Mathematical Journal of Okayama University

Volume 35, *Issue* 1 | 1993 | *Article* 11

JANUARY 1993

On Residually Finite Rings

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Math. J. Okayama Univ. 35(1993), 155-167

ON RESIDUALLY FINITE RINGS

Dedicated to Professor Kazuo Kishimoto on his 60th birthday

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Following K. L. Chew and S. Lawn [3], a ring R is said to be *residually finite* if every nonzero ideal of R is of finite index in R. Obviously all finite rings and all simple rings are residually finite. Other residually finite rings are said to be *proper*. Examples of commutative proper residually finite rings are the ring of algebraic integers of an algebraic number field, the polynomial ring $F[x]$ and the formal power series ring $F[[x]]$ over a finite field F. If R is a residually finite ring and if $n \geq 2$, then the ring $M_n(R)$ of all $n \times n$ matrices over R is a noncommutative residually finite ring. In [3], commutative residually finite rings were investigated very well. In this paper, we study residually finite rings which are not necessarily commutative. A ring R is called a *right residually finite ring* if every nonzero right ideal of R is of finite index in R. Clearly commutative residually finite rings are right residually finite. In §1, we show that a right residually finite ring is a right fully bounded right Noetherian ring, and give some characterizations of such a ring. In §2, we show that certain extensions of a residually finite ring are residually finite. As a result, we obtain many examples of noncommutative residually finite rings. In §3, we study the number of ideals in a residually finite ring. Let R be a residually finite ring, and I a nonzero ideal of R. We define $N(I)$, the norm of I, to be the number of elements in R/I. We show that given a positive integer n, the number of ideals I with $N(I) \leq n$ is finite. As a consequence, we know that the set of ideals of R is either finite or denumerable.

Throughout this paper, all rings have an identity element 1, and subrings of a ring R are assumed to have the same identity element as R.

1. Right residually finite rings. In this section, we consider a special class of residually finite rings which contains commutative residually finite rings. A ring R is said to be *right* (resp. *left*) *residually finite* if every nonzero right (resp. left) ideal of R is of finite index in R.

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Example 1. Let $A = \boldsymbol{Q} \oplus \boldsymbol{Q}i \oplus \boldsymbol{Q}j \oplus \boldsymbol{Q}k$ be the skewfield of quaternions over the field $\boldsymbol{Q}$ of rational numbers, and consider the subring $B = \boldsymbol{Z} \oplus \boldsymbol{Z}i \oplus \boldsymbol{Z}j \oplus \boldsymbol{Z}k$ of A. We claim that B is a right and left residually finite ring. Let $x = a + bi + cj + dk$ be a nonzero element of B with $a, b, c, d \in \boldsymbol{Z}$ and let $\bar{x} = a - bi - cj - dk$. Then it is easily checked that the norm $N(x) = x\bar{x} = \bar{x}x$ ($= a^2 + b^2 + c^2 + d^2$) of x is a positive integer. Let I be a nonzero right ideal of B and let x be a nonzero element of I. Then I contains the nonzero positive integer $N(x)$. Hence B/I is a finitely generated right module over the finite ring $\boldsymbol{Z}/N(x)\boldsymbol{Z}$, and so B/I is finite. This proves that B is a right residually finite ring. Similarly we can prove that B is also a left residually finite ring.

Example 2. Let P be a prime, n a positive integer greater than 1, and $K = GF(p^n)$. Then the map $\theta : K \to K$ defined by $\theta(a) = a^p$ for all a in K is a non-trivial automorphism of K. Now consider the skew polynomial ring $K[X; \theta]$ defined by the relation $Xa = \theta(a)X$ for all $a \in K$. Then we can easily see that $K[X; \theta]$ is a noncommutative right and left residually finite ring.

The proof of the following lemma is similar to that of [3, Lemma 2.1]. However, for the sake of completeness, we shall give the proof.

