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ON EXTENSIONS OF COMPATIBLE PAIRS OF ISOMORPHISMS

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Throughout the present paper, A/B will represent a simple ring extension, namely, $A \supset B$ are (Artinian) simple rings with the same 1. In the previous papers [2], [3], [6], [7] and [8], we studied left locally finite q-Galois extensions. Our principal aim of the present paper is to give a slight generalization of the content of [2 ; § 2] according to the same schedule. As to notations and terminologies used here, we follow in principle the previous papers cited above. We shall begin our study with stating several available results without proof.

1. Available results

In this section, we state without proof several results which will be needed in § 2. One will find them in [2 ; § 1], [4], [5], [6], [7] and [8]. At first, we shall recall the definition of q-Galois extensions. For an arbitrary T in $\mathfrak{R}$ we denote by $G(T, A; B)$ the set of all B-ring isomorphisms of T into A whose images are in $\mathfrak{R}$. If Hom $({}_BT, {}_BA) = \mathfrak{G}(T, A; B)A_R$ for every $T \in \mathfrak{R}^0_{l.f}$ then A/B is said to be *w-q-Galois.* A *w-q*-Galois extension over a regular subring is called a *q-Galois extension.*

Proposition 1. *Assume that A/B is a left locally finite w-q-Galois extension. Let $A' \in \mathfrak{R}$, and $B' \in \mathfrak{R}_{l.f}$.*

(a) *A' contains a subring U such that $U[F]$ is in $\mathfrak{R}_{l.f}$ for every finite subset F of A'.*

(b) *If $B'' \in \mathfrak{R}_{l.f}/B'$ then $\mathfrak{G}(B'', A; B)|B' = \mathfrak{G}(B', A; B)$,* Hom $({}_{B'}B'', {}_{B'}A) = \mathfrak{G}(B'', A; B')A_R$ and $J(\mathfrak{G}(B'', A; B')) = \{x \in B''; x\sigma = x$ *for every σ in $\mathfrak{G}(B'', A; B')\}$ coincides with B'.*

(c) *If σ' is in $\mathfrak{G}(A', A; B)$ then $(A' \cap H)\sigma' = A'\sigma' \cap H$.*

(d) *Let σ be in $\mathfrak{G}(B', A; B)$ and $\mathfrak{H}$ a subset of $\mathfrak{G}(B', A; B)$. If $\sigma A_R \subset \mathfrak{H} A_R$ then $\sigma = \tau \tilde{v}$ for some $\tau \in \mathfrak{H}$ and $v \in V^{\cdot}$.*

We consider here the following conditions:

(i) $J(\mathfrak{G}(T, A; B)) = B$ for every $T \in \mathfrak{R}^0_{l.f}$.

(ii) H/B is Golois and $\mathfrak{G}(B'', A; B)|B' = \mathfrak{G}(B', A; B)$ for every $B'' \supset B'$ in $\mathfrak{R}_{l.f}$.

(iii) B is regular and $\mathfrak{G}(B'', A; B)|B' = \mathfrak{G}(B', A; B)$ for every

$B''\supset B'$ in $\mathfrak{R}_{l.f.}$

(iv) A is $B\cdot V$-A-irreducible.

(v) A is A-$B\cdot V$-irreducible.

(vi) H/B is Galois and $[V_A^2(T): H]_L=[V: V_A(T)]_R$ for every $T\in\mathfrak{R}_{l.f.}^0$.

(vii) B and H are regular and $[V_A^2(T): H]_L=[V: V_A(T)]_R$ for every $T\in\mathfrak{R}_{l.f.}^0$.

(viii) H/B is Golois and $\mathfrak{G}(A', A; B)|H\supset\mathfrak{G}(H/B)$ for every $A'\in\mathfrak{R}^0/H$ with $[A': H]_L<\infty$.

By [3; Remark 2] and [6; Th. 2], one will readily obtain the equivalence in the first assertion of the next:

Proposition 2. *Let A be left locally finite over B.*

(a) *A/B is q-Galois if and only if any of the following equivalent conditions is satisfied*: (Q1) (i)+(iv), (Q2) (i)+(v), (Q3) (i)+(vii), (Q4) (ii)+(iv), (Q5) (ii)+(v), (Q6) (iii)+(vi), (S7) (iv)+(viii) *and* (Q8) (v)+(viii). *If A/B is q-Galois then A is locally finite and right q-Galois over B.*

(b) *Let A be Galois over B. Then, A/B is h-Galois if and only if any of the following equivalent conditions is satisfied*: (H1) *A/B is q-Galois,* (H2) (iv), (H3) (v), (H4) (vii) *and* (H5) *A/B' is Golois for every $B'\in\mathfrak{R}_{l.f.}^0$, H is simple, and $[V_A^2(T): H]_L=[V: V_A(T)]_R$ for every $T\in\mathfrak{R}^0/H$ left finite over H. 1f A/B is h-Galois then A/B is locally finite and right h-Galois.*

Proposition 3. *Assume that A is outer Golois and left algebraic over B. Let A' be an intermediate ring of A/B, and $\mathfrak{H}$ a Galois group of A/B.*

(a) *A' is simple, A/A' is locally finite, and $\mathfrak{G}(A', A; B)=\mathfrak{G}|A'$.*

