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Masayuki Ôhori*

*Shinshu University

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CHAIN CONDITIONS AND QUOTIENT RINGS OF P.P. RINGS

Dedicated to Professor Tosiro Tsuzuku on his 60th birthday

MASAYUKI ÔHORI

The purpose of the present paper is to generalize several results of R. Yue Chi Ming [9]. First, we deal with right hereditary, right Noetherian rings with some conditions (Propositions 1, 2). Second, we present a characterization of duo rings all of whose factor rings are QF (Proposition 3). Besides, a characterization of right Artinian rings by means of nonzero factor rings is given (Proposition 4). Right self-injective, strongly regular rings with nonzero socles are also considered (Proposition 5). Finally, we give a sufficient condition for the classical right quotient ring of a right p.p. ring to be von Neumann regular (Theorem 1).

Throughout this paper we shall assume, unless otherwise specified, all rings under consideration have an identity element. Subrings of a ring R are required to contain the same identity element as R and an ideal of R means a two-sided ideal. The letter R always stands for a ring and all (right or left) R-modules are supposed to be unital. The prime radical of R is denoted by $P(R)$. We call R a *normal ring* if every idempotent of R is central. A ring R is a *right* (resp. *left*) *p.p. ring* if every principal right (resp. left) ideal of R is projective; R is a *p.p. ring* if R is both right and left p.p. For any (right or left) R-module M, $Z(M)$ and $\mathrm{Soc}(M)$ denote the singular submodule and the socle of M respectively. When the right socle $\mathrm{Soc}(R_R)$ and the left socle $\mathrm{Soc}({}_RR)$ of R coincide, we write $\mathrm{Soc}(R)$ for them. A right R-module M is said to be *essentially finitely generated* if M has a finitely generated submodule which is essential in M. For any non-empty subset X of R, $r_R(X)$ (resp. $l_R(X)$) denotes the set of right (resp. left) annihilators of X in R. A right (resp. left) ideal of the form $r_R(X)$ (resp. $l_R(X)$) for some non-empty subset X of R, is called a *right* (resp. *left*) *annihilator*. A ring R is a *right Utumi ring* if R is right non-singular and every closed right ideal of R is a right annihilator. A ring R is a *right Ore ring* if R has a classical right quotient ring. We denote the maximal right quotient ring of R by $Q^r_{\max}(R)$ (or $Q_{\max}$).

In the first place we consider the maximal right quotient ring of a right Ore, normal p.p. ring.

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Lemma 1. *Let R be a right Ore, normal p.p. ring. Then $Q^r_{\max}(R)$ is a strongly regular ring. Consequently, R is a right Utumi ring.*

Proof. Let Q be a classical right quotient ring of R. By [3, Theorem 1], Q is a strongly regular ring. Hence $Q^r_{\max}(Q) = Q^r_{\max}(R)$ is a strongly regular ring [7, XII, Corollary 5.3, p. 254]. The last part follows from [7, XII, Proposition 4.7, p. 251].

The above lemma provides us a slight generalization of [9, Corollary 2.2].

Proposition 1. *Let R be a normal, right Ore, right hereditary ring. Then the following are equivalent:*

1) *R is right Noetherian.*
2) *R satisfies ACC on right annihilators.*
3) *Every essential right ideal of R is essentially finitely generated.*

Proof. 1) ⇨ 2). Trivial.

2) ⇨ 3). First note that R is a right Utumi ring (Lemma 1). Then, by hypothesis, R satisfies ACC on closed right ideals, or equivalently, R_R is finite-dimensional [2, Theorem 3.14, p. 81]. It is now easy to see that every essential right ideal of R is essentially finitely generated.

3) ⇨ 1). This follows from [9, Proposition 2].

If every right (resp. left) ideal of R is an ideal, then R is called a *right* (resp. *left*) *duo ring*. R is a *duo ring* if R is a left duo ring as well as a right duo ring. As is well known, a right duo ring is a normal, right Ore ring.

Lemma 2. *Let R be a right duo, right non-singular ring. If R is a subring of a ring Q and R_R is essential in Q_R, then Q is normal. In particular, if Q is a maximal right quotient ring of R, then Q is strongly regular.*

Proof. For any $q \in Q$, let $(R : q) = \{r \in R \mid qr \in R\}$, which is an essential right ideal of R. Since R is a right duo ring, we have $Rqa \subseteq qaR \subseteq qR$ for every $a \in (R : q)$. Let e be an arbitrary idempotent of Q and let $a \in (R : e)$. From what we have seen above we get, for every $r \in R$, $rea \in eR$ and so $(1-e)rea = 0$. Since $Z(Q_R) = 0$, it follows that $(1-e)re = 0$. Similarly we have $er(1-e) = 0$; hence $er = re$. Thus e is commutative with every element of R, and hence is contained in the center of Q. Therefore Q is normal. The last part is obvious.