Lemma 1. *Let R be a ring and let A, B be right ideals of finite index in R. If $A \cap B$ is finitely generated, then AB is of finite index in R.*

Proof. Let $A \cap B = a_1R + \cdots + a_nR$. Then the map $f : (R/B)^n \to (A \cap B + AB)/AB$ defined by $f(r_1 + B, \cdots, r_n + B) = a_1r_1 + \cdots + a_nr_n + AB$ is an epimorphism. Hence $(A \cap B + AB)/AB$ is a finite set. On the other hand, since $R/A \cap B$ can be embedded in $R/A \oplus R/B$, $R/A \cap B$ is also a finite set. Consequently R/AB is a finite set.

A ring R is called *right fully bounded* if, for each prime ideal P of R, every essential right ideal of R/P contains a nonzero ideal of R/P (cf. [12, Definition in p. 165]).

Theorem 1., *If R is an infinite right residually finite ring, then R is a right fully bounded right Noetherian domain. Consequently an infinite ring R is a right residually finite ring if and only if R is a residually finite ring and every nonzero right ideal of R contains a nonzero ideal.*

Proof. Suppose that R is an infinite right residually finite ring. Clearly R is right Noetherian. By virtue of [6, Theorem 1], every nonzero right ideal of R contains a nonzero ideal. Now let P be a nonzero prime ideal of R. Then R/P is a finite simple ring, whence there is no essential right ideal of R/P except R/P itself. Hence R is right fully bounded. We shall prove that R is a domain. Assume, to the contrary, that there exist two nonzero elements a, b in R such that $ab = 0$. Then the right annihilator $r(a)$ of a in R is nonzero, and hence it contains a nonzero ideal, say I. Then we have $aRI = 0$. Since aR and I are of finite index in R, $aRI = 0$ is of finite index in R by Lemma 1. This is contradictory to the assumption that R is finite. Thus R is a domain. Now the latter assertion is clear.

Trivially, all finite rings and all division rings are right residually finite. The other right residually finite rings are said to be *proper*. Let R be a proper right residually finite ring. By Theorem 1, R is a right Ore domain. Let $Q(R)$ denote the skewfield of fractions of R. It is easy to see that $Q(R)$ is the injective hull of the right R-module R.

Corollary 1. *Let R be a proper right residually finite ring. Let Σ be a set of representatives of all the isomorphic classes of simple right R-modules and let $E(S)$ denote the injective hull of S for each $S \in \Sigma$. Then every injective right R-module is isomorphic to a direct sum of some copies of modules in $\{Q(R)\} \cup \{E(S) | \ S \in \Sigma\}$.*

Proof. Since R is right Noetherian, every injevtive right R-module is a direct sum of indecomposable injective right R-modules by [1, Theorem 25.6]. Now let M be an indecomposable injective right R-module. By Theorem 1, R is a right fully bounded right Noetherian ring. Hence, by [12, Theorem 7.2.1] and its proof, M is isomorphic to a direct summand of the injective hull $E(R/P)$ of R/P for some prime ideal P of R. If $P = 0$, M is isomorpohic to $Q(R)$. If P is nonzero, then R/P is a finite simple ring. In this case, M is isomorphic to $E(S)$ for some $S \in \Sigma$. This proves our assertion.

A polynomial f in the free algebra $Z\langle X_1, X_2, \cdots\rangle$ is said to be *monic* if at least one of the monimials of highest total degree in the support of f has coefficient 1. A *polynomial identity ring*, abbreviated to *P.I. ring*, is defined to be a ring which satisfies some monic polynomial in

$Z\langle X_1, X_2, \cdots\rangle$. It is well known that if an algebra A over a commutative ring R is finitely generated as an R-module, then A is a P. I. ring.

Corollary 2. *Let R be an infinite P.I. ring. Then the following statements are equivalent:*

1) *R is a right residually finite ring.*
2) *R is a left residually finite ring.*
3) *R is a residually finite domain.*

Proof. If R is a right or left residually finite ring, then R is a domain by Theorem 1. Hence it suffices to prove that 3) implies 1). So assume that R is a residually finite domain and let I be a nonzero right ideal of R. By Amitsur's result [9, Corollary 13.2.9], I contains a nonzero ideal of R. Since R is residually finite, this implies that I is of finite index in R.

Of course, there is a proper residually finite domain which is neither right nor left residually finite.

