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Some Torsion-free Subgroups in Group Rings

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SOME TORSION-FREE SUBGROUPS IN GROUP RINGS

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Introduction. Let RG be the group ring of a group G over a commutative ring R with identity. We denote by $\Delta_R(G)$ the augmentation ideal of RG and by $\Delta_R(G,N)$ the kernel of the natural map $RG \to R(G/N)$ if N is a normal subgroup of G. Note that $\Delta_R(G,N) = RG\Delta_R(N)$. Also, for any ideal I of RG, we write $U(1+I) = \{u \in U(RG) | u-1 \in I\}$, where $U(RG)$ is the unit group of RG. Clearly, $U(1+I)$ is a normal subgroup of $U(RG)$.

The purpose of this note is to prove the following two theorems.

Theorem A. *Let R be a commutative ring with identity. Let N be a nilpotent p-group of bounded exponent for some prime p, let A be a central subgroup of N and suppose that p is not a zero divisor in R. Then, for an additive subgroup I of RN,*

$$I \subseteqq \Delta_R(N)\Delta_R(N,A),\ pI \subseteqq I^p \Rightarrow I \subseteqq \textstyle\bigcap_{n=1}^{\infty} p^n \Delta_R(N,A).$$

Theorem B. *Let R be an integral domain of characteristic 0 in which no rational prime is invertible. Let N be a nilpotent normal subgroup of a group G and let A be an abelian normal subgroup of G with $N \supseteqq A$. Assume that one of the following two conditions holds:*

(a) *N is periodic,*

(b) *N is finitely generated.*

Then, $U(1+\Delta_R(G,N)\Delta_R(G,A))$ is torsion-free.

Theorem A was proved by Kmet and Sehgal [7, Theorem 2] for the case $R = Z$, the ring of rational integers. Also they [7, Theorem 1] proved Theorem B for the case where $R = Z$ and N is periodic. The essential result in our arguments is Lemma 1.3 which states that if $N \supseteqq A \supseteqq B$ are normal subgroups of G with A abelian, then there holds

$$\Delta_R(G,N)\Delta_R(G,A) \textstyle\bigcap \Delta_R(G,B) = \Delta_R(G,N)\Delta_R(G,B)$$

for any integral domain R of characteristic 0. This is well-known for $R=Z$. Owing to this lemma, our proofs can be done, as in [7].

Let N be a normal subgroup of G and let $D_{n,R}(N)=N\bigcap(1+\Delta_R(N)^n)$ be the n-th dimension subgroup of N over R. As an application of Theorem B, we show (Proposition 3.2) that if $R=Z$ and if N is periodic or finitely generated, then for each $n\geqq 1$ the factor group

$$U(1+\Delta_R(G,N)^n)/U(1+\Delta_R(G,D_{n,R}(N)))$$

is torsion-free. This is known [2] for the case where $G=N$ (G is arbitrary) and R is an integral domain of characteristic 0.

In this note, unless otherwise stated, R denotes a commutative ring with identity and G denotes an arbitrary group.

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1. Augmentation ideals. In this section, we state some preliminary lemmas concerning augmentation ideals. Let H be a subgroup of G and let T be a right transversal for H in G with $T\ni 1$. Then each element g of G can be written uniquely in the form $g=th, t\in T, h\in H$. Setting $\theta(g)=h$ and extending R-linearly, we have an R-linear map $\theta: RG\to RH$. It is easy to check that θ is a right RH-homomorphism and an identity map on RH; that is, $\theta(\beta\alpha)=\theta(\beta)\alpha$ and $\theta(\alpha)=\alpha$ for $\alpha\in RH$, $\beta\in RG$. From the definition of θ, clearly $\theta(\Delta_R(G))=\Delta_R(H)$. We write $\theta=\theta(G,H,T)$. Recall that the (natural) projection map $\pi: RG\to RH$, which is defined by $\pi(\sum_{g\in G}\alpha(g)g)=\sum_{g\in H}\alpha(g)g$, is also a right RH-homomorphism and an identity map on RH. Note that for any ideal I of RG, $I\subseteqq RG\pi(I)$ (see [9, p.6]).

Lemma 1.1. *Suppose that $N\supseteqq A\supseteqq B$ are normal subgroups of G. Then the following conditions are equivalent.*

(a) $\Delta_R(A)^2\bigcap\Delta_R(A,B)=\Delta_R(A)\Delta_R(A,B)$.

