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## A note on uniform integrability

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## A NOTE ON UNIFORM INTEGRABILITY

SHOZO KOSHI

**1.** Let  $\mathcal{Q}$  be a measure space with a measure  $\mu$ . It is known that in  $L_1(\mathcal{Q}, \mu)$  the space of integrable functions on  $\mathcal{Q}$  with respect to  $\mu$ , the following conditions for a subset  $\mathfrak{U}$  of  $L_1$  are equivalent to each others, provided that the measure  $\mu$  is finite on  $\mathcal{Q}$ .

- (1)  $\mathfrak{U}$  is uniformly integrable ;
- (2)  $\mathfrak{U}$  is bounded (in the sense of  $L_1$ -norm) and for  $\mu(e_i) \downarrow 0$ ,
 
$$\int_{e_i} f d\mu \rightarrow 0 \text{ (as } i \rightarrow \infty \text{) uniformly on } f \in \mathfrak{U} ;$$
- (3)  $\mathfrak{U}$  is relative weakly compact ;
- (4)  $\mathfrak{U}$  is relative weakly sequential compact.

Here, weak topology on  $L_1$  means the weak topology by  $L_\infty$ ; this topology is denoted by  $\sigma(L_\infty)$ .

In this note, we shall generalize the notion of uniform integrability in the case where  $\mu$  is not necessary finite. This situation is also applicable to the cases of measurable function spaces and more general complete vector lattices.

For simplicity, we shall discuss mainly the case of  $L_1(\mathcal{Q}, \mu)$ . The equivalence of (2) and (3) is fully discussed by Nakano and Amemiya ([1], [4]) in the case of complete vector lattice. ( $L_1$  is an example of complete vector lattice). In this case, the condition (2) is reformed in the statement of vector lattice [ $\mathfrak{U}$  is equi-continuous in Nakano's terminology].

Even though (3) and (4) are equivalent in the case of  $L_1$ -spaces, in general measurable function spaces this equivalence does not follow.

**2.** We shall consider the case where  $\mu$  is not necessary finite. Let  $\mathcal{Q} = \bigcup_\alpha S_\alpha$  where  $S_\alpha$  is pairwise disjoint  $\mu$ -measurable set in  $\mathcal{Q}$  and  $\mu$  is  $\sigma$ -finite measure on  $S_\alpha$  for each  $\alpha$  and every finite measurable set of  $\mathcal{Q}$  is contained in countable union of  $S_\alpha$ .

For  $f \in L_1(\mu)$ , we denote by  $[f]$  an operator defined as follows :

$$[f]g = \bigcup_{n=1}^{\infty} (g \cap n|f|) \text{ for } g \geq 0 \text{ and } g \in L_1(\mu)$$

and

$$[f]g = [f]g^+ - [f]g^- \text{ for arbitrary } g \in L_1(\mu).$$

It is easy to see that  $[f]$  is a linear operator on  $L_1(\mu)$  with  $[f] = [|f|]$  and idempotent. If  $\chi_{\{x\}}$  is a characteristic function on a subset  $\{x; f(x) \neq 0, x \in \mathcal{Q}\}$ ,

$$[f]g = \chi_{\{x\}} g \text{ for every } g \in L_1(\mu).$$

A subset  $\mathcal{U} \subset L_1(\mu)$  is called *uniformly integrable with respect to*  $f \in L_1(\mu)$ , if

$$A_n = \sup_{g \in \mathcal{U}} \int_{|g| \geq n|f|} [f]g \, d\mu \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

**Theorem 1** *Let  $\mathcal{U}$  be uniformly integrable with respect to  $f \in L_1(\mu)$ , and  $P_1 \supset P_2 \supset P_3 \supset \dots$  with  $\mu(\bigcap_{i=1}^{\infty} P_i) = 0$ , then  $\sup_{g \in \mathcal{U}} \int_{P_n} [f]|g| \, d\mu \rightarrow 0$  as  $n \rightarrow \infty$  and  $[f]\mathcal{U}$  is bounded. (in the sense of  $L_1$ -norm)*