Example 3. Let $F(z)$ be the field of rational functions over a finite field F, and let $B(z, F(z))$ be the $F(z)$-algebra generated by x, x^{-1}, y and y^{-1} subject to the relation $xy = zyx$. By [7, Theorem 2.1], $B(z, F(z))$ is a simple domain, but it is not a division ring. Now let $a \neq 0$ be a non-invertible element of $B(z, F(z))$ and let $R = F + aB(z, F(z))$. Then $aB(z, F(z))$ is a unique non-trivial ideal of R and is obviously of finite index in R. Hence R is a proper residually finite domain. However $R/a^2B(z, F(z))$ contains the nonzero $F(z)$-subspace $aB(z, F(z))/a^2B(z, F(z))$, and hence it is not finite. Hence R is not right residually finite. Similarly we can show that R is not left residually finite.

Following Michler [10], a right ideal I of a ring R is said to be *prime* if, for each elements s, t of R, $sRt \subset I$ implies that either $s \in I$ or $t \in I$. Now we shall prove a theorem which corresponds to [3, Theorem 2.3 and Corollary 2.4].

Theorem 2. *The following statements are eqivalent:*

1) *R is a right residually finite ring.*
2) *R is a right Noetherian ring and every nonzero prime right ideal of R is of finite index in R.*

3) *Every nonzero prime right ideal of R is finitely geberated and of finite index in R.*

Proof. Obviously 1) implies 2) and the eqivalence of 2) and 3) follows from [10, Theorem 6].

Suppose that 2) holds and let E be the set of nonzero right ideals of R of finite index. We claim that E is empty. Suppose, to the contrary, that E is non-empty. Then, since R is right Noetherian, E has a maximal element, say M. By hypothesis, M is not prime. Hence there are s, t in $R \setminus M$ such that $sRt \subset M$. Put $A = sR + M$ and $B = tR + M$. Then M is strictly contained in both A and B. By the maximality of M, A and B are of finite index in R. Then AB is of finite index in R by Lemma 1. Since $AB \subset M$, this is a contradiction. Thus E is empty as we claimed.

2. Extensions of residually finite rings. In this section, we consider some ring extensions $S \supset R$ and examine when "residual finiteness" go up from R to S or go down from S to R. We first show that "residual finiteness" is a Morita invariant property.

Proposition 1. *Let R be a residually finite ring. If a ring S is Morita eqivalent to R, then S is also residually finite.*

Proof. By [1, Proposition 21.11], there is an isomorphism F between the lattices of ideals of R and S, and R/I and $S/F(I)$ are Morita equivalent for each ideal I of R. Now let I be a nonzero ideal of R. Then R/I is a finite ring. Since $S/F(I)$ is Morita equivalent to R/I, there is a finitely generated projective R/I-generator P such that $S/F(I)$ is isomorphic to $\mathrm{End}_{R/I}(P)$. Therefore $S/F(I)$ is also finite. This proves that S is a residually finite ring.

A ring extension A/B is called an *H-separable extension* if $A \otimes_B A$ is A-A-isomorphic to an A-A-direct summand of a finite direct sum A^n of copies of A. Let R be a commutative ring. By virtue of [4, Theorem 2.3.4], an R-algebra A is Azumaya if and only if A/R is H-separable and A is R-central.

Proposition 2. *Let A/B be an H-separable extension such that ${}_BA$ is projective. If B is residually finite, then A is residually finite.*

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Proof. Suppose that B is residually finite, and let I be a nonzero ideal of A. By Sugano [13, Theorem 3.1], it holds that $I = A(I \cap B)A$. Hence, in particular, we obtain $I \cap B \neq 0$. Hence $B/I \cap B$ is a finite ring. Now by Tominaga [14, Proposition], ${}_BA$ is finitely generated. Therefore, A/I is a finitely generated left $B/I \cap B$-module, whence A/I is a finite ring. This proves our assertion.

For a ring R, $Z(R)$ denotes the center of R. The following theorem yields many examples of noncommutative residually finite rings.