(b) $\Delta_R(G,N)\Delta_R(G,A)\bigcap\Delta_R(G,B)=\Delta_R(G,N)\Delta_R(G,B)$.

Proof. (a) $\Rightarrow$ (b). We need only to prove that the left-hand side is contained in the right-hand side, the reverse inclusion being obvious. Let $\pi_N: RG\to RN$ be the projection map and write I for the left-hand side of (b). Then we have

$$\begin{aligned}\pi_N(I) &\subseteqq \pi_N(\Delta_R(G,N)\Delta_R(G,A))\bigcap \pi_N(\Delta_R(G,B))\\ &= \Delta_R(N)\Delta_R(N,A)\bigcap \Delta_R(N,B).\end{aligned}$$

Since $I \subseteqq RG\pi_N(I)$, it suffices to show that

$$\Delta_R(N)\Delta_R(N,A)\bigcap\Delta_R(N,B) \subseteqq \Delta_R(N)\Delta_R(N,B).$$

To this end, let $\alpha \in \Delta_R(N)\Delta_R(N,A)\bigcap\Delta_R(N,B)$ and let T be a transversal for A in N with $T \ni 1$. Then α can be written as a finite sum of the form $\alpha = \sum_{t\in T} t\alpha_t$ with $\alpha_t \in RA$, and taking the projection map $\pi : RN \to RA$ we see that

$$\alpha_t = \pi(t^{-1}\alpha) \in \pi(\Delta_R(N,B)) = \Delta_R(A,B)$$

for all $t \in T$. Furthermore, under the right RA-homomorphism $\theta = \theta(N,A,T) : RN \to RA$, it follows that

$$\begin{aligned}\sum_{t\in T}\alpha_t = \theta(\alpha) &\in \theta(\Delta_R(N)\Delta_R(N,A))\bigcap\theta(\Delta_R(N,B))\\ &= \Delta_R(A)^2\bigcap\Delta_R(A,B) = \Delta_R(A)\Delta_R(A,B).\end{aligned}$$

So we have $\alpha = \sum_{t\in T}(t-1)\alpha_t + \sum_{t\in T}\alpha_t \in \Delta_R(N)\Delta_R(N,B)$ which completes the proof.

(b) $\Rightarrow$ (a). We have only to verify that the left-hand side is contained in the right-hand side. Let $\theta = \theta(G,N,T_1) : RG \to RN$ be the right RN-homomorphism obtained by a transversal $T_1(\ni 1)$ for N in G. Also, let $\varphi = \theta(N,A,T_2) : RN \to RA$ be the right RA-homomorphism obtained by a transversal $T_2(\ni 1)$ for A in N. Then for any $\alpha \in \Delta_R(A)^2\bigcap\Delta_R(A,B)$,

$$\alpha = \theta(\alpha) \in \theta(\Delta_R(G,N)\Delta_R(G,B)) = \Delta_R(N)\Delta_R(N,B).$$

Thus we have

$$\alpha = \varphi(\alpha) \in \varphi(\Delta_R(N)\Delta_R(N,B)) = \Delta_R(A)\Delta_R(A,B),$$

and hence the result follows.

The next lemma is well-known as "modular law", which will be frequently in the subsequent argument.

Lemma 1.2. *Let X, Y, Z be three subgroups of an additive group. Then*

$$X \supseteqq Y \Rightarrow X \cap (Y+Z) = Y + (X \cap Z).$$

Lemma 1.3. *Let R be an integral domain of characteristic* 0 *and suppose that $N \supseteqq A \supseteqq B$ are normal subgroups of G with A abelian. Then*

$$\Delta_R(G,N)\Delta_R(G,A) \cap \Delta_R(G,B) = \Delta_R(G,N)\Delta_R(G,B).$$

Proof. By Lemma 1.1, it suffices to verify that

$$\Delta_R(A)^2 \cap \Delta_R(A,B) \subseteqq \Delta_R(A)\Delta_R(A,B).$$

We prove this inclusion by considering some special cases.