(b) *A/B is h-Galois and there exists a* 1-1 *dual correspondence between closed subgroups of $\mathfrak{G}$ and intermediate rings of A/B, in the usual sense of Galois theory.*

(c) *If $[A': B]_L<\infty$ then there exists an element $a'\in A'$ such that $A'=B[a']$, and there holds $\sharp(\mathfrak{H}|A')=\sharp\{a'\mathfrak{H}\}=[A': B]$. Conversely, if $\sharp(\mathfrak{H}|A')<\infty$ then $[A': B]<\infty$.*

(d) *If A'/B is Galois then A' is $\mathfrak{G}$-invariant, and conversely.*

Proposition 4. *Assume that A/B is a left locally finite q-Golois extension. Let $A'\in\mathfrak{R}$, and $B'\in\mathfrak{R}_{l.f.}$.*

(a) *If H' is an intermediate ring of H/B that is Galois over B then $A'[H']$ is outer Galois and locally finite over A' and $\mathfrak{G}(A'[H']/A')\simeq\mathfrak{G}(H'/A'\cap H')$ by the contraction map.*

(b) $[B':B]\geqslant[V:V_A(B')]=[V:V_A(B'\sigma)]=[V_A^2(B'):H]=[B':B'\cap H]$, *where* $\sigma\in\mathfrak{G}(B',A;B)$.

(c) *If* A' *is f-regular then* A/A' *is a locally finite q-Galois extension and* $\mathfrak{G}(A'',A;B)|A'\subset\mathfrak{G}(A',A;B)$ *for every* $A''\in\mathfrak{R}/A'$.

(d) *If* $A''\supset A'$ *is an f-regular subring then* $\mathfrak{G}(A'',A;B)|A'=\mathfrak{G}(A',A;B)$ *and* $J(\mathfrak{G}(A'',A;A'))=A'$.

(e) *Every* $(*_f)$*-regular subgroup of* $\mathfrak{G}$ *is f-regular.*

Finally, we state the following:

Proposition 5. *Let A be q-Calois and left locally finite over B, and* $[A:H]_L<\aleph_0$. *If* A' *is an arbitrary f-regular intermediate ring of* A/B, *then* $\mathfrak{G}(A',A;B)=\mathfrak{G}|A'$, $J(\mathfrak{G}(A'))=A'$ *and* A/A' *is a locally finite h-Galois extension.*

2. Extensions of compatible pairs of isomorphisms

In what follows, we assume always that *A/B is a left locally finite q-Galois extension. Let* A_1 be in $\mathfrak{R}$, and A_2 an f-regular intermediate ring of A/B. If $S=A_1\cap A_2$ is a simple ring such that $V_A(S)=V_A(A_2)$ then A_1 is said to be *annexable* to A_2. Evidently, A_1 is annexable to every intermediate ring of H/B (Prop. 3). If A_1 is annexable to A_2 then A is q-Galois and locally finite over the f-regular intermediate ring S (Prop. 4 (c)), and then, noting that $S\subset A_2\subset V_A^2(S)=S[H]$, there holds $A_2=S[A_2\cap H]$ and $A_1[A_2]=A_1[A_2\cap H]$ is in $\mathfrak{R}$ (Prop. 4 (a)), If $\sigma_1\in\mathfrak{G}(A_1,A;B)$ and $\sigma_2\in\mathfrak{G}(A_2,A;B)$ are compatible, namely, if $\sigma_1|S=\sigma_2|S$ then we denote by $\sigma_1\vee\sigma_2$ the (not necessarily single-valued) mapping of $A_1[A_2]$ into A defined as follows;

$$(\textstyle\sum_k a_{k1}^{(1)}a_{k1}^{(2)}\cdots a_{km_k}^{(1)}a_{km_k}^{(2)})(\sigma_1\vee\sigma_2)=$$

$$\textstyle\sum_k(a_{k1}^{(1)}\sigma_1)(a_{k1}^{(2)}\sigma_2)\cdots(a_{km_k}^{(1)}\sigma_1)(a_{km_k}^{(2)}\sigma_2)\quad(a_{kj}^{(i)}\in A_i).$$

For any subset $\mathfrak{H}_i$ of $\mathfrak{G}(A_i,A;B)$, we set $\mathfrak{H}_1\vee\mathfrak{H}_2=\{\sigma_1\vee\sigma_2;\sigma_1\in\mathfrak{H}_1$ and $\sigma_2\in\mathfrak{H}_2$ are compatible$\}$. With those notations, we have the following which is useful in the subsequent study.

Rroposition 6. *Let* $A_1\in\mathfrak{R}$ *be annexable to an f-regular intermedtate ring* A_2 *of* A/B. *We set* $A_0=A_1[A_2]$ *and* $S=A_1\cap A_2$.

(a) $\mathfrak{G}(A_0,A;B)=\Gamma(A_0,A)\cap(\mathfrak{G}(A_1,A;B)\vee\mathfrak{G}(A_2,A;B))$, *where* $\Gamma(A_0,A)$ *denotes the set of all ring isomorphisms of* A_0 *into* A.

(b) *Let* $\sigma\vee\tau$ *be in* $\mathfrak{G}(A_1,A;B)\vee\mathfrak{G}(A_2,A;B)$. *If* $(\sigma_1|A_1')\vee(\sigma_2|A_2')$ *is in* $\Gamma(A_1'[A_2'],A)$ *for every* $A_i'\in\mathfrak{R}_{i,f}$ *contained in* A_i *then* $\sigma\vee\tau$ *is in* $\mathfrak{G}(A_0,A;B)$.

