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# ON SELF MAPS OF HP ${ }^{n}$ FOR $\mathrm{n}=4$ AND 5 

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We determine the cardinality of the set of the homotopy classes of self maps of $\mathrm{HP}^{4}$ with degree 0 . And we shall determine the nilpotency of $\mathrm{HP}^{5}$.


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# ON SELF MAPS OF $\mathbb{H P}^{n}$ FOR $n=4$ AND 5. 

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#### Abstract

We determine the cardinality of the set of the homotopy classes of self maps of $\mathbb{H} \mathrm{P}^{4}$ with degree 0 . And we shall determine the nilpotency of $\mathbb{H} \mathrm{P}^{5}$.


## 1. MAIN RESULTS

Let $\mathbb{H}^{n}$ be the quaternionic projective space of dimension $n$. We shall denote by $j_{n}: S^{4} \rightarrow \mathbb{H} \mathrm{P}^{n}$ the inclusion. Especially, we put $j=j_{\infty}$.

We shall prove the following theorem:
Theorem 1. The cardinality of $\mathscr{H}_{0}\left(\mathbb{H} \mathrm{P}^{4}\right)$ is 2.
Here, for each integer $k, \mathscr{H}_{k}\left(\mathbb{H P}^{n}\right)$ is the totalities of the homotopy classes $[f]$ of maps $f: \mathbb{H} \mathrm{P}^{n} \rightarrow \mathbb{H} \mathrm{P}^{n}$ such that $f \circ j_{n}: S^{4} \rightarrow \mathbb{H} \mathrm{P}^{n}$ has degree $k$.

For $n=2,3$, the cardinalities $K(n, k)$ of $\mathscr{H}_{k}\left(\mathbb{H}^{n}\right)$, if it is not 0 , are determined in [2] so as to $K(2,2 k)=1, K(2,2 k+1)=2, K(3,2 k)=2$ and $K(3,2 k+1)=4$.

Next, for all based spaces $X$, we shall denote by $Z_{\infty}(X)$ (c.f., [1]) the totalities of the homotopy classes $\alpha \in[X, X]$ with the property that $\pi_{n}(\alpha)=$ $0: \pi_{n}(X) \rightarrow \pi_{n}(X)$ for all integer $n \geqslant 0$. Clearly, $Z_{\infty}\left(\mathbb{H} \mathrm{P}^{n}\right) \subseteq \mathscr{H}_{0}\left(\mathbb{H} \mathrm{P}^{n}\right)$ holds.

In [1], the nilpotency $t_{\infty}(X)$ of a based space $X$ is defined to be the least natural number $k$ such that $x_{1} \circ \cdots \circ x_{k}=0$ holds for all $x_{1}, \cdots, x_{k} \in Z_{\infty}(X)$, if such $k$ exists. If not, we put $t_{\infty}(X)=\infty$. It was proved in [1] that $t_{\infty}\left(\mathbb{H P}^{i}\right)=1$ for $i=1,2,3$ and $t_{\infty}\left(\mathbb{H} \mathrm{P}^{4}\right)=2$. We shall prove:

Theorem 2. $t_{\infty}\left(\mathbb{H} \mathrm{P}^{5}\right)=2$

## 2. Proof of the theorems

We shall consider in the category $\mathscr{T}_{0}$ of based topological spaces and based maps. We denote by $[f]$ the homotopy class of each map $f$ in $\mathscr{T}_{0}$. We denote by 0 the homotopy class of any trivial maps. We put $\Sigma^{n} X=S^{n} \wedge X$.

By $h_{n}: S^{4 n+3} \rightarrow \mathbb{H} \mathrm{P}^{n}$, denotes the Hopf fiber map, and by $q_{n}: \mathbb{H} \mathrm{P}^{n} \rightarrow$ $S^{4 n}$ the canonical quotient map. We shall put $r_{n}=\Sigma^{n-4}\left(q_{1} \circ h_{1}\right): S^{n+3} \rightarrow$ $S^{n}$. We shall put, for each $m, n, k$ with $0<k \leqslant m \leqslant n \leqslant \infty, \mathbb{H P}_{k}^{n}:=$ $\mathbb{H} \mathrm{P}^{n} / \mathbb{H} \mathrm{P}^{k-1}, q_{k}^{n}: \mathbb{H} \mathrm{P}^{n} \rightarrow \mathbb{H} \mathrm{P}_{k}^{n}$ to be the quotient map and $i_{k}^{m, n}: \mathbb{H} \mathrm{P}_{k}^{m} \rightarrow$
$\mathbb{H} \mathrm{P}_{k}^{n}$ induced from the inclusion $i^{m, n}: \mathbb{H} \mathrm{P}^{m} \rightarrow \mathbb{H} \mathrm{P}^{n}$. Especially, we shall put $i_{n}:=i^{n, n+1}$.