First assume that A/B is cyclic. Let $\overline{}: RA \to R(A/B)$ be the natural map. Then $\overline{A} = \langle \overline{a} \rangle$ for some $a \in A$, so $\Delta_R(\overline{A}) = R\overline{A}(\overline{a}-1)$. Thus we have $\Delta_R(A) = \Delta_R(A,B) + RA(a-1)$ and hence

$$\Delta_R(A)^2 = \Delta_R(A)\Delta_R(A,B) + \Delta_R(A)(a-1).$$

Because $\Delta_R(A,B) \supseteqq \Delta_R(A)\Delta_R(A,B)$, we see from Lemma 1.2 that

$$\begin{aligned}\Delta_R(A)^2 \cap \Delta_R(A,B) &= \Delta_R(A,B) \cap (\Delta_R(A)\Delta_R(A,B) + \Delta_R(A)(a-1)) \\ &= \Delta_R(A)\Delta_R(A,B) + (\Delta_R(A,B) \cap \Delta_R(A)(a-1)).\end{aligned}$$

Thus it remains to show that

$$\Delta_R(A,B) \cap \Delta_R(A)(a-1) \subseteqq \Delta_R(A)\Delta_R(A,B).$$

To do this, let $\alpha \in \Delta_R(A,B) \cap \Delta_R(A)(a-1)$. Then $\alpha = \beta(a-1)$ for some $\beta \in \Delta_R(A)$ and $0 = \overline{\beta}(\overline{a}-1)$. Therefore, $\overline{\beta}$ is in the annihilator $l(\Delta_R(\overline{A}))$ of $\Delta_R(\overline{A})$ in $R\overline{A}$. In case $\overline{A}$ is infinite, $l(\Delta_R(\overline{A})) = 0$ so that $\overline{\beta} = 0$ (see [10, p.95]). Thus $\beta \in \Delta_R(A,B)$ and so we have $\alpha \in \Delta_R(A)\Delta_R(A,B)$. In case $\overline{A}$ is finite,

$$l(\Delta_R(\overline{A})) = R\overline{A}(1 + \overline{a} + \cdots + \overline{a}^{n-1}),$$

where $|\overline{A}| = n$, and so we can write β as

$$\beta = \gamma(1 + a + \cdots + a^{n-1}) + \delta \ (\gamma \in RA, \delta \in \Delta_R(A,B)).$$

Then, under the augmentation map $\varepsilon : RA \to R$ we have $0 = \varepsilon(\gamma)n$, and by the hypothesis on R, $\varepsilon(\gamma) = 0$ i.e. $\gamma \in \Delta_R(A)$. Since $a^n \in B$, we obtain

$$\alpha = \gamma(a^n - 1) + \delta(a - 1) \in \Delta_R(A)\Delta_R(A, B)$$

and the result follows here.

Next assume that A is finitely generated. Then since A/B is finitely generated, A has a series

$$A = A_1 \supseteqq A_2 \supseteqq \cdots \supseteqq A_{n+1} = B$$

of subgroups of A with each A_i/A_{i+1} cyclic. By the foregoing,

$$\Delta_R(A_i)^2 \bigcap \Delta_R(A_i, A_{i+1}) = \Delta_R(A_i)\Delta_R(A_i, A_{i+1})$$

and hence, by Lemma 1.1, we deduce that

$$\Delta_R(A)\Delta_R(A, A_i) \bigcap \Delta_R(A, A_{i+1}) = \Delta_R(A)\Delta_R(A, A_{i+1}) \text{ for } 1 \leq i \leq n.$$

Now we use induction on i to show that

$$\Delta_R(A)^2 \bigcap \Delta_R(A, B) \subseteqq \Delta_R(A)\Delta_R(A, A_i) \text{ for } 1 \leq i \leq n + 1,$$

the case $i = 1$ being obvious. Suppose that this inclusion holds for some i with $1 \leq i < n + 1$. Then we have

$$\begin{aligned} \Delta_R(A)^2 \bigcap \Delta_R(A, B) &\subseteqq \Delta_R(A)\Delta_R(A, A_i) \bigcap \Delta_R(A, B) \\ &\subseteqq \Delta_R(A)\Delta_R(A, A_i) \bigcap \Delta_R(A, A_{i+1}) \\ &= \Delta_R(A)\Delta_R(A, A_{i+1}) \end{aligned}$$

and hence the induction is complete. Consequently the result follows because $A_{n+1} = B$.