Theorem 3. *Let $S \supset R$ be prime rings with $Z(S) \supset Z(R)$, let K denote the field of fractions of $Z(R)$, and suppose that $S \otimes_{Z(R)} K$ is finite dimensional over K. If R is residually finite, then so is S.*

Proof. Put $Z = Z(R)$.. By hypothesis there exist $s_1, \cdots, s_m$ in S such that $S \otimes_Z K = s_1(R \otimes_Z K) + \cdots + s_m(R \otimes_Z K)$. Let us put $M = s_1R + \cdots + s_mR$. Then M is a Z-R-subbimodule of S. Let a be a nonzero element of Z. Then M/aM is a finitely generated R/aR-module. Since R/aR is a finite ring, the number of elements of M/aM is finite, say n. First we claim that S/aS has at most n^n elements. This can be proved similarly as in the proof of [3, Theorem 4.1], but for the sake of completeness we prove this. Since M/aM has n elements, we have

$$(1) \qquad (a^nS \cap M) + aM = (a^{n+1}S \cap M) + aM = \cdots.$$

Let b be an arbitrary element of S. Since $S \otimes_Z K = M \otimes_Z K$, we can write $b = c/d$ with $c \in M$ and $0 \neq d \in Z$. Since R/dR is finite, there exists a positive integer k such that

$$(2) \qquad a^kR + dR = a^{k+1}R + dR = \cdots.$$

Hence we have $a^k = a^{k+1}x + dr$ for some $x, r \in R$, whence $1 = ax + dr/a^k$ in $S \otimes_Z K$. Thus we have

$$(3) \qquad b = a(c/d)x + cr/a^k \equiv cr/a^k \pmod{aS}.$$

We show that $b \equiv u/a^{n-1} \pmod{aS}$ for some u in M. In view of (3), we may assume that $n \leq k$. Since $cr = a^k(cr/a^k) \in a^kS \cap M$, by (1) we have $cr = a^{k+1}s + at$ for some $s \in S$ and $t \in M$. It follows that $c/d \equiv cr/a^k = as + t/a^{k-1} \equiv t/a^{k-1} \pmod{aS}$. Continuing this process, we obtain

$$c/d \equiv t/a^{k-1} \equiv \cdots \equiv u/a^{n-1} \pmod{aS},$$

where $t, \cdots, u$ are elements of M.

Let $x_1, x_2, \cdots, x_n$ be the complete representatives of the distinct cosets of aM in M. Then $u = x_{1'} + ax_{2'} + \cdots + a^{n-1}x_{n'} + a^n y$, where $1', 2', \cdots, n'$ belong to $\{1, 2, \cdots, n\}$ and $y \in M$. Thus we obtain

$$b \equiv u/a^{n-1} \equiv x_{1'}/a^{n-1} + x_{2'}/a^{n-2} + \cdots + x_{n'} \pmod{aS}.$$

Therefore S/aS has at most n^n elements.

Now let I be a nonzero ideal of S. Since $S \otimes_Z K$ is a simple Artinian ring, $I \otimes_Z K = S \otimes_Z K$. Clearly this implies that $I \cap Z \neq 0$. Hence, by the result proved above, we conclude that S/I is a finite ring. This completes the proof.

In Example 1, we showed that $\boldsymbol{Z} \otimes \boldsymbol{Z}i \otimes \boldsymbol{Z}j \otimes \boldsymbol{Z}k$ is a right and left residually finite domain. By virtue of Corollary 2 and Theorem 3, we know that any subring of a finite dimensional division $\boldsymbol{Q}$-algebra is a right and left residually finite ring. More generally we have

Corollary 3. *Let R be a commutative residually finite domain, and K the field of fractions of R. Let D be a finite dimensional division K-algebra. Then all subrings of D containing R are right and left residually finite rings.*

Using Theorem 3, we can construct many noncommutative residually finite rings.

Example 4. Let R be a commutative residually finite domain and let I, J be two nonzero ideals of R. By virtue of Theorem 3, we can easily see that

$$\begin{pmatrix} R & I \\ J & R \end{pmatrix}$$

is a residually finite ring.

Let R be a subring of a ring S. We say that S is a *finite normalizing extension* of R if there exist $a_1, \cdots, a_n$ in S such that $S = Ra_1 + \cdots + Ra_n$ and $Ra_i = a_iR$ for each i.