For all spaces $X, Y \in \mathscr{T}_{0}$, the symbol $\mathbf{F}(X, Y)$ denotes the totalities of the maps $X \rightarrow Y$ in $\mathscr{T}_{0}$, endowed with the compact open topology and with the trivial map as its base point. We shall denote by $\mathbf{F}_{0}\left(\mathbb{H} \mathrm{P}^{n}, \mathrm{~B} S^{3}\right)$ the totality of the maps $\mathbb{H}^{n} \rightarrow \mathrm{~B} S^{3}$ with degree 0 endowed with the relative topology induced from $\mathbf{F}\left(\mathbb{H P}^{n}, \mathrm{~B} S^{3}\right)$.

Proof of the theorem 1. The theorem 1 consists of following two lemmas:
Lemma 1. The cardinality of $Z_{\infty}\left(\mathbb{H} \mathrm{P}^{4}\right)$ is 2 and the only non-trivial homotopy class of the set is $\left[j_{4}\right] \circ\left(\Sigma \nu^{\prime}\right) \circ \mu_{7} \circ\left[q_{4}\right]$.
Lemma 2. $Z_{\infty}\left(\mathbb{H P}^{4}\right)=\mathscr{H}_{0}\left(\mathbb{H} \mathrm{P}^{4}\right)$ holds.
Proof of Lemma 1. Consider the following exact sequence:

$$
\pi_{16}\left(\mathbb{H P}^{4}\right) \xrightarrow[q_{4}^{*}]{ }\left[\mathbb{H P}^{4}, \mathbb{H} \mathrm{P}^{4}\right] \underset{i_{3}^{*}}{ }\left[\mathbb{H P}^{3}, \mathbb{H P}^{4}\right] \xrightarrow[h_{3}^{*}]{\longrightarrow} \pi_{15}\left(\mathbb{H P}^{4}\right)
$$

Let $i_{*}:\left[\mathbb{H}^{3}, \mathbb{H} \mathrm{P}^{3}\right] \rightarrow\left[\mathbb{H}^{3}, \mathbb{H} \mathrm{P}^{4}\right]$ be induced from $i_{3}$. From Lemma 5.9 and Theorem 5.10 of [1], the image of $Z_{\infty}\left(\mathbb{H P}^{4}\right)$ by the map $i_{*}^{-1} \circ i_{3}^{*}$ : $\left[\mathbb{H P}^{4}, \mathbb{H P}^{4}\right] \rightarrow\left[\mathbb{H P}^{3}, \mathbb{H P}^{3}\right]$ is contained in $Z_{\infty}\left(\mathbb{H} \mathrm{P}^{3}\right)=\{0\}$. Hence, $Z_{\infty}\left(\mathbb{H P}^{4}\right)$ is contained in the image of the map $q_{4}^{*}: \pi_{16}\left(\mathbb{H P}^{4}\right) \rightarrow\left[\mathbb{H P}^{4}, \mathbb{H} \mathrm{P}^{4}\right]$.

Conversely, let $\alpha \in \pi_{16}\left(\mathbb{H} \mathrm{P}^{4}\right)$. Then, since $\pi_{16}\left(\mathbb{H P}^{4}\right)=\mathbb{Z}_{2}\left\{\left[j_{4}\right] \circ\left(\Sigma \nu^{\prime}\right) \circ\right.$ $\left.\mu_{7}\right\} \oplus \mathbb{Z}_{2}\left\{\left[j_{4}\right] \circ\left(\Sigma \nu^{\prime}\right) \circ \eta_{7} \circ \varepsilon_{8}\right\}$ holds by taking adjoint of $\pi_{15}\left(S^{3}\right)([6])$, $\alpha \circ\left[q_{4}\right] \circ\left[h_{4}\right]=\alpha \circ\left(4\left[r_{16}\right]\right)=0$ holds in the group $\pi_{19}\left(\mathbb{H P}^{4}\right)$, and clearly $\alpha \circ\left[\left.q_{4}\right|_{S^{4}}\right]=0$. Hence, from the Proposition 5.4 of $[1], \alpha \circ\left[q_{4}\right] \in Z_{\infty}\left(\mathbb{H P}^{4}\right)$. Therefore, $Z_{\infty}\left(\mathbb{H P}^{4}\right)=q_{4}^{*}\left(\pi_{16}\left(\mathbb{H} \mathrm{P}^{4}\right)\right)$.