Finally, let A be arbitrary, and let $\alpha \in \Delta_R(A)^2 \bigcap \Delta_R(A, B)$. Then clearly there exists a finitely generated subgroup A^* of A and a subgroup B^* of A^* such that $\alpha \in \Delta_R(A^*)^2 \bigcap \Delta_R(A^*, B^*)$. Thus by the previous case we conclude that

$$\alpha \in \Delta_R(A^*)\Delta_R(A^*, B^*) \subseteqq \Delta_R(A)\Delta_R(A, B).$$

This completes the proof of the lemma.

We note that the above argument also proves the following result.

Lemma 1.4. *Suppose that $N \supseteqq A \supseteqq B$ are normal subgroups of G. If A is an abelian p-group for some prime p and p is not a zero divisor in R, then*

$$\Delta_R(G,N)\Delta_R(G,A)\bigcap\Delta_R(G,B) = \Delta_R(G,N)\Delta_R(G,B).$$

The proof of the next lemma is straightforward.

Lemma 1.5. *Let $\mathscr{S}$ be a set of normal subgroups of G such that $N_1\bigcap N_2 \in \mathscr{S}$ whenever N_1, $N_2 \in \mathscr{S}$ Then*

$$\bigcap_{N\in\mathscr{S}} \Delta_R(G,N) = \Delta_R(G, \bigcap_{N\in\mathscr{S}} N).$$

Proof. It is clear that the right-hand side is contained in the left-hand side. To show the reverse inclusion, let $\alpha \in \bigcap_{N\in\mathscr{S}}\Delta_R(G,N)$ and set $M = \bigcap_{N\in\mathscr{S}}N$. Then, choosing a transversal T for M in G, α can be written uniquely in the form $\alpha = \sum_{i=1}^{n} t_i\alpha_i$, $\alpha_i \in RM$, $t_i \in T$. By the property of $\mathscr{S}$, we may pick some $N \in \mathscr{S}$ with $\{t_1^{-1}t_i | 1 \leq i \leq n\}\bigcap N = \{1\}$. Then, under the projection map $\pi : RG \rightarrow RN$, we have

$$\alpha_1 = \pi(t_1^{-1}\alpha) \in \pi(\Delta_R(G,N)) = \Delta_R(N),$$

so $\alpha_1 \in \Delta_R(N)\bigcap RM = \Delta_R(M)$. Similarly, we see that all α_i's are in $\Delta_R(M)$ so that $\alpha \in \Delta_R(G,M)$. This completes the proof.

2. Proofs of Theorem A and B. In order to prove our theorems we need furthermore a few lemmas. For a subgroup H of G, we denote by $H^G = \langle g^{-1}Hg \mid g \in G\rangle$ the normal closure of H in G.

Lrmma 2.1. *Let H be a subgroup of G and suppose that H^G is nilpotent. If H is of bounded exponent, then so is H^G.*

Proof. Clearly, $H^G/\gamma_2(H^G)$ is of bounded exponent, where $\gamma_2(H^G)$ is the derived subgroup of H^G. Since H^G is nilpotent, we see that H^G is of bounded exponent, too (see e.g. [5, p.266, 2.13 Satz]).

Lemma 2.2. *Let N be a normal subgroup of G. If N is a nilpotent p-group of bounded exponent for some prime p, then $\bigcap_{n=1}^{\infty}\Delta_R(G,N)^n \subseteqq p^l(RG)$ for all $l \geq 1$.*

Proof. Let us fix $l \geq 1$ and set $S = R/p^l R$. Then, since $\bigcap_{n=1}^{\infty} p^n S = 0$, it follows from [4, Theorem E] that $\bigcap_{n=1}^{\infty} \Delta_S(N)^n = 0$ (see also [8, p. 84, 2.11 Theorem]). Let $f : RG \to SG$ be the ring homomorphism induced by the natural homomorphism $R \to S$. Then

$$f(\textstyle\bigcap_{n=1}^{\infty} \Delta_R(G,N)^n) \subseteqq \bigcap_{n=1}^{\infty} \Delta_S(G,N)^n = SG(\bigcap_{n=1}^{\infty} \Delta_S(N)^n) = 0.$$

Since $Ker\ f = p^l(RG)$, the result follows.