(c) *If* $\mathfrak{G}(A_0,A;B)=\mathfrak{G}(A_1,A;B)\vee\mathfrak{G}(A_2,A;B)$ *then* $\mathfrak{G}(A_0,A;B)|A_1$

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$=\mathfrak{G}(A_1, A; B)$, $\mathfrak{G}(A_0, A; A_2)|A_1=\mathfrak{G}(A_1, A; S)$ *and* $\mathfrak{G}(A_0, A; A_1)|A_2=\mathfrak{G}(A_2, A; S)$.

(d) *Assume that* $\mathfrak{G}(A_0, A; B)=\mathfrak{G}(A_1, A; B)\vee\mathfrak{G}(A_2, A; B)$. *If* $\mathfrak{G}(A_1, A; B)|S=\mathfrak{G}(S, A; B)$ *then* $\mathfrak{G}(A_0, A; B)|A_2=\mathfrak{G}(A_2, A; B)$, *and conversely.*

Proof. (a) If ρ is in $\mathfrak{G}(A_0, A; B)$ then $\rho|A_2\in\mathfrak{G}(A_2, A; B)$ (Prop. 4 (c)), and then $V_A(A_1\rho)=V_A(A_1[A_2\cap H]\rho)=V_A(A_0\rho)$ is simple. Hence, $\mathfrak{G}(A_0, A; B)\subset\mathfrak{G}(A_1, A; B)\vee\mathfrak{G}(A_2, A; B)$. Conversely, if $\sigma\vee\tau$ is an isomorphism then $A_1\sigma\in\mathfrak{R}$ and $(A_2\cap H)\tau\subset H$ yield $A_0(\sigma\vee\tau)=A_1[A_2\cap H](\sigma\vee\tau)=(A_1\sigma)[(A_2\cap H)\tau]\in\mathfrak{R}$ (Prop. 4 (a)).

(b) This is an easy consequence of (a) and Prop. 1 (a).

(c) If σ is in $\mathfrak{G}(A_1, A; B)$ then $\sigma|S=\tau|S$ for some $\tau\in\mathfrak{G}(A_2, A; B)$ (Prop. 4 (c) and (d)), and then $\sigma\vee\tau\in\mathfrak{G}(A_0, A; B)$ is an extension of σ. Conbining this with (a), we obtain at once the first equality. The others will be almost evident.

(d) This can be proved by the similar method as in (c). Conversely, by (c) and Prop. 4 (d), we obtain $\mathfrak{G}(A_1, A; B)|S=(\mathfrak{G}(A_0, A; B)|A_1)|S=(\mathfrak{G}(A_0, A; B)|A_2)|S=\mathfrak{G}(A_2, A; B)|S=\mathfrak{G}(S, A; B)$.

Lemma 1. *Assume* $[A:H]_L\leqslant\aleph_0$. *Let* A_1 *and* A_2 *be f-regular intermediate rings of* A/B, *and* $S=A_1\cap A_2$.

(a) *The following conditions are equivalent*: (1) *Every compatible pair* (σ_1, σ_2) $(\sigma_i\in\mathfrak{G}(A_i, A; B))$ *has a common extension in* $\mathfrak{G}$, (2) $\mathfrak{G}(S)=\mathfrak{G}(A_2)\cdot\mathfrak{G}(A_1)$, (2′) $\mathfrak{G}(S)=\mathfrak{G}(A_1)\cdot\mathfrak{G}(A_2)$, (3) $\mathfrak{G}(A_1)|A_2=\mathfrak{G}(A_2, A; S)$, *and* (3′) $\mathfrak{G}(A_2)|A_1=\mathfrak{G}(A_1, A; S)$.

(b) *If* $V_A(S)=V_A(A_2)$ *then* A_1 *is annexable to* A_2, *and so* A *is h-Galois and locally finite over* S *and any of the conditions* (1)-(3′) *in* (a) *is equivalent to the following*: (4) $\mathfrak{G}(A_1[A_2], A; B)=\mathfrak{G}(A_1, A; B)\vee\mathfrak{G}(A_2, A; B)$.

Proof. (a) (1) $\Rightarrow$ (2): If σ is in $\mathfrak{G}(S)$ then there exists some $\tau\in\mathfrak{G}(A_1)$ such that $\tau|A_2=\sigma|A_2$. Obviously, $\sigma=(\sigma\tau^{-1})\tau$ and $\sigma\tau^{-1}\in\mathfrak{G}(A_2)$. (2) $\Rightarrow$ (3): By Prop. 5, $\mathfrak{G}(A_2, A; S)=\mathfrak{G}(S)|A_2=\mathfrak{G}(A_2)\mathfrak{G}(A_1)|A_2=\mathfrak{G}(A_1)|A_2$. (3) $\Rightarrow$ (1): Let (σ_1, σ_2) be compatible $(\sigma_i\in\mathfrak{G}(A_i, A; B))$. By Prop. 5, there exists some $\tau_1\in\mathfrak{G}$ such that $\sigma_1=\tau_1|A_1$. Since $\gamma=\sigma_2\tau_1^{-1}\in\mathfrak{G}(A_2, A; S)=\mathfrak{G}(A_1)|A_2$, $\gamma=\tau|A_2$ with some $\tau\in\mathfrak{G}(A_1)$, and so $\sigma=\tau\tau_1$ is an extension requested.

