant vector bundles over a τ-space $(X,\ t)$ and let $h:\xi\to\eta$ an isomorphism of equivariant vector bundles. Let $(J,\ \tau'\oplus c_{r,s})$ be a weakly real structure of η. Then a weakly real structure $(h^*J,\ \tau\oplus c_{r,s})$ of ξ is induced by the isomorphism

$$h\oplus id:\xi\oplus(X\times R^{r,s},\ p_1,\ X,\ t\times c_{r,s})$$
$$\cong \eta\oplus(X\times R^{r,s},\ p_1,\ X,\ t\times c_{r,s})$$

of equivariant vector bundles, where $h^*J=(h\oplus id)^{-1}J(h\oplus id)$.

Often we denote the equivariant vector bundle $(X\times R^{r,s},\ p_1,\ X,\ t\times c_{r,s})$ simply by $(\underline{R}^{r,s},\ c_{r,s})$ or $\underline{R}^{r,s}$, and denote the real vector bundle $(X\times R^{m,m},\ p_1,\ X,\ id\times I_m,\ t\times c_m)$ simply by $(\underline{R}^{m,m},\ I_m,\ c_m)$ or $\underline{R}^{m,m}$.

2. *R*-manifolds. By an m-dimensional *τ-manifold* $(M,\ t)$ we mean an m-dimensional compact C^∞-manifold M together with a C^∞-involution $t:M\to M$. As for τ-manifolds we have the following imbedding theorem.

Proposition 2.1. *Any m-dimensional τ-manifold $(M,\ t)$ without boundary can be equivariantly imbedded as a closed subset of $R^{n,n}$ for some positive integer n.*

This is deduced from a more general situation [6], Theorem 4.1, in the category of C^∞-manifolds with G-action and equivariant C^∞-maps.

Furthermore, in virtue of the imbedding theorem, we have the collaring theorem, approximation theorems and the straightening angles in our cate-

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gory.

Let (M, t) be a τ-manifold of dimension m and let $\tau(M)$ be its tangent bundle. When there exists a weakly real structure $(J, dt \oplus c_{r,s})$ of the equivariant vector bundle $\tau(M)$, an equivalence class $\Phi = \{(J, dt \oplus c_{r,s})\}$ of the weakly real structure $(J, dt \oplus c_{r,s})$ is called an *R-structure* of the τ-manifold (M, t). An *R-manifold* (M, Φ) is a pair consisting of a τ-manifold (M, t) and an R-structure Φ of (M, t). Often we denote an R-manifold (M, Φ) simply by M.

Let (M, Φ) be an R-manifold and $(J, dt \oplus c_{r,s})$ a representative of its R-structure Φ. Then $(J \oplus (-I_1), dt \oplus c_{r,s} \oplus c_{1,1})$ is also a weakly real structure of the tangent bundle $\tau(M)$. Now, define $-(M, \Phi)$ by $-\Phi = \{(J \oplus (-I_1), dt \oplus c_{r,s} \oplus c_{1,1})\}$ and $-(M, \Phi) = (M, -\Phi)$.

Let (M, Φ) be an R-manifold with boundary ∂M and $(J, dt \oplus c_{r,s})$ a representative of its R-structure Φ. In virtue of the equivariant collaring theorem, there exists an isomorphism of equivariant vector bundles

$$h : \tau(\partial M) \oplus \underline{R}^{0,1} \to \tau(M) | \partial M,$$

where the positive unit vector of $\underline{R}^{0,1}$ corresponds to an inward unit normal vector in $\tau(M)|\partial M$ by h. Hence $(h^*(J|\partial M), d(t|\partial M) \oplus c_{r,s+1})$ is a weakly real structure of $\tau(\partial M)$. Let us define $\partial(M, \Phi)$ by $\partial\Phi = \{(h^*(J|\partial M), d(t|\partial M) \oplus c_{r,s+1})\}$ and $\partial(M, \Phi) = (\partial M, \partial\Phi)$.