We denote by TG the set of torsion elements in G. Also, for a prime p, $T_p(G)$ denotes the set of p-elements in G. We say here that R is *G-adapted* if R is an integral domain of characteristic 0 in which no element $g \neq 1$ of G has order invertible. The next result is an extension of [7, Lemma 3].

Lemma 2.3. *Let p be a prime and let N be a nilpotent normal subgroup of G such that $T_p(N) = \{1\}$. Assume that one of the following two conditions holds:*

(a) *R is an integral domain of characteristic* 0 *in which no rational prime is invertible.*

(b) *G is polycyclic-by-finite and R is G-adapted.*

Then $T_p(U(1 + \Delta_R(G,N))) = \{1\}$.

Proof. Assume first (b). Let $u = \sum_{g\in G} u(g)g \in T(1 + \Delta_R(G,N))$. We have only to show that $u = 1$ if $u^p = 1$. Consider the natural map $\overline{} : RG \to R(G/TN)$. Then, $\overline{u} \in TU(1 + \Delta_R(\overline{G},\overline{N}))$. However, since $\overline{N}$ is torsion-free nilpotent, we know from [3, Lemma 1.2] that $TU(1 + \Delta_R(\overline{G},\overline{N})) = \{1\}$. Thus, $\overline{u} = 1$ i.e. $u - 1 \in \Delta_R(G, TN)$. Therefore we may assume that N is periodic. Since $u - 1 \in \Delta_R(G,N)$, we can write $u - 1$ in the form

$$u - 1 = \sum_{i=1}^{n} \lambda_i g_i(x_i - 1)\ (\lambda_i \in R, g_i \in G, x_i \in N).$$

Then, setting $H = \langle x_1, \cdots, x_n \rangle$, we see that $u - 1 \in \Delta_R(G, H^G)$. Because H is finite, H^G is of bounded exponent by Lemma 2.1. Thus we may further assume that N is of bounded exponent. We proceed by induction on the exponent $exp(N)$ of N. The cace $exp(N) = 1$ is trivial, so let $exp(N) > 1$ and let q be a prime divisor of $exp(N)$. Then we get $N = T_q(N) \times N_1$ so that $exp(N/T_q(N)) < exp(N)$. Since R is also $G/T_q(N)$-adapted, by considering $R(G/T_q(N))$, we have $u - 1 \in \Delta_R(G, T_q(N))$ by induction.

Now, set $B = T_q(N)$ and $I = RG(u-1)RG$. Then $I \subseteqq \Delta_R(G, B)$, and we obtain $pI \subseteqq I^p$, since $p(u-1) \in (u-1)^p RG$ (see e.g. the proof of [1, Lemma 3.4]). On the other hand, the additive group $\Delta_R(B)/\Delta_R(B)^2$ is a q-group and hence, so is each $\Delta_R(G,B)^n/\Delta_R(G,B)^{n+1}$ (see [6, p. 23]). Therefore, $I \subseteqq \Delta_R(G,B)^n$ for all $n \geqq 1$, because $p \neq q$. Thus by Lemma 2.2, $I \subseteqq q(RG)$ so that $u(1) - 1 \in qR$. Since q is a nonunit in R, we have $u(1) \neq 0$ and so it follows from [10, p. 45, Corollary 1.4] that $u = 1$.

Under condition (a), any element u of $TU(1 + \Delta_R(G))$ with $u(1) \neq 0$ is necessarily the identity ([10, p.45, Corollary 1.2]). Thus by the same argument as above, the result follows.

Lemma 2.4. *Let A be a normal subgroup of a group N. Suppose that A is a nilpotent p-group of bounded exponent for some prime p and that p is not a zero divisor in R. Then, for an additive subgroup I of $\Delta_R(N, A)$,*

$$pI \subseteqq I^p,\ I \subseteqq p\Delta_R(N) \Rightarrow I \subseteqq p^l \Delta_R(N, A) \quad \textit{for all} \quad l \geqq 1.$$

Proof. We have $I \subseteqq \Delta_R(N,A) \bigcap p\Delta_R(N) = p\Delta_R(N,A)$ and $pI \subseteqq I^p \subseteqq p^p \Delta_R(N,A)^p$, so that $I \subseteqq p\Delta_R(N,A)^p$ since p is not a zero divisor in R. Hence by induction on n, we see that $I \subseteqq p\Delta_R(N,A)^n$ for all $n \geqq 1$. So, by Lemma 2.2, $I \subseteqq p^l(RN)$ for all $l \geqq 1$. Thus the result follows, because $I \subseteqq \Delta_R(N,A) \bigcap p^l(RN) = p^l \Delta_R(N,A)$.