(b) Noting that $J(\mathfrak{G}(A_i))=A_i$ by Prop. 5, it follows at once $J(\mathfrak{G}(S))=S$, which means that $\mathfrak{G}(S)$ is $(*_f)$-regular. Hence, S is f-regular by Prop. 4 (e). Now, the latter assertions are easy by Props. 5 and 6 (a).

The next contains partial extensions of several results obtained in [1] and [9].

Proposition 7. *Assume $[A:H]_L \leqslant \aleph_0$. Let A_1 and A_2 be f-regular intermediate rings of A/B such that $V_A(S)=V_A(A_2)$, where $S=A_1\cap A_2$.*

(a) *If every compatible pair (σ_1, σ_2) $(\sigma_i \in \mathfrak{G}(A_i, A; B))$ has a common extension in $\mathfrak{G}$ then A_1 is linearly disjoint from A_2.*

(b) *Assume that A_1 is linearly disjoint from A_2. If $A'\in \mathfrak{R}/S$ is a subring of A_1 left finite over S then $\mathfrak{G}(S)|A'=\mathfrak{G}(A_2)|A'$.*

(c) *Assume that A_1 is left finite over S. In order that every compatible pair (σ_1, σ_2) $(\sigma_i \in \mathfrak{G}(A_i, A; B))$ have a common extension in $\mathfrak{G}$, it is necessary and sufficient that A_1 be linearly disjoint from A_2.*

(d) *Let $\mathfrak{S}$ be the group of all S-ring automorphisms of A_2. If $J(\mathfrak{S})=S$ then every compatible pair $(\sigma_1, \sigma_2)(\sigma_i \in \mathfrak{G}(A_i, A; B))$ has a common extension in $\mathfrak{G}$.*

Proof. By Lemma 1 (b), A is h-Galois and locally finite over the f-regular intermediate ring S.

(a) Let A' be an arbitrary intermediate ring of A_1/S left finite over S. Then, by Lemma 1 (a), $\mathrm{Hom}({}_SA', {}_SA)=\mathrm{Hom}({}_SA, {}_SA)|A'=\mathfrak{G}(S)A_R|A'=\mathfrak{G}(A_1)\cdot\mathfrak{G}(A_2)A_R|A'=\mathfrak{G}(A_2)A_R|A'\subset \mathrm{Hom}({}_{A_2}A, {}_{A_2}A)|A'$, which proves evidently our assertion.

(b) Since A/A_2 is h-Golois and locally finite (Prop. 5), our assumption yields $\mathrm{Hom}\,({}_SA', {}_SA)=\mathrm{Hom}\,({}_{A_2}A_2\cdot A', {}_{A_2}A)|A'=\mathfrak{G}(A_2)A_R|A'=\sum_{\sigma\in\mathfrak{G}(A_2)}(\sigma|A')A_R$. Now, let τ be an arbitrary element of $\mathfrak{G}(S)$. Then, by Prop. 1 (d), $\tau|A'=\sigma\tilde{v}|A'$ for some $\sigma\in\mathfrak{G}(A_2)$ and $v\in V_A(S)^{\cdot}$. Hence, $\tau|A'$ is contained in $\mathfrak{G}(A_2)|A'$, namely, $\mathfrak{G}(S)|A'=\mathfrak{G}(A_2)|A'$.

(c) This is an easy consequence of (a), (b) and Lemma 1 (a).

(d) We set $A_0=A_1[A_2]$. Since $S\subset A_2\subset V_A^2(S)$ and A_2/S is Galois, we obtain $\mathfrak{G}(A_0/A_1)|A_2=\mathfrak{S}=\mathfrak{G}(A_2, A; S)$ (Props. 3 (d), 4 (a) and 5). Noting that $V_A(A_0)=V_A(A_1)$, Props. 3 (d) and 5 yield $\mathfrak{G}(A_0/A_1)=\mathfrak{G}(A_1)|A_0$, whence together with the above it follows $\mathfrak{G}(A_1)|A_2=\mathfrak{G}(A_2, A; S)$. Now, our assertion is immediate by Lemma 1 (a).

Corollary 1. *Let A_1 and A_2 be intermediate rings of H/B. If one of the subrings A_1 and A_2 is Galois over $A_1\cap A_2$ then $\mathfrak{G}(A_1[A_2], A; B)=\mathfrak{G}(A_1, A; B)\vee\mathfrak{G}(A_2, A; B)$.*

Proof. By Prop. 1(c), $\mathfrak{G}(A_i, A; B)=\mathfrak{G}(A_i, H; B)$ and $\mathfrak{G}(A_1[A_2], A; B)=\mathfrak{G}(A_1[A_2], H; B)$. Now, our assertion is evident by Props. 3 (a) and 7 (d).