*Proof.* For every  $\varepsilon > 0$ , there exists an integer  $N > 0$  such that

$$A_n = \sup_{g \in \mathcal{U}} \int_{|g| \geq n|f|} [f]|g| \, d\mu \leq \varepsilon \text{ for } n \geq N.$$

Since  $P_1 \supset \dots \supset P_n \supset \dots$  with  $\mu(\bigcap_{i=1}^{\infty} P_i) = 0$ , we have

$$\int_{P_n} N|f| \, d\mu \rightarrow 0 \text{ as } n \rightarrow \infty \text{ for a fixed number } N.$$

$$\text{Hence } \int_{P_n} [f]|g| \, d\mu \leq \int_{P_n} N|f| \, d\mu + \int_{|g| \geq N|f|} [f]|g| \, d\mu \leq 2\varepsilon$$

for sufficient large  $n$  and for every  $g \in \mathcal{U}$ .

It is easy to see that  $[f]\mathcal{U}$  is norm bounded.

**Theorem 2** *Let  $\mathcal{U}$  be a subset of  $L_1(\mu)$  such that  $\sup_{g \in \mathcal{U}} \int_{P_n} [f]|g| \, d\mu \rightarrow 0$  for  $P_1 \supset P_2 \supset \dots \supset P_n \supset \dots$  with  $\mu(\bigcap_{n=1}^{\infty} P_n \cap \{x; f(x) \neq 0\}) = 0$  and  $[f]\mathcal{U}$  is norm bounded. Then  $\mathcal{U}$  is uniformly integrable with respect to  $f$ .*

*Proof.* If  $\mathcal{U}$  is not uniformly integrable with respect to  $f$ , then there exists  $\varepsilon > 0$  such that

$$A_n = \sup_{g \in \mathcal{U}} \int_{|g| \geq n|f|} [f]|g| \, d\mu \geq \varepsilon > 0$$

for  $n = 1, 2, \dots$

Since  $[f]\mathcal{U}$  is norm bounded, there exists a number  $M$  and  $g_n \in \mathcal{U}$  with

$$M \geq \int_{|g_n| \geq n|f|} [f]|g_n| \, d\mu \geq \frac{\varepsilon}{2} (n = 1, 2, \dots) \text{ i. e.}$$

$$M \geq \int_{P_n} n|f| \, d\mu = n \int_{P_n} |f| \, d\mu$$

where  $P_n = \{x; |g_n(x)| \geq n|f(x)|\}$ . Hence  $\int_{P_n} |f| \, d\mu \rightarrow 0$  as  $n \rightarrow \infty$  and

$\sup_{g \in \mathcal{U}} \int_{Q_n} [f]|g| \, d\mu \geq \frac{\varepsilon}{2}$  for a suitable  $Q_n \subset \{x; f(x) \neq 0\}$  decreasing

with  $\mu(\bigcap_{n=1}^{\infty} Q_n) = 0$ ; it means that  $\mathcal{U}$  does not satisfy the assumption of Theorem 2.

**Theorem 3** *If  $[f_1] = [f_2]$ , and  $\mathcal{U}$  is uniformly integrable with respect to  $f_1$ , then  $\mathcal{U}$  is uniformly integrable with respect to  $f_2$ .*

**Proof.** Since  $[f_1] = [f_2]$  is equivalent to  $\{x : f_1(x) \neq 0\} = \{x : f_2(x) \neq 0\}$  except  $\mu$ -measure zero set, Theorem 3 is a direct consequence of Theorem 1 and 2.

**Theorem 4.** *If  $\mathfrak{L}$  is uniformly integrable with respect to every  $f \in L_1(\mu)$ , then  $\mathfrak{L}$  is norm-bounded.*