Next, it is well known by [4] that $\left[j_{4}\right] \circ\left(\Sigma \nu^{\prime}\right) \circ \mu_{7} \circ\left[q_{4}\right] \neq 0$, in the set $\left[\mathbb{H P}^{4}, \mathbb{H}^{4}\right]$. Hence $([1]) \operatorname{card}\left(Z_{\infty}\left(\mathbb{H} \mathrm{P}^{4}\right)\right) \geqslant 2$.

We shall express the co-action (c.f., chapter III of [7]) by the element $\alpha$ of $\pi_{16}\left(\mathbb{H} \mathrm{P}^{4}\right)$ on the element $x$ of $\left[\mathbb{H P}^{4}, \mathbb{H} \mathrm{P}^{4}\right]$ by the symbol $x \dot{+} \alpha$. Then, from the Puppe theorem, to prove Lemma 1, we have only to prove the following lemma:

Lemma 3. $\left[j_{4}\right] \circ\left(\Sigma \nu^{\prime}\right) \circ \eta_{7} \circ \varepsilon_{8} \circ\left[q_{4}\right]=0$ holds in $\left[\mathbb{H P}^{4}, \mathbb{H P}^{4}\right]$.
Proof. Let $h_{3}^{\prime}=q_{3} \circ h_{3}: S^{15} \rightarrow S^{12}$, and $q^{\prime}: \mathbb{H P}_{3}^{4} \rightarrow S^{16}$ the quotient map. Then, we obtain the following exact sequence:

$$
\pi_{13}\left(\mathrm{~B} S^{3}\right) \xrightarrow[\left(\Sigma h_{3}^{\prime}\right)^{*}]{ } \pi_{16}\left(\mathrm{~B} S^{3}\right) \longrightarrow q^{\prime *}\left[\mathbb{H} \mathrm{P}_{3}^{4}, \mathrm{~B} S^{3}\right]
$$

It is well known that $\left[h_{3}^{\prime}\right] \equiv \pm 3\left[r_{12}\right] \equiv \pm 3 \nu_{12}\left(\bmod . \Sigma^{9} \nu^{\prime}\left(=2 \nu_{12}\right)\right)$ and from [6], $\pi_{12}\left(S^{3}\right)=\mathbb{Z}_{2}\left\{\mu_{3}\right\} \oplus \mathbb{Z}_{2}\left\{\eta_{3} \circ \varepsilon_{4}\right\}$ holds. Now, since from [5], $\mu_{3} \circ \nu_{12}=$ $\nu^{\prime} \circ \eta_{6} \circ \varepsilon_{7}$ and $\eta_{3} \circ \varepsilon_{4} \circ \nu_{12}=\varepsilon_{3} \circ \eta_{11} \circ \nu_{12}=\varepsilon_{3} \circ\left(\Sigma^{8} \nu^{\prime}\right) \circ \eta_{14}=\varepsilon_{3} \circ\left(2 \nu_{11}\right) \circ \eta_{14}=0$
holds. Therefore, $\left(\Sigma h_{3}^{\prime}\right)^{*}\left(\pi_{13}\left(\mathrm{~B} S^{3}\right)\right)=\left\{0,[j] \circ\left(\Sigma \nu^{\prime}\right) \circ \eta_{7} \circ \varepsilon_{8}\right\}$ holds, hence $[j] \circ\left(\Sigma \nu^{\prime}\right) \circ \eta_{7} \circ \varepsilon_{8} \circ\left[q_{4}\right]=[j] \circ\left(\Sigma \nu^{\prime}\right) \circ \eta_{7} \circ \varepsilon_{8} \circ\left[q^{\prime}\right] \circ\left[q_{3}^{4}\right]=0$ holds in the set $\left[\mathbb{H P}^{4}, \mathrm{~B} S^{3}\right]$.

Proof of Lemma 2. We shall use the notations in [2]: Let $d_{4,0}: \pi_{12}\left(\mathrm{~B} S^{3}\right) \rightarrow$ $\pi_{15}\left(\mathrm{~B} S^{3}\right)$ be the composition of $j_{3,0}: \pi_{12}\left(\mathrm{~B} S^{3}\right) \rightarrow \pi_{0}\left(\mathbf{F}_{0}\left(\mathbb{H}^{3}, \mathrm{~B} S^{3}\right)\right)$ and the map $\partial_{4,0}: \pi_{0}\left(\mathbf{F}_{0}\left(\mathbb{H} \mathrm{P}^{3}, \mathrm{~B} S^{3}\right)\right) \rightarrow \pi_{15}\left(\mathrm{~B} S^{3}\right)$ induced from $h_{3}$. As in the proof of Proposition 1.3 of $[3], d_{4,0}(l)= \pm 3 l \circ\left[r_{12}\right]$ holds for $l \in \pi_{12}\left(\mathrm{~B} S^{3}\right)$. Since $\pi_{12}\left(\mathrm{~B} S^{3}\right)=\mathbb{Z}_{2}\left\{j_{*} \varepsilon_{4}\right\}([6]), d_{4,0}\left(j_{*} \varepsilon_{4}\right)=[j] \circ \varepsilon_{4} \circ\left[r_{12}\right]=[j] \circ \varepsilon_{4} \circ \nu_{12} \neq 0$ in the set $\pi_{15}\left(\mathrm{~B} S^{3}\right)$.