Lemma 2. *Let A' be in $\mathfrak{R}$, and H^* an intermediate ring of H/B. If $A'\cap H^*=A'\cap H$ then $\mathfrak{G}(A'[H^*], A; B)=\mathfrak{G}(A', A; B)\vee\mathfrak{G}(H^*, A; B)=\mathfrak{G}(A'[H^{**}], A; B)\vee\mathfrak{G}(H^*, A; B)$ for every intermediate ring H^{**} of $H^*/A'\cap H^*$.*

Proof. Although this was shown in [2], for the sake of completeness, we shall give here a shorter proof. Since $A'[H^{**}]\cap H=H^{**}=(A'[H^{**}]\cap H)\cap H^*=A'[H^{**}]\cap H^*$ by Props. 3 and 4 (a), it suffices to prove that every $\sigma\vee\tau$ in $\mathfrak{G}(A', A; B)\vee\mathfrak{G}(H^*, A; B)$ is an isomorphism (Prop. 6 (a)). By Prop. 4 (a), $\mathfrak{G}(A'[H]/A')|H=\mathfrak{G}(H/A'\cap H)=\mathfrak{G}(H/A'\cap H^*)$, and so $\mathfrak{G}(H^*, A; A'\cap H^*)=\mathfrak{G}(H/A'\cap H^*)|H^*=\mathfrak{G}(A'[H]/A')|H^*\subset\mathfrak{G}(A'[H^*], A; A')|H^*$(Props. 1 (c) and 3 (a)). Hence, $\mathfrak{G}(H^*, A; A'\cap H^*)=\mathfrak{G}(A'[H^*], A; A')|H^*$ and $\mathfrak{G}(A'[H^*], A; A')=\{(1|A')\vee\rho'; \rho'\in\mathfrak{G}(H^*, A; A'\cap H^*)\}$. We shall prove first our lemma for the case that A' and H^* are finite over B. Let N^* be a $\mathfrak{G}(H/B)$-invariant shade of H^*, and S^* in $\mathfrak{R}_{l,f}/A'[N^*]$. Then, $\tau=\tau_0|H^*$ for some $\tau_0\in\mathfrak{G}(N^*/B)$ and $\sigma=\sigma^*|A'$ for some $\sigma^*\in\mathfrak{G}(S^*, A; B)$ (Props. 1 (b) and 3). By Props. 1 (c), 3 (a) and the remark stated at the beginning of this proof, we obtain $\mathfrak{G}(A'[H^*], A; A')\sigma^*|H^*=\mathfrak{G}(H^*, A; A'\cap H^*)\sigma^*\subset\mathfrak{G}(H^*, A; A'\cap H^*)\tau_0$. Since $\sharp\mathfrak{G}(H^*, A; A'\cap H^*)<\infty$ by Prop. 3, we have $\mathfrak{G}(A'[H^*], A; A')\sigma^*|H^*=\mathfrak{G}(H^*, A; A'\cap H^*)\tau_0\ni\tau$. Hence, $\tau=\tau'\sigma^*=((1|A')\vee\tau')\sigma^*|H^*$ for some $\tau'\in\mathfrak{G}(H^*, A; A'\cap H^*)$, whence it follows $\sigma\vee\tau=((1|A')\vee\tau')\sigma^*\in\Gamma(A'[H^*], A)$. We shall proceed now in the general case. Let $A''\in\mathfrak{R}_{l,f}$ be a subring of A', H^{**} an intermediate ring of H^*/B with $[H^{**}: B]<\infty$, and $H^{*\prime}$ an intermediate ring of $H^*/H^{**}[A''\cap H]$ with $[H^{*\prime}: B]<\infty$. By Prop. 4 (c), $\sigma|A''\in\mathfrak{G}(A'', A; B)$ and $\tau|H^{*\prime}\in\mathfrak{G}(H^{*\prime}, A; B)$. Since $\sigma|A''$ and $\tau|H^{*\prime}$ are compatible and $A''\cap H^{*\prime}=A''\cap H$, the first step implies $(\sigma|A'')\vee(\tau|H^{*\prime})\in\mathfrak{G}(A''[H^{*\prime}], A; B)$. Hence, we obtain $(\sigma|A'')\vee(\tau|H^{**})=(\sigma|A'')\vee(\tau|H^{*\prime})|A''[H^{**}]\in\Gamma(A''[H^{**}], A)$. Our assertion is therefore a consequence of Prop. 6 (b).

Corollary 2. *Let A' be in $\mathfrak{R}$, and H^* an intermediate ring of H/B. If one of the subrings H^* and $A'\cap H$ is Galois over $A'\cap H^*$ then* $\mathfrak{G}(A'[H^*], A; B)=\mathfrak{G}(A', A; B)\vee\mathfrak{G}(H^*, A; B)$.

Proof. Let $\sigma\vee\tau$ be in $\mathfrak{G}(A', A; B)\vee\mathfrak{G}(H^*, A; B)$. Since $\sigma|A'\cap H\in\mathfrak{G}(A'\cap H, A; B)$ by Prop. 4 (c), there holds $(\sigma|A'\cap H)\vee\tau\in\mathfrak{G}(A'\cap H, A; B)\vee\mathfrak{G}(H^*, A; B)=\mathfrak{G}((A'\cap H)[H^*], A; B)$ (Cor. 1). Moreover, noting that $A'\cap(A'\cap H)[H^*]=A'\cap H$, we obtain $\mathfrak{G}(A'[H^*], A; B)=\mathfrak{G}(A', A; B)\vee\mathfrak{G}((A'\cap H)[H^*], A; B)$ (Lemma 2). Hence, $\sigma\vee((\sigma|A'\cap H)\vee\tau)$ is in $\mathfrak{G}(A'[H^*], A; B)$ and coincides with $\sigma\vee\tau$, and then our assertion is clear by Prop. 6 (a).