**Proof.** Suppose that  $\mathfrak{L}$  is not norm bounded. Then there exists a sequence of  $g_n \in \mathfrak{L}$  ( $n=1, 2, \dots$ ) such that  $\int |g_n| d\mu \geq n$ . Since  $S_\alpha \cap \{x : g_n(x) \neq 0\} \neq \phi$  only for countable  $\alpha$  for each  $n$ , there exists at most countable  $S_{\alpha_i}$  for which  $S_{\alpha_i} \cap \{x : g_n(x) \neq 0\} \neq \phi$  for some  $n$  ( $i=1, 2, \dots$ ). Since  $\mu$  is  $\sigma$ -finite on  $S_{\alpha_i}$ , there exists a function  $f \in L_1(\mu)$ , whose support is exactly  $\bigcup_{i=1}^{\infty} S_{\alpha_i}$ . Because  $\mathfrak{L} \supset \{g_n\}$  is uniformly integrable with respect to  $f$ ,  $[f]\mathfrak{L} \supset \{g_n\}$  is norm bounded ; this is a contradiction. It is easy to see :

**Theorem 5.** *If  $\mathfrak{L}$  is norm bounded and  $\int_{P_n} |g| d\mu \rightarrow 0$  uniform on  $g \in \mathfrak{L}$  for  $P_1 \supset P_2 \supset \dots \supset P_n \supset \dots$  with  $\mu(\bigcap_{n=1}^{\infty} P_n) = 0$ , then  $\mathfrak{L}$  is uniformly integrable with respect to all  $f \in L_1(\mu)$ . We shall say that  $\mathfrak{L}$  is uniformly integrable if  $\mathfrak{L}$  is uniformly integrable with respect to all  $f \in L_1$ .*

**Theorem 6.** *If  $\mathfrak{L}$  has the property :  $\int_{P_n} |g| d\mu \rightarrow 0$  uniform on  $g \in \mathfrak{L}$  for  $P_1 \supset P_2 \supset \dots \supset P_n \supset \dots$  with  $\mu(\bigcap_{n=1}^{\infty} P_n) = 0$ , and  $\mathcal{Q}$  is non-atomic with respect to  $\mu$ , then  $\mathfrak{L}$  is norm bounded.*

**Proof.** Suppose that  $\mathfrak{L}$  is not norm bounded. There exists a sequence of elements  $g_n \in \mathfrak{L}$  with  $\int |g_n| d\mu \geq 2n$ . We can find a  $\mu$ -measurable finite set  $P_n$  with  $\int_{P_n} |g_n| d\mu \geq n$ . Put  $Q_n = P_1 \cup P_2 \cup \dots \cup P_n$  and  $Q_0 = \bigcup_{n=1}^{\infty} Q_n$ . Then,  $Q_0 - Q_n$  is decreasing with  $\bigcap_{n=1}^{\infty} (Q_0 - Q_n) = \phi$ . Hence, observing  $\int |g| d\mu \rightarrow 0$  as  $n \rightarrow \infty$ , there exists  $Q = Q_m$  with  $\int_Q |g_n| d\mu \geq n/2$ . Since  $\mu(Q) < +\infty$  and  $\mathcal{Q}$  is non-atomic, we can decompose  $Q$  into  $Q_{1,1}$  and  $Q_{1,2}$  with  $Q = Q_{1,1} \cup Q_{1,2}$ ;  $Q_{1,1} \cap Q_{1,2} = \phi$ ;  $\mu(Q_{1,1}) = \frac{\mu(Q)}{2} = \mu(Q_{1,2})$ . We can select  $R_1 = Q_{1,1}$  or  $Q_{1,2}$  such that  $\int_{R_1} |g_n| d\mu$  ( $n=1, 2, \dots$ ) is not bounded. By the same

way and by induction, we can find  $R_1 \supset R_2 \supset \dots \supset R_n \supset \dots$  with  $\mu(R_n) = \frac{\mu(Q)}{2^n}$  and for all  $m \int_{R_m} |g_n| d\mu (n=1, 2, \dots)$  is not bounded. But  $\mu(\bigcap_{n=1}^{\infty} Q_n) = 0$ , and so this is a contradiction.

**Theorem 7.** *If  $\mu$  is non-atomic on  $[f]L_1(\mu)$ , then  $\mathfrak{U}$  is uniform integrable with respect to  $f$  if and only if  $[f]\mathfrak{U}$  has the property of Theorem 6.*

Normal hull of  $\mathfrak{U}$  is a subset of all  $f$ , to which there exists  $g$  with  $|f| \leq |g|$  for some  $g \in \mathfrak{U}$ .