We are now in a position to prove our theorems.

2.5. Proof of Theorem A. In view of Lemma 2.4, it suffices to prove that $I \subseteqq p\Delta_R(N, A)$. To do this, we first assume that A is finite and proceed by induction on the order of A. The case $|A| = 1$ is trivial, so let $|A| > 1$. Then there exists a subgroup W of A with $|W| = p$. Let $\overline{}$: $RN \to R(N/W)$ be the natural map. Then, clearly, $\overline{I} \subseteqq \Delta_R(\overline{N})\Delta_R(\overline{N}, \overline{A})$ and $p\overline{I} \subseteqq \overline{I}^p$. Hence by induction, we have $\overline{I} \subseteqq p\Delta_R(\overline{N}, \overline{A})$. So it follows from Lemma 2.4 that $\overline{I} \subseteqq p^l \Delta_R(\overline{N}, \overline{A})$ i.e. $I \subseteqq \Delta_R(N, W) + p^l \Delta_R(N, A)$ for all $l \geqq 1$. Now, let $|A| = p^l$. Then $p^l \Delta_R(N, A) \subseteqq \Delta_R(N)\Delta_R(N, A)$ (see e.g. [6, p.23]). Thus by Lemma 1.2 and Lemma 1.4,

$$\begin{aligned} I &\subseteqq \Delta_R(N)\Delta_R(N,A) \bigcap (\Delta_R(N,W) + p^l \Delta_R(N,A)) \\ &= (\Delta_R(N)\Delta_R(N,A) \bigcap \Delta_R(N,W)) + p^l \Delta_R(N,A) \\ &= \Delta_R(N)\Delta_R(N,W) + p^l \Delta_R(N,A). \end{aligned}$$

We use induction on n to show the following:

$$(*) \qquad I \subseqq \Delta_R(N)^n \Delta_R(N,W) + p\Delta_R(N) \text{ for all } n \geqq 1.$$

The case $n = 1$ is assured by the foregoing, so assume that $(*)$ holds for some $n \geq 1$. Note here that since $|W| = p$, $\Delta_R(N,W)^p = p\Delta_R(N,W)$ (see [1, Lemma 3.4]). Furthermore, W is central in N, and therefore it follows that

$$\begin{aligned} pI &\subseteqq I^p \subseteqq (\Delta_R(N)^n \Delta_R(N,W) + p\Delta_R(N))^p \\ &\subseteqq (\Delta_R(N)^n \Delta_R(N,W))^p + p\Delta_R(N)^{n+1}\Delta_R(N,W) + p^2\Delta_R(N) \\ &\subseteqq p\Delta_R(N)^{n+1}\Delta_R(N,W) + p^2\Delta_R(N). \end{aligned}$$

Thus, $I \subseteqq \Delta_R(N)^{n+1}\Delta_R(N,W) + p\Delta_R(N)$ and hence $(*)$ is established. By considering $(R/pR)N$, we deduce from Lemma 2.2 and $(*)$ that $I \subseteqq p(RN)$. Hence, $I \subseteqq \Delta_R(N,A) \bigcap p(RN) = p\Delta_R(N,A)$.

Next, let A be infinite, and let $\mathscr{S}$ be the set of all subgroups of finite index in A. Then the previous case shows that for any $B \in \mathscr{S}$, $\overline{I} \subseteqq p\Delta_R(\overline{N},\overline{A})$ under the natural map $\overline{} : RN \to R(N/B)$, and hence $I \subseteqq \Delta_R(N,B) + p\Delta_R(N,A)$. Thus, under the natural homomorphism $f : RN \to SN$ where $S = R/pR$, we have $f(I) \subseteqq \Delta_R(N,B)$. Since $\mathscr{S}$ has a property that $B_1 \bigcap B_2 \in \mathscr{S}$ for any B_1, $B_2 \in \mathscr{S}$, it follows from Lemma 1.5 that

$$f(I) \subseteqq \bigcap_{B \in \mathscr{S}} \Delta_S(N,B) = \Delta_S(N, \bigcap_{B \in \mathscr{S}} B).$$

However, since A is an abelian group of bounded exponent, it is a direct sum of finite cyclic groups, and so we readily see that $\bigcap_{B \in \mathscr{S}} B = \{1\}$. (That is, A is residually finite.) Hence $f(I) = 0$, so $I \subseteqq p(RN)$. Thus we conclude that $I \subseteqq p\Delta_R(N,A)$, which completes the proof.