Now, we can prove the following:

Theorem 1. *Let $A_1\in\mathfrak{R}$ be annexable to an f-regular intermediate ring A_2 of A/B. If one of the subrings A_2 and $A_1\cap V_A^2(A_2)$ is Galois over $S=A_1\cap A_2$ then* $\mathfrak{G}(A_0, A; B)=\mathfrak{G}(A_1, A; B)\vee\mathfrak{G}(A_2, A; B)$, $\mathfrak{G}(A_0, A; B)|A_1=\mathfrak{G}(A_1, A; B)$, $\mathfrak{G}(A_0, A; A_2)|A_1=\mathfrak{G}(A_1, A; S)$ and $\mathfrak{G}(A_0, A; A_1)|A_2=\mathfrak{G}(A_2,$

$A;S)$, *where* $A_0=A_1[A_2]$.

Proof. We set $H_2=A_2\cap H$. If A_2/S is Galois, then so is $H_2/S\cap H$ (Prop. 1 (c)). On the other hand, if $A_1\cap V_A^2(S)/S$ is Galois then $\mathfrak{G}(A_1\cap V_A^2(S)/S)=\mathfrak{G}(V_A^2(S)/S)|A_1\cap V_A^2(S)$ by Prop. 3, and so by Pror. 4 (a) $\mathfrak{G}(A_1\cap V_A^2(S)/S)|A_1\cap H=\mathfrak{G}(H/S\cap H)|A_1\cap H$, whence we obtain $(A_1\cap H)\mathfrak{G}(A_1\cap V_A^2(S)/S)\subset(A_1\cap V_A^2(S))\cap H=A_1\cap H$, which implies that $A_1\cap H/S\cap H$ is Galois. Hence, noting that $A_1\cap H_2=S\cap H$, Cor. 2 yields $\mathfrak{G}(A_0, A;B)=\mathfrak{G}(A_1[H_2], A;B)=\mathfrak{G}(A_1, A;B)\vee\mathfrak{G}(H_2, A;B)$. Now, let $\sigma\vee\tau$ be in $\mathfrak{G}(A_1, A;B)\vee\mathfrak{G}(A_2, A;B)$. Then, $\sigma\vee(\tau|H_2)$ is in $\mathfrak{G}(A_0, A;B)$ and coincides with $\sigma\vee\tau$. Hence, we obtain our first assertion (Prop. 6 (a)). Accordingly, the others are valid by Prop. 6 (c).

Lemma 3. *Let* $B'\in\mathfrak{R}_{l.f}$. *If* $s=[B'\cap H:B]$ *and* $t=[V:V_A(B')]$ *then* $\mathfrak{G}(B', A;B)=\bigcup_{i=1}^s \sigma_i\widetilde{V}$ *(direct union) and* $\sigma_i\widetilde{V}A_R=\bigoplus_{j=1}^t \sigma\tilde{v}_{ij}A_R$ *with some* $v_{ij}\in V^\cdot$.

Proof. At first, we shall prove $\mathfrak{G}(B', A;B'\cap H)=\widetilde{V}|B'$. Let B^* be in $\mathfrak{R}_{l.f}^0/B'$, and $H^*=B^*\cap H$. Then, $B'\cap H^*=B'\cap H$. Hence, by Prop. 1 (b) and the proof of Lemma 2, we have $\mathfrak{G}(B^*, A;B')|B'[H^*]=\mathfrak{G}(B'[H^*], A;B')=\{(1|B')\vee\tau;\ \tau\in\mathfrak{G}(H^*, A;B'\cap H^*)\}$. Accordingly, $\mathfrak{G}(B'[H^*], A;B')=\{\tau_1^*|B'[H^*],\cdots,\tau_{s^*}^*|B'[H^*]\}$, where $\tau_i^*\in\mathfrak{G}(B^*, A;B')$ and $s^*=\#\mathfrak{G}(H^*, A;B'\cap H^*)=[H^*:B'\cap H^*]$ (Prop. 3). Recalling here that $[B^*:H^*]=[V:V_A(B^*)]=[V:V_A(B^*\tau_i^*)]$ (Prop. 4 (b)) and $[\tau_i^*\widetilde{V}A_R:A_R]_R=[V:V_A(B^*\tau_i^*)]$, we readily obtain $\mathfrak{G}(B^*, A;B'\cap H)=\mathfrak{G}(B^*, A;B'\cap H^*)=\bigcup_1^{s^*}\tau_i^*\widetilde{V}$ (Prop. 1 (b)). It follows therefore $\mathfrak{G}(B', A;B'\cap H)=\mathfrak{G}(B^*, A;B'\cap H)|B'=\bigcup_1^{s^*}\tau_i^*\widetilde{V}|B'=\widetilde{V}|B'$ (Prop. 1 (b)), Now, we shall prove our assertion. If σ, ρ are in $\mathfrak{G}(B', A;B)$ and $\sigma|B'\cap H=\rho|B'\cap H$ then $\rho=\sigma\varepsilon$ with some $\varepsilon\in\mathfrak{G}(B'\sigma, A;(B'\cap H)\sigma)$. Since $(B'\cap H)\sigma=B'\sigma\cap H$ (Prop. 1 (c)), the proposition stated at the beginning proves $\varepsilon=\tilde{v}|B'\sigma$. We have seen thus $\mathfrak{G}(B', A;B)=\bigcup_1^s \sigma_i\widetilde{V}$ (direct union), where $\mathfrak{G}(B'\cap H, A;B)=\{\sigma_1|B'\cap H,\cdots,\sigma_s|B'\cap H\}$ (Prop. 3). If $\{v_{i1},\cdots,v_{it}\}\subset V^\cdot$ is a right free $V_A(B'\sigma_i)$-basis of V (cf. Prop. 4 (b)), then we readily obtain $\sigma_i\widetilde{V}A_R=\sum_{j=1}^t \sigma_i\tilde{v}_{ij}A_R$ and Hom $({}_BB', {}_BA)=\mathfrak{G}(B', A;B)A_R=\sum_{i,j}\sigma_i\tilde{v}_{ij}A_R$. Combining this with $s\cdot t=s\cdot[B':B'\cap H]=[B':B]$ (Prop. 4 (b)), we obtain at once Hom $({}_BB', {}_BA)=\bigoplus_{i=1}^s\bigoplus_{j=1}^t\sigma_i\tilde{v}_{ij}A_R$.