**Theorem 8.** *If  $\mathfrak{U}$  is uniformly integrable with respect to all  $f \in L_1(\mu)$ , then  $\bigcup_{g \in \mathfrak{U}} \{x; g(x) \neq 0\}$  is contained in at most countable union of  $S_\alpha$ . Hence, there exists  $f_0 \in L_1(\mu)$  such that  $[f_0]\mathfrak{U} = \mathfrak{U}$ .*

*Proof.* If  $\mathfrak{U}$  is uniformly integrable with respect to  $f$ , then the normal hull  $\tilde{\mathfrak{U}}$  of  $\mathfrak{U}$  is uniformly integrable with respect to  $f$ . Suppose that  $\bigcup_{g \in \mathfrak{U}} \{x; g(x) \neq 0\}$  is not contained in any countable union of  $S_\alpha$ . Then, we can find an element  $f_n \in L_1(\mu)$ , whose supports are pairwise disjoint and  $\|f_n\| = \int |f_n| d\mu \geq \varepsilon > 0$  for some number  $\varepsilon$  for every integer  $n$ . Considering  $g_n = \frac{1}{2^n \|f_n\|} f_n$  and  $f = \sum_{n=1}^{\infty} g_n$ , we see  $f \in L_1(\mu)$  and  $\{g_n\}$  is not uniformly integrable with respect to  $f$ . Hence,  $\tilde{\mathfrak{U}}$  and  $\mathfrak{U}$  are not uniformly integrable with respect to  $f$ . This is a contradiction.

**Theorem 9.**  *$\mathfrak{U}$  is relative weakly compact if and only if  $\mathfrak{U}$  is uniformly integrable with respect to all  $f \in L_1(\mu)$ .*

*Proof.* Let  $\mathfrak{U}$  be uniformly integrable. Then the normal hull  $\tilde{\mathfrak{U}}$  of  $\mathfrak{U}$  is also uniformly integrable. We define then the semi-norm in  $L_\infty$  such that

$$\|g\|_{\mathfrak{U}} = \sup_{f \in \tilde{\mathfrak{U}}} |\langle f, g \rangle| = \sup_{f \in \tilde{\mathfrak{U}}} \left| \int f \cdot g d\mu \right|.$$

(\*) For every  $P_n \supset P_{n+1} \supset \dots$  with  $\mu(\bigcap_{n=1}^{\infty} P_n) = 0$ , we have  $\|\chi_{P_n} \cdot g\|_{\mathfrak{U}} \rightarrow 0$ .

We have also  $\|\chi_{P_\lambda} \cdot g\|_{\mathfrak{U}} \rightarrow 0$  by Theorem 8 for any directed system  $P_\lambda$  with  $\mu(\bigcap P_\lambda) = 0$ .

$\tau$  is a linear topology defined by all semi-norm  $\|\cdot\|_{\mathfrak{U}}$  where  $\mathfrak{U}$  is uniformly integrable subset of  $L_1(\mu)$ .

Any  $\tau$ -continuous linear functional on  $L_\infty$  is considered as an element of  $L_1(\mu)$  by (\*) and conversely any  $f \in L_1(\mu)$  is a  $\tau$ -continuous linear functional on  $L_\infty(\mu)$  i. e.  $L'_\infty[\tau] = L_1$ . Hence, every uniformly integrable subset  $\mathfrak{U}$  of  $L_1$  is equi-continuous by  $\tau$ . By Alaoglu-Bourbaki's theorem,  $\mathfrak{U}$  is a relative weakly compact set.

Conversely, if  $\mathfrak{U}$  is not uniformly integrable, then we can find a sequence of measurable sets  $P_1 \supset P_2 \supset \dots$  with  $\mu(\bigcap_{n=1}^{\infty} P_n) = 0$  and  $f_n$  such that

$$\int_{P_n} |f_n| d\mu \geq \varepsilon.$$

Suppose that  $\mathfrak{U}$  is relative weakly compact. By Smulian's theorem and Eberlein's theorem  $\mathfrak{U}$  is relative weakly sequentially compact. There exists a subsequence  $\{f_{n_i}\}$  of  $\{f_n\}$  such that  $f_{n_i} \rightarrow f$  (weak). By Nikodym's theorem, the set functions  $\lambda_i(P_n) = \int_{P_n} f_{n_i} d\mu$  tends to 0 uniformly on  $i=1, 2, \dots$ .