Theorem 4. *Let R be a right Noetherian prime ring and let S be a prime finite normalizing extension of R. Then S is residually finite if and only if R is residually finite.*

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Proof. Clearly S is also right Noetherian. Suppose that S is residually finite and let P be a nonzero prime ideal of R. By [9, Theorem 10.2.9], there is a prime ideal I of S such that P is a minimal prime over $I \cap R$. Since 0 is a prime ideal of R, $I \cap R$ must be nonzero and hence $I \neq 0$. Therefore $R/I \cap R$ can be considered as a subring of the finite ring S/I. Thus $R/I \cap R$ and hence R/P is finite. Therefore R is residually finite by [3, Theorem 2.3].

Next suppose that R is residually finite. Let P be a nonzero prime ideal of S. Since S is right Noetherian and since 0 is a prime ideal of S, we get $P \cap R \neq 0$ by [9, Propositions 10.2.12 and 10.2.13]. Hence $R/P \cap R$ is finite. Since S/P is finitely generated as an $R/P \cap R$-module, S/P is also finite. This completes the proof.

Corollary 4. *Let R be a prime P.I. ring and let S be a prime finite normalizing extension of R. Then S is residually finite if and only if R is residually finite.*

Proof. By virtue of Theorem 4, it suffices to prove that R is right Noetherian. If R is residually finite, then R satisfies the ascending chain condition on two-sided ideals by [3, Theorem 2.3]. Then R is right Noetherian by [9, Theorem 13.6.15]. On the other hand, if S is residually finite, then S satisfies the ascending chain condition on two-sided ideals by [3, Theorem 2.3]. By the way S is also a *P.I.* ring by [9, Corollary 13.4.9], whence S is right Noetherian by [9, Theorem 13.6.15]. By Formanek and Jategaonkar [5, Theorem 4], we conclude that R is right Noetherian.

In the rest of this section, we consider when "residual finiteness" go down from a ring S to a subring R.

Proposition 3. *Let S be a residually finite ring and let R be a subring of S. If R contains a nonzero ideal I of S, then R is residually finite.*

Proof. Ovbiously we may assume that S is a proper residually finite ring. Let J be a nonzero ideal of R. Since S is a prime ring by [3, Corollary 2.2], IJI is a nonzero ideal of S contained in J. Since IJI is of finite index in S, J is of finite index in R. Therefore R is residually finite.

Example 5. Let F be a finite field and consider the formal power series ring $F[[x]]$. We can easily see that $F[[x]]$ is residually finite. By the routine argument on cardinality, we obtain an element y of $F[[x]]$ such that x and y are algebraically independent over F. Then the subring $F[x, y]$ of $F[[x]]$ is not residually finite. However the subring $F[y] + xF[[x]]$ is residually finite by Proposition 3.

Let A be an integral domain and let K be the field of fractions of A. Then A is said to be *completely integrally closed* if, for $k \in K$ and $a \in A$ with $a \neq 0$, $ak^n \in A$ for all $n > 0$ implies $k \in A$. It is well known that if A is completely integrally closed then A is integrally closed, and the converse of this is true if A is Noetherian.

Theorem 5. *Let R be a residually finite prime P.I. ring and let F be the field of fractions of the center $Z(R)$ of R. If C is a completely integrally closed subring of F containing $Z(R)$, then C is a residually finite Dedekind domain and is the center of a residually finite Dedekind prime ring which contains R.*

Proof. By Posner-Formanek-Rowen theorem [11, Theorem 1.7.9], the ring $Q(R)$ of central quotients of R is Artinian simple. By the same way as in the proof of [2, Lemma 2.1.1], we can show that RC is integral over C. Since C is integrally closed, the center of RC is C. By [8, Corollary VII.3.4], there exists a maximal C-order A of $Q(R)$ containing RC with $Z(A) = C$. By [2, Proposition 2.1.2 a)], A is a maximal order of $Q(R)$. Since $Q(R) \supset A \supset R$, A is residually finite by Theorem 3. In particular, A is a Noetherian ring (see the proof of Corollary 4), and the classical Krull dimension of A is equal to or less than 1. Then, by [9, Theorem 13.9.14], C is a Dedekind domain and A is a maximal classical C-order and a Dedekind prime ring. Since A is integral over C, the pair (A, C) satisfies "Lying over" by [9, Theorem 13.8.14]. That is, if $\mathfrak{p}$ is a prime ideal of C, then there is a prime ideal P of A such that $P \cap C = \mathfrak{p}$. If $\mathfrak{p} \neq 0$, then $P \neq 0$, and hence $C/\mathfrak{p}$ is a subring of the finite ring A/P. Hence $C/\mathfrak{p}$ is a finite ring. By [3, Theorem 2.3], this implies that C is residually finite.