Proposition 8. *Assume that* $A'\in\mathfrak{R}$ *is Galois over B. Then,* $\mathfrak{G}(B', A;B)=\mathfrak{G}(A', A;B)|B'=\mathfrak{G}(A'/B)\widetilde{V}|B'$ *for every* $B'\in\mathfrak{R}_{l.f}$ *contained in* A', *and* $\mathfrak{G}(A'/B)\widetilde{V}$ *is dense in* $\mathfrak{G}(A', A; B)$ (*in the finite topology*). *In particular, if* A' *contains* V *then* A'/B *is h-Galois,* $\mathfrak{G}(A'/B)$ *is dense in* $\mathfrak{G}(A', A; B)$ *and* $\mathfrak{G}(A', A; B) = \mathfrak{G}(A', A'; B)$.

Proof. Obviously, $A'\cap H$ is outer Galois over B and $\mathfrak{G}(A'/B)|A'\cap H$ is dense in $\mathfrak{G}(A'\cap H/B)$ by Props. 1 (c) and 3. Accordingliy, noting that $A'\cap H$ is $\mathfrak{G}(H/B)$-invariant (Prop. 3 (d)), we obtain $\mathfrak{G}(B', A; B)|B'\cap H = \mathfrak{G}(B'\cap H, A; B) = \mathfrak{G}(H/B)|B'\cap H = (\mathfrak{G}(H/B)|A'\cap H)|B'\cap H = \mathfrak{G}(A'\cap H/B)|B'\cap H = (\mathfrak{G}(A'/B)|A'\cap H)|B'\cap H = \mathfrak{G}(A'/B)|B'\cap H$ (Props. 1 (b) and 3). Hence, $\mathfrak{G}(B', A; B) = \mathfrak{G}(A'/B)\widetilde{V}|B' = \mathfrak{G}(A', A; B)|B'$ by Lemma 3 and Prop. 4 (c), and so $\mathfrak{G}(A'/B)\widetilde{V}$ is dense in $\mathfrak{G}(A', A; B)$ by Prop. 1 (a). Next, assume that $A'\supset V$. Since A is $B\cdot V$-A-irreducible (Prop. 2 (a)), one will easily see that A' is $B\cdot V$-A'-irreducible. Hence, A'/B is h-Calois by Prop. 2 (b). Now, the latter assertion will be obvious by the former.

Finally, we shall prove the following:

Theorem 1. *Let* $A_1\in\mathfrak{R}$ *be annexable to an f-regular intermediate ring* A_2 *of* A/B, $A_0 = A_1[A_2]$, *and* $S = A_1\cap A_2$.

(a) *Assume that one of the subrings* A_2 *and* $A_1\cap V_A^2(S)$ *is Galois over S. Tnen, the contraction maps* $\varphi: \bar{\sigma} \longrightarrow \bar{\sigma}|A_1$ *of* $\mathfrak{G}(A_0, A; A_2)$ and $\psi: \bar{\tau} \longrightarrow \bar{\tau}|A_2$ *of* $\mathfrak{G}(A_0, A; A_1)$ *are* 1–1 *onto* $\mathfrak{G}(A_1, A; S)$ *and* 1–1 *onto* $\mathfrak{G}(A_2, A; S)$, *respectively, and* $J(\mathfrak{G}(A_0, A; A_1)) = A_1$. *If* $\mathfrak{G}(A_1, A; S)|A_1' = \mathfrak{G}(A_1', A; S)$ *for every* $A_1'\in\mathfrak{R}/S$ *contained in* A_1 *such that* $[A_1':S]_L < \infty$, *then* $J(\mathfrak{G}(A_0, A; A_2)) = A_2$.

(b) *If* A_1 *is Galois over S and contains* $V_A(S)$ *then* $A_1\cap V_A^2(S)/S$ *is outer Galois,* A_1/A_2 *is h-Golois and* φ *induces an equivalence between* $\mathfrak{G}(A_0/A_2)$ *and* $\mathfrak{G}(A_1/S)$.