This is a contradiction.

**Theorem 10.** *Let  $\mu$  be a finite measure on  $\Omega$ . A subset  $\mathfrak{U}$  of  $L_1(\mu, \Omega)$  is uniformly integrable if and only if there exists a convex function  $\phi(t) \geq 0$  for  $t \geq 0$  with  $\lim_{t \rightarrow \infty} \frac{\phi(t)}{t} = +\infty$  such that  $\sup_{f \in \mathfrak{U}} \int \phi(|f|) d\mu < +\infty$ .*

Even if this theorem is famous and easy, I can not find a detail proof of this theorem. We shall sketch a proof of this theorem.

*Proof.* Let  $\mathfrak{U}$  be uniformly integrable. For every integer  $i \geq 1$ , there exists  $n_i \uparrow \infty$  with  $\sup_{f \in \mathfrak{U}} \int_{|f| \geq n_i} |f| d\mu \leq \frac{1}{2^i}$ . At first, for  $i=1$ , we choose a point  $(n_1, \sqrt{2}n_1) = (m_1, \sqrt{2}m_1)$  in the plane. Next for  $i \geq 2$  we find

$$\frac{(\sqrt{2})^i n_i - \sqrt{2}n_1}{n_i - n_1} \geq (\sqrt{2}). \quad \text{Hence, } \min_{i \geq 2} \frac{(\sqrt{2})^i n_i - \sqrt{2}n_1}{n_i - n_1} \text{ exists and we}$$

put  $m_2 = n_i$  and  $i = i_2$  such that for  $n_i$ ,  $\frac{(\sqrt{2})^i n_i - \sqrt{2}m_1}{n_i - m_1}$  takes the minimum value. For  $i \geq i_2$ , we find  $\frac{(\sqrt{2})^i n_i - (\sqrt{2})^2 m_2}{n_i - m_2} \geq (\sqrt{2})^2$  and put

$$m_3 = n_i \text{ and } i = i_3 \text{ such that for } n_i = m_3, \frac{(\sqrt{2})^i n_i - (\sqrt{2})^2 m_2}{n_i - m_2} \text{ takes the minimum}$$

value. By induction, we find  $\frac{(\sqrt{2})^i n_i - (\sqrt{2})^{j-1} m_{j-1}}{n_i - m_{j-1}}$  takes the minimum value for  $m_j = n_i$  and  $i = i_j$ .

Now, we draw a polygon combining  $(0, 0)$ ,  $(m_1, \sqrt{2}m_1)$ ,  $(m_2, (\sqrt{2})^2 m_2)$ ,  $\dots$ ,  $(m_n, (\sqrt{2})^n m_n)$ ,  $\dots$ . This polygon defines a convex function  $\psi(t)$  for  $t \geq 0$  and  $\lim_{t \rightarrow \infty} \frac{\psi(t)}{t} = +\infty$ . Let  $\eta(t) = (\sqrt{2})^i n_i$  for  $n_{i-1} < t \leq n_i$ , then  $t \leq$

$$\psi(t) \leq \eta(t). \text{ For } f \in \mathfrak{U}, \int_{\Omega} \eta(|f|) d\mu \leq \sum_{i=0}^{\infty} \frac{1}{(\sqrt{2})^i} < +\infty. \text{ Hence we have}$$

$$\sup_{f \in \mathfrak{U}} \int \psi(|f|) d\mu < +\infty.$$

Conversely, if a convex function  $\phi$  satisfies  $\lim_{t \rightarrow \infty} \frac{\phi(t)}{t} = +\infty$  and  $\sup_{f \in \mathfrak{U}} \int \phi(|f|) d\mu = M < +\infty$ , then for every  $n \geq 1$ , there exists  $m$  such that  $u(x) \geq m$  implies  $\phi(|u(x)|) \geq n|u(x)|$ .

$$M \geq \int_{|u| \geq m} \phi(|u|) d\mu = n \int_{|u| \geq m} |u(x)| d\mu.$$

i. e.  $\frac{M}{n} \geq \int_{|u| \geq m} |u(x)| d\mu$  for all  $u \in \mathfrak{U}$ . This means that  $\mathfrak{U}$  is uniformly integrable.