Proposition 2.2. *If a τ-manifold (M, t) of dimension m has an R-structure Φ, then the fixed point set F of M by the involution t has a uniform dimension.*

Proof. Let $(J, dt \oplus c_{r,s})$ be a representative of Φ. For any $x \in F$, $((\tau(M) \oplus \underline{R}^{r,s})|x, J)$ is a complex vector space and $dt \oplus c_{r,s}$ is a conjugate linear involution on it. Therefore the dimension of the fixed point set of it is equal to $m+r+s/2$. Hence $\dim F\tau(M)|x = m+r-s/2$, where $F\tau(M)|x$ means the fixed point set of $\tau(M)|x$. This depends only on the class Φ. q. e. d.

3. Bordism groups of *R*-manifolds. Let $\mathfrak{D}_{p,q}$ denote the family of all $p+q$ dimensional R-manifolds with a weakly real structure of the type $(J, dt \oplus c_{q+r,p+r})$ for some integer r as a representative of the R-structure.

Fix a pair (X, A) consisting of a τ-space X and a τ-subspace A. A *(p, q)-singular R-manifold* in (X, A) is a triple (M, Φ, f) consisting of a $p+q$ dimensional R-manifold (M, Φ) in $\mathfrak{D}_{p,q}$ and a τ-map $f:(M, \partial M) \to (X, A)$. If $A=\phi$, then of course $\partial M=\phi$ also.

A (p, q)-singular R-manifold (M, Φ, f) in (X, A) is said to *bord* if and only if there is a $p+q+1$ dimensional R-manifold (W, Ψ) in $\mathfrak{D}_{p,q+1}$ and a τ-map $F: W \to X$ for which

i) M is contained in ∂W as a regular τ-submanifold, and

ii) $\partial\Psi|M=\Phi$, $F|M=f$ and $F(\partial W-M) \subset A$.

From two (p, q)-singular R-manifolds (M_1, Φ_1, f_1) and (M_2, Φ_2, f_2) a disjoint union $(M_1 \cup M_2, \Phi_1 \cup \Phi_2, f_1 \cup f_2)$ is defined, where $M_1 \cap M_2=\phi$, $\Phi_1 \cup \Phi_2|M_i=\Phi_i$ and $f_1 \cup f_2|M_i=f_i$. Define $-(M, \Phi, f)=(M, -\Phi, f)$. A pair (M_1, Φ_1, f_1) and (M_2, Φ_2, f_2) of (p, q)-singular R-manifolds in (X, A) are *bordant* if and only if the disjoint union $(M_1 \cup M_2, \Phi_1 \cup -\Phi_2, f_1 \cup f_2)$ bords in (X, A).

Lemma 3.1. *Let (M, Φ) be an R-manifold in $\mathfrak{D}_{p,q}$. If two τ-maps f, $g:(M, \partial M) \to (X, A)$ are homotopic as τ-maps, then (p, q)-singular R-manifolds (M, Φ, f) and (M, Φ, g) in (X, A) are bordant.*

Proof. There is a τ-homotopy $F:(M\times I, \partial M\times I) \to (X, A)$ such that $F(x, 0)=f(x)$ and $F(x, 1)=g(x)$ for $x \in M$, where the involution acts trivially on $I=[0, 1]$. Let $p: M\times I \to M$ be the projection. Then there exists an isomorphism

$$\lambda: \tau(M\times I) \to p^*\tau(M) \oplus \underline{R}^{0,1}$$

of equivariant vector bundles. Let $(J, dt \oplus c_{r,s})$ be a representative of the R-structure Φ. Then a weakly real structure $(J', d(t\times 1) \oplus c_{r+1,s})$ of $\tau(M\times I)$ is induced by isomorphisms

$$\tau(M\times I) \oplus \underline{R}^{r+1,s} \xrightarrow{\lambda \oplus 1} p^*\tau(M) \oplus \underline{R}^{0,1} \oplus \underline{R}^{r+1,s} \xrightarrow{\alpha} p^*\tau(M) \oplus \underline{R}^{r,s} \oplus \underline{R}^{1,1}$$

where $J'=(\lambda \oplus 1)^{-1}\alpha^{-1}(p^*J \oplus I_1)\alpha(\lambda \oplus 1)$. Now, let us define $\Psi=\{(J', d(t\times 1) \oplus c_{r+1,s})\}$. Then we may show that

1) $M \cup M$ is a regular τ-submanifold of $\partial(M\times I)$,

2) $F(\partial(M\times I)-M\times 0 \cup M\times 1) \subset A$,

3) $F|M\times 0 \cup M\times 1=f \cup g$, and

4) $\partial\Psi|M\times 0=\Phi$ and $\partial\Psi|M\times 1=-\Phi$.