Corollary 1. *Let R be a duo, right non-singular ring. Then the maximal right and the maximal left quotient rings of R coincide.*

Proof. Obvious from the above lemma and [7, XII, Proposition 5.5, p. 255].

A right R-module M is said to be *p-injective* if, for any principal right ideal A of R, every R-homomorphism of A to M can be extended to an R-homomorphism of R to M. For divisible modules and torsion-free modules, the reader is referred to [6].

By the aid of Lemma 2 we can prove the following proposition which contains [9, Remark 3].

Proposition 2. *Let R be a duo ring. Then the following are equivalent:*

1) *R is right hereditary and (right) Noetherian.*

2) *R is right p.p. and every p-injective right R-module is injective.*

3) *Every divisible singular right R-module and every divisible torsion-free quasi-injective right R-module are injective.*

Proof. Assume 1). By [7, XV, Proposition 4.7, p. 289], R has a classical right quotient ring Q which is right Artinian. By hypothesis, the maximal right quotient ring $Q_{\max}$ of R is a strongly regular ring (Lemma 2). Being a subring of $Q_{\max}$, Q is reduced and hence is a finite direct sum of division rings. Q is also a classical left quotient ring of R, because R is a duo ring. Since every p-injective right R-module is divisible, the implications 1) $\Rightarrow$ 2) and 1) $\Rightarrow$ 3) now follow from [6, Theorem 3.4]. 2) $\Rightarrow$ 1) is [9, Remark 1], while 3) $\Rightarrow$ 1) follows from [9, Corollary 5.1].

A ring R is said to satisfy the *regularity condition* if an element a of R is a non-zero-divisor of R whenever $a+P(R)$ is a non-zero-divisor of $R/P(R)$. R is said to satisfy the *full regularity condition* if an element a of R is a non-zero-divisor of R if and only if $a+P(R)$ is a non-zero-divisor of $R/P(R)$. We call R a *right T-Goldie ring* if R/T_k as well as R is a right Goldie ring for every positive integer k, where $T_k = P(R) \cap l_R(P(R)^k)$. We cite here the following theorem of L. W. Small [5, 4.1 Small's theorem, p. 26].

The following are equivalent:

1) R has a classical right quotient ring which is right Artinian.

2) R is a right T-Goldie ring satisfying the regularity condition.

3) R is a right T-Goldie ring satisfying the full regularity condition.

As an application of Small's theorem, we get the following

Lemma 3. *Let R be a right T-Goldie ring such that $R/P(R)$ is Artinian. Then R is right Artinian.*

Proof. It is easy to see that R satisfies the regularity condition. In fact, if a is an element of R such that $a+P(R)$ is a non-zero-divisor of $R/P(R)$, then a is invertible in R. Hence by Small's theorem R has a classical right quotient ring Q which is right Artinian. But then R satisfies the full regularity condition. Therefore every non-zero-divisor of R is invertible in R, that is, $R=Q$.

Following R. Yue Chi Ming [9], a right R-module M is said to be *YJ-injective* if, for any nonzero $a \in R$, there exists a positive integer n such that $a^n \neq 0$ and any R-homomorphism of a^nR to M can be extended to an R-homomorphism of R to M. If R_R is YJ-injective, then R is called a *right YJ-injective ring*. Left YJ-injective rings are defined similarly.

A finite set of orthogonal idempotents $e_1, \ldots, e_n$ of R is said to be *complete* if $e_1+\cdots+e_n=1$.

We are now ready to prove the following proposition which contains [9, Remark 5(1)].

Proposition 3. *For a duo ring R, the following are equivalent:*

1) *Every factor ring of R is QF.*

2) *Every factor ring of R is left YJ-injective, left Goldie as well as right YJ-injective, right Goldie.*

Proof. 1) $\Rightarrow$ 2). Trivial.