We shall consider the case where  $\mu$  is not finite on  $\Omega$ . For this, we need another definition of uniformly integrable subsets.

**Theorem 11.** *A subset  $\mathfrak{U}$  of  $L_1(\mu)$  is uniformly integrable with respect to  $f$ , if and only if there exists a sequence of  $a_n(x) \in L_1(\mu)$  with  $0 \leq a_n(x) \uparrow \infty$  and  $[a_n] = [f]$  such that  $\sup_{u \in \mathfrak{U}} \int_{|u| \geq |a_n|} |u(x)| d\mu \rightarrow 0$  as  $n \rightarrow \infty$ .*

Proof. By the same argument of Theorem 1, if  $\mathfrak{U}$  satisfies the conditions of the present theorem,  $\mathfrak{U}$  is uniformly integrable with respect to  $f$ . Let  $\mathfrak{U}$  be uniformly integrable with respect to  $f$ . Suppose that there exists  $g_n$  ( $n=1, 2, \dots$ ) with

$$M \geq \int_{|g_n| \geq |a_n|} [f] |g_n| d\mu \geq \varepsilon > 0,$$

because  $[f]\mathfrak{U}$  is bounded by Theorem 2.

$$M \geq \int_{P_n} a_n(x) a_n^\mu$$

where  $P_n = \{x \in \Omega; |g_n(x)| \geq a_n(x)\} \cap \{x; f(x) \neq 0\}$ .

Since  $[a_n] = [f]$  and  $a_n(x) \uparrow \infty$ , and

$$\int_{P_n} a_n(x) d\mu \rightarrow 0 \text{ as } n \rightarrow \infty, \text{ we have } \sup_{Q \in \mathfrak{U}} \int_{Q_n} [f] g d\mu \geq \frac{\varepsilon}{2} \text{ for some } \mu\text{-}$$

measurable  $Q_n$  with  $\mu(\bigcap_{n=1}^{\infty} Q_n) = 0$ . This is a contradiction.

Now, we have the following theorem.

**Theorem 11.**  *$\mathfrak{U}$  is uniformly integrable if and only if there exists a function  $\psi(t, x)$  of two variables  $x \in \Omega$  and real  $t \geq 0$  such that  $\psi(t, x)$  is a convex function of  $t$  for a fixed  $x \in \Omega$ , a measurable function of  $x$  for a fixed  $t \geq 0$ , and  $a_n(x) \uparrow \infty$ ,  $\psi(a_n(x), x) \geq n a_n(x)$  with  $[a_1] = [a_2] = \dots = [a_n] = \dots$ ,  $[a_1]\mathfrak{U} = \mathfrak{U}$  and*

$$\sup_{f \in \mathfrak{U}} \int \psi(|f(x)|, x) d\mu < +\infty.$$

Proof. Let  $\mathfrak{U}$  be uniformly integrable. Then, there exists  $0 \leq f \in L_1(\mu)$  with  $[f]\mathfrak{U} = \mathfrak{U}$  and  $\mathfrak{U}$  is uniformly integrable with respect to  $f$  by Theorem

8. By Theorem 3, we can assume that  $f$  is a step function whose values may be countably many. For every integer  $i=1$ , there exists  $n_1 < n_2 < \dots < n_i < \dots$  with

$$\sup_{g \in \mathfrak{U}} \int_{|g| \leq n_i f} |g| d\mu \geq \frac{1}{2^i}.$$

By the method discussed in the proof of Theorem 10, we find a convex function  $\phi(t)$  ( $t \geq 0$ ) such that for infinitely many  $i$ ,

$$\begin{aligned} \phi(n_i) &= (\sqrt{2})^i n_i \text{ and } \phi(t) \geq \phi(t); \\ \text{where } \phi(t) &= (\sqrt{2})^i n_i \text{ for } n_{i-1} < t \leq n_i, \end{aligned}$$