This shows that (M, Φ, f) and (M, Φ, g) are bordant. q. e. d.

By a parallel argument to the complex bordism theory we may show that the bordism relation is an equivalence relation. Denote the *bordism class* of (M, Φ, f) by $[M, \Phi, f]$, and the collection of all such bordism classes by $\mathfrak{M}\mathfrak{R}_{p,q}(X, A)$. Then $\mathfrak{M}\mathfrak{R}_{p,q}(X, A)$ is also an abelian group by the disjoint union. Given a τ-map $\psi: (X, A) \to (Y, B)$, there is associated a natural homomorphism $\psi_* : \mathfrak{M}\mathfrak{R}_{p,q}(X, A) \to \mathfrak{M}\mathfrak{R}_{p,q}(Y, B)$ given by $\psi_*([M, \Phi, f]) = [M, \Phi, \psi f]$. There is also a homomorphism $\partial: \mathfrak{M}\mathfrak{R}_{p,q}(X, A) \to \mathfrak{M}\mathfrak{R}_{p,q-1}(A)$ given by $\partial([M, \Phi, f]) = [\partial M, \partial\Phi, f|\partial M]$. Hence we have

Theorem 1. *For any integer p, $\mathfrak{M}\mathfrak{R}_{p,*}(\ ,\ )$ is a generalized homology theory on the category of pairs of τ-spaces and τ-maps of pairs.*

4. Poincaré duality. Let (M, Φ), (N, Ψ) be R-manifolds and $(J_M, dt_M \oplus c_{p,q})$, $(J_N, dt_N \oplus c_{r,s})$ representatives of Φ, Ψ respectively. Let

$$f: M \to N$$

be a τ-imbedding. A real structure (J, e) of the normal bundle $\nu(f)$ is called a *proper real structure* if and only if two real vector bundles $(\nu(f), J, e) \oplus (\tau(M) \oplus \underline{R}^{p,q}, J_M, dt_M \oplus c_{p,q})$ and $(f^*\tau(N) \oplus \underline{R}^{r,s}, J_N, dt_N \oplus c_{r,s})$ are stably equivalent.

Proposition 4.1. *Let (M, Φ), (N, Ψ) be R-manifolds and $f: M \to N$ a τ-imbedding. If there is a proper real structure of $\nu(f)$, then the real structure is unique up to stable equivalence.*

Proof. Let (J_1, e_1) and (J_2, e_2) be two proper real structures of $\nu(f)$. Then there exist representatives $(J_M, dt_M \oplus c_{p,q})$ and $(J_N, dt_N \oplus c_{r,s})$ of Φ and Ψ respectively, such that

$$\begin{aligned}
&(\nu(f), J_1, e_1) \oplus (\tau(M) \oplus \underline{R}^{p,q}, J_M, dt_M \oplus c_{p,q}) \\
&\quad \cong (f^*\tau(N) \oplus \underline{R}^{r,s}, J_N, dt_N \oplus c_{r,s}) \\
&(\nu(f), J_2, e_2) \oplus (\tau(M) \oplus \underline{R}^{p,q}, J_M, dt_M \oplus c_{p,q}) \\
&\quad \cong (f^*\tau(N) \oplus \underline{R}^{r,s}, J_N, dt_N \oplus c_{r,s})
\end{aligned}$$

as real vector bundles. Since M is compact, there exist a real vector bundle ξ over M and a positive integer k satisfying

$$(\tau(M) \oplus \underline{R}^{p,q}, J_M, dt_M \oplus c_{p,q}) \oplus \xi \cong (\underline{R}^{k,k}, I_k, c_k)$$

as real vector bundles (cf. [10], Prop. 1.4, 1.7). Hence we have

$$(\nu(f) \oplus \underline{R}^{k,k},\ J_1 \oplus I_k,\ e_1 \oplus c_k) \cong (\nu(f) \oplus \underline{R}^{k,k},\ J_2 \oplus I_k,\ e_2 \oplus c_k)$$

as real vector bundles. q. e. d.