Corollary 5. *Let R be a residually finite prime P.I. ring and let F be the field of fractions of the center $Z(R)$ of R. If C is a Noetherian subring of F containing $Z(R)$, then C is residually finite. In particular, $Z(R)$ is residually finite if and only if $Z(R)$ is Noetherian.*

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Proof. Let $\tilde{C}$ denote the integral closure of C in F. Since C is Noetherian, $\tilde{C}$ is completely integrally closed. Then $\tilde{C}$ is residually finite by Theorem 5, and hence C is residually finite by [3, Theorem 4.2].

Remark 1. This corollary also follows from [11, Corollary 5.1.4], Theorem 3 and Corollary 4.

3. The number of ideals in a residually finite rings. Let R be a residually finite ring and let I be a nonzero ideal of R. The number of elements in R/I we shall call the *norm* of I and denote it by $N(I)$. The following is well known for rings of algebraic integers.

Proposition 4. *Let R be a residually finite Asano order. Then, for any nonzero ideals I, J of R, we have $N(IJ)=N(I)N(J)$.*

Proof. Let I be a nonzero ideal of R. By [9, Theorem 5.2.9], we can write $I = M_1^{n_1}M_2^{n_2}\cdots M_t^{n_t}$ for some distinct maximal ideals $M_1, M_2, \cdots, M_t$ and some positive integers $n_1, n_2, \cdots, n_t$. Using Chinese remainder theorem, we have $N(I) = N(M_1^{n_1})\cdots N(M_t^{n_t})$. Now let M be a maximal ideal of R and let k be any positive integer. Then M^k is a progenerator as a left or right R-module. Hence the functor $N_R \mapsto N \otimes_R M^k$ provides a category equivalence from Mod-R to Mod-R. Since $R/M \otimes_R M^k$ is isomorphic to M^k/M^{k+1}, M^k/M^{k+1} has the same lemgth as R/M. However both of them are modules over the finite simple ring R/M. Hence they have the same number of elements. From this, we can easily show $N(M^k) = N(M)^k$. Therefore we obtain $N(I) = N(M_1)^{n_1}\cdots N(M_t)^{n_t}$. Now the assertion in this proposition is obvious.

We shall show that given positive integer n, the number of ideals I of a residually finite ring R satisfying $N(I) \leqq n$ is finite. To do this, we need the following.

Lemma 2. *Let R be a residually finite ring and let I be a nonzero ideal of R satisfying $N(I) \leqq n$. Then we have $x^n \equiv x^{n+n!}$ (mod I) for all $x \in R$.*

Proof. By hypothesis, R/I is a finite ring and the number of elements in R/I is equal to or less than n. Let a be an arbitrary element of R/I. Then we have $a^i = a^j$ for some i, j with $1 \leqq i < j \leqq n+1$. Hence

$a^n(1 - a^{j-i}) = 0$, and so $a^n(1 - a^{n!}) = 0$. Therefore R/I satisfies the identity $x^n - x^{n+n!} = 0$. This proves our lemma.

Theorem 6. *Let R be a residually finite ring and let n be a positive integer. Then the number of ideals of R satisfying $N(I) \leqq n$ is finite.*

Proof. If R is a finite ring, then there is nothing to prove. Hence, in view of [3, Corollary 2.2], we may assume that R is an infinite prime ring. Let J denote the intersection of all ideals I satisfying $N(I) \leqq n$. If $J \neq 0$, then R/J is a finite ring, whence the assertion is trivial. Suppose that $J = 0$. Then R satisfies the identity $X^n - X^{n+n!} = 0$. This implies that R is a prime $P.\ I.$ ring and a periodic ring. In particular, the center of R is a periodic field. Then R is simple by [11, Corollary 1.6.28]. This proves the theorem.