2) $\Rightarrow$ 1). It suffices to show that R is QF. By hypothesis $R/P(R)$ is a right Goldie ring and is a divisible right module over itself [9, Lemma 6]. Consequently $R/P(R)$ is Artinian. By Lemma 3 we see that R is right and left Artinian. Since $\bar{R}=R/P(R)$ is duo and hence is normal, we have

$$\bar{R}=\bar{e}_1\bar{R}\oplus\cdots\oplus\bar{e}_n\bar{R},$$

where $\bar{e}_1, \ldots, \bar{e}_n$ is a complete set of orthogonal idempotents and $\bar{e}_i\bar{R}$ is a division ring for each i. Then $\bar{e}_1, \ldots, \bar{e}_n$ can be lifted to a complete set of orthogonal idempotents $e_1, \ldots, e_n$ of R. Note that $\bar{e}_1\bar{R}, \ldots, \bar{e}_n\bar{R}$ is a complete representative system of isomorphism classes of simple right R-modules and $\mathrm{Hom}_R(\bar{e}_i\bar{R}, e_jR)=0$ for $i\neq j$. For every i we have

$$\begin{aligned}\mathrm{Hom}_R(\bar{e}_i\bar{R}, R) &= \mathrm{Hom}_R(\bar{e}_i\bar{R}, \oplus_{j=1}^n e_jR) \simeq \oplus_{j=1}^n \mathrm{Hom}_R(\bar{e}_i\bar{R}, e_jR)\\ &= \mathrm{Hom}_R(\bar{e}_i\bar{R}, e_iR).\end{aligned}$$

As is easily seen, any minimal right subideal of e_iR is isomorphic to $\bar{e}_i\bar{R}$. We claim that each e_iR has a unique minimal right subideal. Suppose the contrary and let A_1, A_2 be distinct minimal right subideals of e_iR. Let $\phi: A_1 \to A_2$ be an R-isomorphism. Since R is right YJ-injective and A_1 is minimal, ϕ can be extended to an endomorphism of R_R. Hence there exists a $b \in R$ such that for any $a \in A_1$, $\phi(a) = ba$, which belongs to A_1 by hypothesis. We then have $A_1 = A_2$, contradiction. Therefore

$$\mathrm{Hom}_R(\bar{e}_i\bar{R},\ e_iR) \simeq \mathrm{Hom}_R(\bar{e}_i\bar{R},\ \bar{e}_i\bar{R}) \simeq \bar{e}_i\bar{R},$$

which is a simple left R-module. Similarly the R-dual of any simple left R-module is a simple right R-module. Consequently by [1, 24.4 Theorem and Definition (c), p. 204], R is QF.

If R is a semiprime ring, then the right socle and the left socle of R coincide. Moreover, the homogeneous components of the socles coincide and these are simple rings [4, IV, § 3, Theorem 1, p. 65]. By the aid of these remarks and Lemma 3, we are able to prove that [9, Remark 5(2)] holds for non-commutative rings.

Proposition 4. *A ring R is right Artinian if and only if every nonzero factor ring of R is a right Goldie ring with nonzero right socle.*

Proof. It suffices to prove the if part. By hypothesis $\bar{R} = R/P(R)$ is a right Goldie ring with nonzero (right) socle. The socle of $\bar{R}$ has finitely many homogeneous components $\bar{R}_1, \ldots, \bar{R}_n$ and these are simple Artinian rings. Let $\bar{e}_i$ be the identity element of $\bar{R}_i$ for each i. Then $\bar{e}_1, \ldots, \bar{e}_n$ are orthogonal central idempotents of $\bar{R}$. Put $\bar{e} = \bar{e}_1 + \cdots + \bar{e}_n$. If $\bar{e} \neq \bar{1}$ $(= 1 + P(R))$, then by hypothesis the right socle of $(\bar{1} - \bar{e})\bar{R}$ is nonzero, which yields a contradiction. Therefore $\bar{e} = \bar{1}$, and hence $\bar{R}$ is Artinian. By Lemma 3 we conclude that R is right Artinian.

If R is a commutative ring which contains an injective maximal ideal, then R is self-injective [9, Remark 10(4)]. As an extension of this to normal rings, we have the following

Lemma 4. *Let R be a normal ring.*

(1) *If U is a non-nilpotent minimal right ideal of R, then U_R is injective.*

(2) *If R contains an injective maximal right ideal, then R is right self-injective.*

Proof. (1) By hypothesis there exists an idempotent e of R such that $U = eR$. Since R is normal, U is an ideal of R. Let A be a nonzero right ideal of R and let $\phi: A \to U$ be a nonzero R-homomorphism. Since ϕ is surjective and U_R is projective, there exists a right ideal B of R such that $A = B \oplus \mathrm{Ker}\ \phi$. Let b be an element of B such that $\phi(b) = e$. Then $B = bU \subseteq U$, and so $B = U$. Evidently $e \cdot \mathrm{Ker}\ \phi = 0$, that is, $\mathrm{Ker}\ \phi \subseteq (1-e)R$. Since ϕ induces an automorphism of U_R, there exists an $x \in U$ such that $\phi(u) = xu$ for every $u \in U$. We then have $\phi(a) = xa$ for every $a \in A$, showing that U_R is injective.