2.6. Proof of Theorem B. We first claim that it suffices to consider the case where A is central in N. Assume the theorem to be true in this case. For the general case, we set $(A,_1 N) = (A,N)$ and, inductively, $(A,_n N) = ((A,_{n-1} N), N)$ for $n \geq 2$, where for any two subgroups X and Y of G, (X,Y) is the subgroup generated by all commutators $x^{-1}y^{-1}xy$, $x \in X$, $y \in Y$. Then, $(A,_n N) = \{1\}$ for some $n \geqq 1$, because N is nilpotent. We proceed by induction on n. The case $n = 1$ is assured by our hypothesis, so let $n \geqq 2$. Set $B = (A,_{n-1} N)$ and let $\overline{} : RG \to R(G/B)$ be the natural map. Then $(\overline{A},_{n-1} \overline{N}) = \{1\}$ and hence $U(1+\Delta_R(\overline{G},\overline{N})\Delta_R(\overline{G},\ \overline{A}))$ is torsion-free by induction. Therefore, if $u \in TU(1+\Delta_R(G,N)\Delta_R(G,A))$, then $\overline{u}=1$, that is, $u-1 \in \Delta_R(G,B)$. So it follows from Lemma 1.3 that

$$u-1 \in \Delta_R(G,N)\Delta_R(G,A)\bigcap\Delta_R(G,B) = \Delta_R(G,N)\Delta_R(G,B).$$

Since $(A,_n N)=\{1\}$, B is central in N and thus, by our hypothesis, $U(1+\Delta_R(G,N)\Delta_R(G,B))$ is torsion-free. Therefore $u=1$. This substantiates our claim.

Now, turning the proof of the theorem, we assume (a). Then, by the above argument, we may assume that A is central in N. Let $u=\sum_{g\in G} u(g)g \in U(1+\Delta_R(G,N)\Delta_R(G,A))$. It suffices to show that $u=1$ if $u^p=1$ for some prime p. Because N is a periodic nilpotent group, we have $N=N_1\times N_2$ where N_1 is a p-group and N_2 is a p'-group. Then the natural map $\overline{}: RG \to R(G/N_2)$ yields a group homomorphism

$$f: U(1+\Delta_R(G,N)\Delta_R(G,A)) \to U(1+\Delta_R(\overline{G},\overline{N})\Delta_R(\overline{G},\overline{A}))$$

and since $Ker\ f \subseteqq U(1+\Delta_R(G,N_2))$, we obtain $T_p(Ker\ f)=\{1\}$ by Lemma 2.3. Therefore, replacing N by $\overline{N}$, we may assume that N is a p-group. Observe that $u-1$ can be written in the form

$$u-1=\sum_{i=1}^{n}\lambda_i g_i(x_i-1)(a_i-1)\ (\lambda_i \in R, g_i \in G, x_i \in N, a_i \in A),$$

and as in Lemma 2.3, set $H=\langle x_1,\cdots,x_n,a_1,\cdots,\ a_n\rangle$. Then

$$u-1 \in \Delta_R(G,H^G)\Delta_R(G,H^G\bigcap A)$$

and H^G is of bounded exponent by Lemma 2.1. Thus we may assume here that N is a nilpotent p-group of bounded exponent. Set $I=RG(u-1)RG$ so that $pI \subseteqq I^p$, and let $\pi: RG \to RN$ be the projection map. Then since I is an ideal of RG, it is readily verified that $\pi(I^n)\subseteqq \pi(I)^n$ for all $n \geqq 1$. Hence we have

$$p\pi(I)=\pi(pI)\subseteqq \pi(I^p)\subseteqq \pi(I)^p.$$

Since $\pi(I)\subseteqq \Delta_R(N)\Delta_R(N,A)$, we deduce from Theorem A that $I \subseteqq RG\pi(I)\subseteqq p(RG)$. Thus $u-1\in p(RG)$ so that $u(1)\neq 0$. Therefore, $u=1$ by [10, p.45, Corollary 1.2].