We define

$$\begin{aligned} \phi(t, x) &= f(x) \phi\left(\frac{t}{f(x)}\right) && \text{for } f(x) \neq 0 \\ &= t^2 && \text{for } f(x) = 0. \end{aligned}$$

Then,  $\phi(t, x)$  is a convex function of  $t \geq 0$  for  $x \in \Omega$ , and measurable function of  $t$  for  $x \in \Omega$ .

$$\phi(n_i f(x), x) = (\sqrt{2})^i n_i f(x) \text{ for infinitely many of } i.$$

Hence, putting  $a_n(x) = n_i f(x)$  for suitable  $n_i = n$ , then

$$\phi(a_n(x), x) \geq n a_n(x) \text{ and } a_n(x) \in L_1(\mu) \text{ and } [a_n] = [f].$$

We have also,

$$\int \phi(|g(x)|, x) d\mu \leq \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i < \infty.$$

Conversely, if there exists  $\phi(t, x)$  with the conditions of Theorem for  $n \geq 1$ , there exists  $m$  such that

$$\begin{aligned} u(x) \geq a_m(x) \text{ implies } \phi(|u(x)|) &\geq n u(x); \\ M &\geq \int \phi(|u(x)|, x) d\mu \geq n \int_{u(x) \geq a_m} |u(x)| d\mu \quad \text{for all } u \in \mathfrak{U}. \end{aligned}$$

i. e.

$$\frac{M}{n} \geq \int_{|u| \geq a_m} |u| d\mu \text{ for } u \in \mathfrak{U}.$$

Hence, we see that  $\mathfrak{U}$  is uniformly integrable by Theorem 10.

3. The same argument discussed in 2 for  $L_1(\mu)$  is also applicable for the case of complete vector lattice which is order dual of some complete vector lattice. For example,  $L_1(\mu)$  is order dual of  $L_\infty(\mu)$ .

A subset  $\mathfrak{U}$  of  $L$ , which is a complete vector lattice of order dual of a complete vector lattice  $C$  is *uniformly integrable with respect to  $f$*  if

$$\langle s, [(|g| - nf)^+] g \rangle \rightarrow 0 \text{ uniformly on } g \in \mathfrak{U} \text{ for every } 0 \leq s \in C.$$

Then,  $\mathfrak{U}$  is uniformly integrable (for all  $f \in L$ ) if and only if  $\mathfrak{U}$  is rela-



tive weakly compact (i. e. relative compact by (C)). But, essentially the characterization of relative weakly compact sets is obtained by Nakano [4].

If  $L$  is a Banach lattice and order dual of a complete vector lattice  $C$  and  $C$  is conjugate space of  $L$  as Banach space, then  $\mathfrak{U}$  is relative weakly compact if and only if  $\mathfrak{U}$  is relative weakly sequentially compact.

But the equivalence of relative weakly compact sets and relative weakly sequentially compact sets is not true in general.

We shall show that in  $L_\infty(\mu)$ , there exists  $\mathfrak{U}$  which is uniformly integrable (=relative weakly compact), but  $\mathfrak{U}$  is not relative weakly sequentially compact.

Let  $\Omega$  be a set of density of real number space and  $\mu$  be discrete measure on  $\Omega$  (i. e. every point measure is one).

We shall consider a countable set  $\{f_i\} \subset L_\infty(\mu)$  such that the totality of sequence  $f_i(x) (i=1, 2, \dots) x \in \Omega$  is the totality of the sequence whose coordinate is 1 or  $-1$ . For every subsequence  $\{f_{i_j}\} \subset \{f_i\}$ , we find an element  $x$  such that  $\{j; f_{i_j}(x)=1\}$  and  $\{j; f_{i_j}(x)=-1\}$  are infinitely many. If  $g_j$  is weakly convergent to some  $g$ , then  $g_j(x) \rightarrow g(x)$  for every  $x \in \Omega$ . Hence the sequence of elements  $f_{i_j}$  is not weakly convergent.

The set  $f_i$  is relative weakly compact, but not relative weakly sequentially compact. Weak topology is here  $\sigma(L_1(\mu))$ .

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