(2) Let I be an injective maximal right ideal of R and let U be a right ideal of R such that $R = I \oplus U$. Then U is a non-nilpotent minimal right ideal. By (1) we see that U_R is injective. This completes the proof.

A right ideal A of R is said to be *von Neumann regular* in R if for every $a \in A$, there exists a $b \in R$ such that $aba = a$.

The following proposition contains [9, Remark 12].

Proposition 5. *For a normal ring R, the following are equivalent:*

1) *R is a right self-injective, von Neumann regular ring with non-zero socle.*

2) *R contains an injective maximal right ideal which is von Neumann regular in R.*

3) *R contains a non-singular, injective maximal right ideal.*

Proof. 1) ⇨ 2). Trivial.

2) ⇨ 3). Let I be an injective maximal right ideal which is von Neumann regular in R. For any $a \in Z(I_R)$, there exists a $b \in R$ such that $aba = a$. Then we have $r_R(a) = r_R(ba) = (1-ba)R$, which is an essential right ideal as well as a direct summand of R_R. Hence $r_R(a) = R$, that is, $a = 0$. Thus $Z(I_R) = 0$.

3) ⇨ 1). Since R is right self-injective (Lemma 4(2)), R is von Neumann regular [9, Theorem 13]. It is almost obvious that the socle of R is nonzero.

The following lemma seems to be well known, so we omit the proof.

Lemma 5. *Let R be a semiprime ring and let A be an ideal of R. If A is essential in R as an ideal, then A is essential in R both as a right ideal and as a left ideal.*

Lemma 6. *Let R be a right non-singular ring and let A be an ideal of R. If A is essential as a right ideal, then the centralizer $C_R(A)$ of A in R is the center of R.*

Proof. Suppose that $C_R(A)$ is not the center of R and let x be an element of $C_R(A)$ such that $xr \neq rx$ for some $r \in R$. By hypothesis there exists an $a \in A$ with $(xr-rx)a \neq 0$. But in view of $ra \in A$ we have $(xr-rx)a = x(ra)-(ra)x = 0$, contradiction. This completes the proof.

Following R. Yue Chi Ming [8], a ring R is called a *right CM-ring* if for any maximal essential right ideal A of R and any complement right subideal B of A, we have $AB \subseteq B$.

Let R be a right p.p. ring with classical right quotient ring Q. Then Q is a right p.p. ring [3, Proposition 1]. If moreover Q is right p-injective, then Q is von Neumann regular [9, Theorem 14]. We now consider the case where Q is a right CM-ring.

Theorem 1. *Let R be a right p.p. ring with classical right quotient ring Q. If Q is a right CM-ring, then Q is von Neumann regular.*

Proof. First note that Q is a semiprime ring [8, Lemma 1.6 (3)].

Case 1. $\mathrm{Soc}(Q)$ is a direct summand of ${}_QQ_Q$. We write $Q = \mathrm{Soc}(Q) \oplus I$, where I is an ideal of Q. In this case $\mathrm{Soc}(Q)$ is a semisimple Artinian ring and I is a semiprime ring. By [8, Lemma 1.6 (4), (5)] we see that I is a right non-singular, right CM-ring and that either the socle of I is nonzero or I is reduced. Since $\mathrm{Soc}(I) = \mathrm{Soc}(I_Q) = 0$, I must be reduced. Now I is a normal p.p. ring in which every non-zero-divisor is invertible, and so I is a strongly regular ring [3, Theorem 1]. Therefore Q is a von Neumann regular ring.

Case 2. $\mathrm{Soc}(Q)$ is not a direct summand of ${}_QQ_Q$. Let I be an ideal of Q such that $K = \mathrm{Soc}(Q) + I = \mathrm{Soc}(Q) \oplus I$ is essential in ${}_QQ_Q$. By Lemma 5 we see that K is essential in Q_Q. If K contains a nonzero nilpotent element, then K is a direct summand of Q_Q [8, Lemma 1.6 (1)], and hence $K = Q$, contradiction. Consequently K is reduced. We now wish to show that Q is normal. For this, let e be an idempotent of Q and let a be an arbitrary element of K. Then $ea-eae \in K$ and $(ea-eae)^2 = 0$, whence $ea = eae$. Similarly we have $ae = eae$. Thus e is commutative with every element of K and hence is contained in the center of Q (Lemma 6). Therefore Q is normal. In view of the fact that Q is a normal p.p. ring in which every non-zero-

divisor is invertible, we see that Q is a strongly regular ring [3, Theorem 1]. This completes the proof.

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DEPARTMENT OF MATHEMATICS
SHINSHU UNIVERSITY

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