Next assume (b). Then N is a finitely generated nilpotent group and hence is polycyclic. We argue by induction on the Hirsch number

$h(N)$ of N. If $h(N) = 0$, then N is finite and the result is assured by the case (a). Let $h(N) > 0$. Then N is infinite, and so there exists a characteristic infinite torsion-free abelian subgroup B of N (see [9, p.425]). Let $u \in TU(1 + \Delta_R(G,N)\Delta_R(G,A))$ and consider the natural map $\overline{}$: $RG \to R(G/B)$. Then since $h(\overline{N}) < h(N)$, we have $\overline{u} = 1$ by induction. So $u - 1 \in \Delta_R(G,B)$. Furthemore, $TU(1+\Delta_R(G,B)) = \{1\}$ by [3, Lemma 1.2], and hence $u = 1$. This completes the proof of the theorem.

3. Application. We shall omit the subscript Z from $\Delta_Z(G)$ and $\Delta_Z(G,N)$, which are denoted by $\Delta(G)$ and $\Delta(G,N)$, respectively. Also, we write $D_n(G)$ for the n-th dimension subgroup of G over Z, that is, $D_n(G) = G \bigcap (1 + \Delta(G)^n)$. (We do not know whether the next lemma holds in a more general coefficient ring.)

Lemma 3.1. *Let N be a normal subgroup of G and suppose $D_n(N) = \{1\}$ for some $n \geqq 2$. Then*

$$\Delta(G,N)^n \bigcap \Delta(G, D_{n-1}(N)) = \Delta(G,N)\Delta(G, D_{n-1}(N)).$$

Proof. We have only to verify that the left-hand side is contained in the right-hand side, the oppsite inclusion being trivial. Denote by I the left-hand side and let $\pi : RG \to RN$ be the projection map. Then $\pi(I) \subseteqq \Delta(N)^n \bigcap \Delta(N, D_{n-1}(N))$ and so it suffices to prove that

$$\Delta(N)^n \bigcap \Delta(N, D_{n-1}(N)) \subseteqq \Delta(N)\Delta(N, D_{n-1}(N)).$$

To this end, let $\alpha \in \Delta(N)^n \bigcap \Delta(N, D_{n-1}(N))$. Then, as is well-known, $\alpha \equiv x - 1 \pmod{\Delta(N)\Delta(N, D_{n-1}(N))}$ for some $x \in D_{n-1}(N)$ (see [10, p. 76]). However, since $\alpha \in \Delta(N)^n$, we have $x - 1 \in \Delta(N)^n$ so that $x = 1$. Therefore, $\alpha \in \Delta(N)\Delta(N, D_{n-1}(N))$, as required.

Proposition 3.2. *Let N be a periodic or finitely generated normal subgroup of G. Then for each $n \geqq 1$ the factor group*

$$U(1 + \Delta(G,N)^n)/U(1 + \Delta(G, D_n(N)))$$

is torsion-free.

Proof. By considering $G/D_n(N)$, it suffices to prove that if $D_n(N) = \{1\}$ then $TU(1+\Delta(G,N)^n) = \{1\}$. To do this, we proceed by induction on n, the case $n = 1$ being trivial. Let $n \geqq 2$ and let $\overline{} : ZG \to Z(G/D_{n-1}(N))$ be the natural map. Then, $D_{n-1}(\overline{N}) = \{1\}$ and so $TU(1+\Delta(\overline{G},\overline{N})^{n-1}) = \{1\}$ by induction. Therefore, if $u \in TU(1+\Delta(G,N)^n)$, then we obtain $\overline{u} = 1$ so that

$$u - 1 \in \Delta(G,N)^n \bigcap \Delta(G, D_{n-1}(N)).$$

Moreover, since $D_n(N) = \{1\}$, $u-1 \in \Delta(G,N)\Delta(G, D_{n-1}(N))$ by Lemma 3.1. Note here that N is nilpotent and that $D_{n-1}(N)$ is central in N. Thus by Theorem B, we have $u = 1$, which completes the proof.

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