Proposition 4.2. *Let (M, Φ) and (N, Ψ) be R-manifolds in $\mathfrak{D}_{p,q}$ and $\mathfrak{D}_{m,n}$ respectively, and let $f: M \to N$ be a τ-imbedding. If $m-n=p-q$, then there is a positive integer k such that, for a τ-imbedding*

$$f': M \to R^{k,k} \times N$$

defined by $f'(x) = (0, f(x))$ for $x \in M$, the normal bundle $\nu(f')$ has a proper real structure.

Proof. Since $m-n=p-q$ by the assumption, we may choose $(J_M, dt_M \oplus c_{q+s,p+s})$ and $(J_N, dt_N \oplus c_{n+r,m+r})$ satisfying $m+r=p+s$ and $n+r=q+s$ as representatives of Φ and Ψ respectively. In virtue of [10], Propositions 1.4 and 1.7, there exist a real vector bundle $\xi = (E(\xi), J_\xi, e_\xi)$ over M and a positive integer k such that

$$\begin{aligned}(E(\xi), J_\xi, e_\xi) \oplus (\tau(M) \oplus \underline{R}^{q+s,p+s},\ J_M,\ dt_M \oplus c_{q+s,p+s})\\ \cong (\underline{R}^{k,k},\ I_k,\ c_k)\end{aligned}$$

as real vector bundles. Then we have isomorphisms

$$\begin{aligned}\nu(f') &\cong \underline{R}^{k,k} \oplus \nu(f)\\ &\cong E(\xi) \oplus \tau(M) \oplus \underline{R}^{q+s,p+s} \oplus \nu(f)\\ &\cong E(\xi) \oplus \nu(f) \oplus \tau(M) \oplus \underline{R}^{q+s,p+s}\\ &\cong E(\xi) \oplus f^*\tau(N) \oplus \underline{R}^{n+r,m+r}\end{aligned}$$

of equivariant vector bundles. Hence a real structure of $\nu(f')$ is induced by the above isomorphisms from the real structure $(J_\xi \oplus J_N, e_\xi \oplus dt_N \oplus c_{n+r,m+r})$ of $E(\xi) \oplus f^*\tau(N) \oplus \underline{R}^{n-r,m+r}$. This real structure of $\nu(f')$ is a proper real structure and the proposition follows. q. e. d.

Let (X, Ψ) be an R-manifold without boundary and belong to $\mathfrak{D}_{m,n}$. Let $MR^{r,s}(X)$ be the real cobordism group introduced in [10]. A natural duality homomorphism

$$D_R: \mathfrak{MR}_{p,q}(X) \to MR^{m-p,n-q}(X)$$

is defined by the Pontrjagin-Thom construction as follows: Let $[M, \Phi, f]$ be any element in $\mathfrak{MR}_{p,q}(X)$. We may assume that $f: M \to X$ is a C^∞-τ-

map. In virtue of Proposition 2.1, there exists a τ-imbedding

$$h : M \to R^{l,l}.$$

Then, defining a τ-map

$$f' : M \to R^{l,l} \times X$$

by $f'(x) = (h(x), f(x))$ for $x \in M$, f' is also a τ-imbedding.