Finally, from Theorem 2 of [2], the cardinality of the set $\mathscr{H}_{0}\left(\mathbb{H P}^{3}\right)(\cong$ $\left.\pi_{0}\left(\mathbf{F}_{0}\left(\mathbb{H P}^{3}, \mathrm{~B} S^{3}\right)\right)\right)$ is 2. Therefore, $\partial_{4,0}$ is injection, hence $i_{3}^{*}\left(\mathscr{H}_{0}\left(\mathbb{H P}^{4}\right)\right) \subseteq$ $\operatorname{Ker}\left(\partial_{4,0}\right)=0$. Hence $\mathscr{H}_{0}\left(\mathbb{H} \mathrm{P}^{4}\right) \subseteq q_{4}^{*}\left(\pi_{16}\left(\mathbb{H}^{4}\right)\right)=Z_{\infty}\left(\mathbb{H P}^{4}\right)$. This completes the proof of Lemma 2 so that Theorem 1 holds.

Proof of theorem 2. The non trivial class $\xi:=\left[j_{4}\right] \circ\left(\Sigma \nu^{\prime}\right) \circ \mu_{7} \circ\left[q_{4}\right] \in Z_{\infty}\left(\mathbb{H} \mathrm{P}^{4}\right)$ is represented by the restriction of a map $f: \mathbb{H} \mathrm{P}^{5} \rightarrow \mathbb{H} \mathrm{P}^{5}$ which represents a non trivial class $\alpha \in\left[\mathbb{H P}^{5}, \mathbb{H P}^{5}\right]$, because of $\xi \circ\left[h_{4}\right]=0$ and of the fact that $\mathbb{H} \mathrm{P}^{5}$ is the mapping cone of the map $h_{4}: S^{19} \rightarrow \mathbb{H} \mathrm{P}^{4}$. Let $x \dot{+} \gamma$ be the co-action on $x \in\left[\mathbb{H P}^{5}, \mathbb{H}^{5}\right]$ by $\gamma \in \pi_{20}\left(\mathbb{H} \mathrm{P}^{5}\right)$.

Take two elements $x, y \in Z_{\infty}\left(\mathbb{H} \mathrm{P}^{5}\right)$. Then $x$ has the form $x=0 \dot{+} \gamma=\gamma \circ q_{5}$ or the form $x=\alpha \dot{+} \gamma$ for some $\gamma \in \pi_{20}\left(\mathbb{H}^{5}\right)$. For the former case, it is trivial that $y \circ x=0$. For the latter case, $y \circ x=y \circ \alpha \dot{+} y \circ \gamma=y \circ \alpha$. Therefore, it is enough that we can take $\alpha$ so as to be factored through $S^{15}$ (or $S^{4}$ ).

By $[5],\left[j_{5}\right] \circ\left(\Sigma \nu^{\prime}\right) \circ \mu_{7}=\left[j_{5}\right] \circ\left(\Sigma \mu^{\prime}\right) \circ \eta_{15}$, and $\eta_{15} \circ\left[q_{4}\right] \circ\left[h_{4}\right]=\eta_{15} \circ\left(4 \nu_{16}\right)=0$. Therefore, there exists $\alpha^{\prime}: \mathbb{H P}_{4}^{5} \rightarrow S^{15}$ such that $\alpha^{\prime} \circ\left[i_{4}^{4,5}\right]=\eta_{15}$. Hence $\left[j_{5}\right] \circ\left(\Sigma \mu^{\prime}\right) \circ \eta_{15} \circ\left[q_{4}\right]=\left[j_{5}\right] \circ\left(\Sigma \mu^{\prime}\right) \circ \alpha^{\prime} \circ\left[i_{4}^{4,5}\right] \circ\left[q_{4}\right]=\left[j_{5}\right] \circ\left(\Sigma \mu^{\prime}\right) \circ \alpha^{\prime} \circ\left[q_{4}^{5}\right] \circ\left[i_{4}\right]$. We can put $\alpha=\left[j_{5}\right] \circ\left(\Sigma \mu^{\prime}\right) \circ \alpha^{\prime} \circ\left[q_{4}^{5}\right]$.

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