As an immediate consequence of this theorem, we have

Corollary 6. *The set of ideals of a residually finite ring is either finite or enumerable.*

In view of Theorem 1, right residually finite rings are right fully bounded right Noetherian rings. Moreover, as shown in the proof of Corollary 4, residually finite $P.\ I.$ rings are right fully bounded right Noetherian.

Proposition 5. *Let R be a right fully bounded right Noetherian and proper residually finite ring. Then the set of ideals of R is enumerable.*

Proof. By [3, Corollary 2.2], R is prime. Suppose, on the contrary, that R has only finitely many ideals. Let J denote the intersection of all nonzero ideals of R. Since R is prime, J is nonzero. Since R is right fully bounded, J coincides with the intersection of all essential right ideals of R. By [1, Proposition 9.7], we have $Soc(R_R) = J \neq 0$. Since R is a right Noetherian prime ring, $Soc(R_R)$ is generated by a central idempotent of R, whence we conclude that $Soc(R_R) = R$.

We conclude this paper with the following

Example 6. Any simple ring is trivially a residually finite ring with exactly two ideals. Given a positive integer $n \geqq 3$, we shall construct a proper residually finite ring R with exactly n ideals. For let F be a finite field and let S be F if $n = 3$ and for $n \geqq 4$ let S be the F-subalgebra of $M_{n-2}(F)$ generated by

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ & 0 & 1 & \ddots & \vdots \\ & & \ddots & \ddots & 0 \\ & & & \ddots & 1 \\ 0 & & & & 0 \end{pmatrix}$$

Then S is a finite ring having exactly n-1 ideals. Let R be the set of countable matrices over F of the form

$$\begin{pmatrix} A_k & & & & 0 \\ & a & & & \\ & & a & & \\ & & & \ddots & \\ 0 & & & & \ddots \end{pmatrix}$$

where $a \in S$ and A_k is an arbitrary $k \times k$ matrix over the ring $M_{n-2}(F)$ and k is allowed to be any integer. Then we can easily see that R is a proper residually finite ring with exactly n ideals.

REFERENCES

[1] F. ANDERSON and K. FULLER: Rings and Categories of Modules, Springer-Verlag, Berlin, 1974.

[2] M. CHAMARIE: Sur les orderes maximaux au sens d'Asano, Vorlesungen aus dem Fachbereich Mathematik der Universität Essen, Heft 3, 1979.

[3] K. L. CHEW and S. LAWN: Residually finite rings, Canad. J. Math. **22**(1970), 92–101.

[4] F. DEMEYER and E. INGRAHAM: Separable Algebras over Commutative Rings, Lecture Notes in Math. 181, Springer-Verlag, New York-Berlin, 1970.

[5] E. FORMANEK and A. V. JATEGAONKAR: Subrings of Noetherian rings, Proc. Amer. Math. Soc. **46**(1974), 181–186.

[6] Y. HIRANO: On extensions of rings with finite additive index, Math. J. Okayama Univ. **32**(1990), 93–95.

[7] V. A. JATEGAONKAR: A multiplicative analog of the Weyl algebra, Comm. Algebra **12**(1984), 1669–1688.

[8] G. MAURY and J. RAYNAUD: Ordres Maximaux au Sens de K. Asano, Lecture Notes in Math. 808, Springer-Verlag, Berlin-Heidelberg-New York, 1980.

[9] J. McConnell and J. C. Robson: Noncommutative Noetherian Rings, Wiley-Interscience, New York, 1987.
[10] G. O. Michler: Prime right ideals and right Noetherian rings, Ring Theory-Edited by R. Gordon, Academic Press, New York-London, 1972, 251-255.
[11] L. H. Rowen: Polynomial Identities in Ring Theory, Academic Press, New York-London, 1980.
[12] B. Stenström: Rings of Quotients, Springer-Verlag, New York-Berlin, 1975.
[13] K. Sugano: On projective H-separable extensions, Hokkaido Math. J. **5**(1976), 44–54.
[14] H. Tominaga: A note on H-separable extensions, Proc. Japan Acad. **50**(1974), 446–447.

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(*Receiver September 30, 1992*)