Let us assume $m-n = p-q$ $(m-p = n-q)$. Then, by Proposition 4.2, there exists a positive integer k such that, for a τ-imbedding

$$f'' : M \to R^{k,k} \times R^{l,l} \times X$$

defined by $f''(x) = (0, h(x), f(x))$ for $x \in M$, the normal bundle $\nu(f'')$ has a proper real structure. Since the dimension of the real vector bundle $\nu(f'')$ is $k+l+n-q$, there are a real bundle map

$$g : \nu(f'') \to \gamma^{k+l+n-q}$$

and an induced map $T(\nu(f'')) \to MU(k+l+n-q)$ of real Thom spaces (cf. [10], Proposition 1.5). There is then the composite map $d(f)$ given by

$$\begin{aligned}\textstyle\sum^{k+l,k+l} \wedge X^+ &\to R^{k,k} \times R^{l,l} \times X / R^{k,k} \times R^{l,l} \times X - Int\, D(\nu(f'')) \\ &\to T(\nu(f'')) \to MU(k+l+n-q),\end{aligned}$$

and $d(f)$ represents an element of $MR^{m-p,n-q}(X)$, where $D(\nu(f''))$ is the total space of the associated unit disk bundle of $\nu(f'')$. The correspondence $[M, \Phi, f] \to \{d(f)\}$ is well defined and a homomorphism as usual.

In the case of $m-n \neq p-q$, we may choose two positive integers a and b such that $p-q = (m+a)-(n+b)$. Therefore, considering a τ-map

$$\tilde{f} : M \to R^{a,b} \times X$$

defined by $\tilde{f}(x) = (0, f(x))$ for $x \in M$ instead of f, we can obtain a duality homomorphism

$$D_R : \mathfrak{M}\mathfrak{R}_{p,q}(X) \to \widetilde{M}R^{m+a-p,n+b-q}(\textstyle\sum^{a,b} \wedge X^+) = MR^{m-p,n-q}(X).$$

Generally the above homomorphism D_R is not isomorphic, however we have the following theorem deduced from a general situation [13], lemma 3.2, on the equivariant transversal regularity.

Theorem 2. *For any R-manifold (X, Ψ) in $\mathfrak{D}_{m,n}$ without boundary, the duality homomorphism*

$$D_R : \mathfrak{M}\mathfrak{R}_{p,q}(X) \to MR^{m-p,n-q}(X)$$

is an isomorphism whenever $p > q$ *and is an epimorphism whenever* $p = q$.

Let $MU_*(\quad)$ and $MU^*(\quad)$ be the complex bordism theory and cobordism theory, respectively. In these theories there exists the duality isomorphism of Atiyah-Poincaré type

$$D : MU_r(X) \to MU^{s-r}(X)$$

for any compact weakly complex s-manifold X without boundary.

Let

$$\rho_* : \mathfrak{M}\mathfrak{R}_{p,q}(X) \to MU_{p+q}(X)$$
$$\rho^* : MR^{m,n}(X) \to MU^{m+n}(X)$$

be the natural homomorphisms obtained by ignoring the involutions. Then, for any R-manifold (X, Ψ) in $\mathfrak{D}_{m,n}$ without boundary, we have the following commutative diagram

$$\begin{array}{ccc} \mathfrak{M}\mathfrak{R}_{p,q}(X) & \xrightarrow{D_R} & MR^{m-p,n-q}(X) \\ \rho_* \downarrow & & \rho^* \downarrow \\ MU_{p+q}(X) & \xrightarrow{D} & MU^{m+n-p-q}(X). \end{array}$$

Therefore, $ker\ D_R$ is contained in $ker\ \rho_*$. Especially, in the case of $X = \text{pt}$ and $p = q \gg 0$, we have

$$ker\ D_R = ker\ \rho_*,$$

because $\rho^* : MR^{-p,-p}(\text{pt}) \to MU^{-2p}(\text{pt})$ is isomorphic (cf. [2], Theorem 4.6). Hence we have

Proposition 4.4. *For any integer* $p \gg 0$, *isomorphisms*

$$\bar{D}_R : \mathfrak{M}\mathfrak{R}_{p,p}(\text{pt})/ker\ \rho_* \to MR^{-p,-p}(\text{pt})$$
$$\bar{\rho}_* : \mathfrak{M}\mathfrak{R}_{p,p}(\text{pt})/ker\ \rho_* \to MU_{2p}(\text{pt})$$

are induced by D_R *and* ρ_*, *respectively.*

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