(c) *If* A_1 *and* A_2 *are Galois over B and* A_1 *contains V that* A_0/B *is h-Galois and* $\mathfrak{G}(A_0/B) = \mathfrak{G}(A_1/B) \vee \mathfrak{G}(A_2\cap H/B)$.

(d) *If* A_1 *and* A_2 *are Galois over S and* A_1 *contains* $V_A(S)$ *then* A_0/S *is h-Galois and* $\mathfrak{G}(A_0/S) = \mathfrak{G}(A_1/S)\vee\mathfrak{G}(A_2/S)$ *is equivalent to the direct product* $\mathfrak{G}(A_1/S)\times\mathfrak{G}(A_2/S)$.

Proof. (a) By Th. 1, φ and ψ are evidently onto and 1–1. Next, we shall prove the last part. By Prop. 4 (c), A/S is q-Galois and locally finite. Hence, A_1 contains a subring $U\in\mathfrak{R}/S$ left finite over S such that $U[F]\in\mathfrak{R}$ for every finite subset F of A_1 (Prop. 1 (a)). If a is in A_0/A_2 then we can find regular intermediate rings A_i' of A_i/S left finite over S such that $a\in A_0' = A_1'[A_2']$. Obviously, A_1' is annexable to A_2', and then by Prop. 1 (b), we have $J(\mathfrak{G}(A_0', A; A_2')) = A_2'$. Accordingly, $a\rho'\neq a$ for some $\rho'\in\mathfrak{G}(A_0', A; A_2')$. Since $\rho'|A_1'\in\mathfrak{G}(A_1', A; S)$ by Prop. 1 (b), our assumption implies that $\rho'|A_1' = \rho|A_1'$ for some $\rho\in\mathfrak{G}(A_1, A; S)$. Then, $\bar{\rho} = \rho\vee$

$(1|A_2)$ is in $\mathfrak{G}(A_0, A; A_2)$ (Th. 1) and $a\bar{\rho}=a((\rho|A_1')\vee(1|A_2'))=a\rho'\neq a$, which means $J(\mathfrak{G}(A_0, A; A_2))=A_2$. Finally, the validity of Prop. 4 (d) enables us to apply a similar argument to prove the remainder.

(b) Since A/S is q-Galois and locally finite, $A_1\cap V_A^2(S)/S$ is Galois (Props. 1 (c) and 3) and then φ is 1-1 and onto $\mathfrak{G}(A_1, A; S)$ by (a). Moreover, we obtain $\mathfrak{G}(A_0, A; A_2)=\{\sigma\vee(1|A_2);\ \sigma\in\mathfrak{G}(A_1, A; S)\}$ (Th. 1) and $\mathfrak{G}(A_1, A; S)=\mathfrak{G}(A_1, A_1; S)$ (Prop. 8). Hence, we see that an element $\bar{\rho}$ in $\mathfrak{G}(A_0, A; A_2)$ is an automorphism if and only if so is $\bar{\rho}|A_1$. If $A_1'\in\mathfrak{R}/S$ is a subring of A_1 left finite over S then $\mathfrak{G}(A_1', A; S)=\mathfrak{G}(A_1, A; S)|A_1'=\mathfrak{G}(A_1/S)|A_1'$(Prop. 8). Hence, for every $a\in A_0\backslash A_2$ we can find some $\bar{\rho}\in\mathfrak{G}(A_0, A; A_2)$ such that $\bar{\rho}|A_1\in\mathfrak{G}(A_1/S)$ and $a\bar{\rho}\neq a$ (cf. the proof of (a)). Then, $\bar{\rho}$is an automorphism by the above remark, which means that A_0/A_2 is Galois. Hence, A_0/A_2 and A_1/S are h-Galois by Props. 4 (c) and 8. Finally, the equivalence will be easily seen.

(c) If $H_2=A_2\cap H$ then $A_2=S[H_2]$ and $A_0=A_1[H_2]$. Accordingly, noting that H_2/B is Galois (Prop. 1 (c)), we may assume from the beginning that A_2 is contained in H. By Th. 1, $\mathfrak{G}(A_0, A; B)=\mathfrak{G}(A_1, A; B)\vee\mathfrak{G}(A_2, A; B)$. From $\mathfrak{G}(A_2, A; B)=\mathfrak{G}(H/B)|A_2=\mathfrak{G}(A_2/B)$ (Props. 1 (c) and 3) and $\mathfrak{G}(A_1, A; B)=\mathfrak{G}(A_1, A_1; B)$ (Prop. 8), it follows then $\mathfrak{G}(A_0, A; B)=\mathfrak{G}(A_1, A_1; B)\vee\mathfrak{G}(A_2/B)$. Hence, $\bar{\rho}\in\mathfrak{G}(A_0, A; B)$ is an automorphism if and only if $\bar{\rho}|A_1$ is an automorphism. Therefore, $\mathfrak{H}=\mathfrak{G}(A_1/B)\vee\mathfrak{G}(A_2/B)$ is the group of all B-ring automorphisms of A_0. Since $\mathfrak{G}(A_1/B)|S\subset\mathfrak{G}(S, H; B)=\mathfrak{G}(A_2/B)|S$ by Props. 1 (c) and 3, we have $\mathfrak{H}|A_1=\mathfrak{G}(A_1/B)$. Accordingly, noting that A_0/A_1 is Galois by Prop. 4 (a), we readily see that $J(\mathfrak{H})=B$, namely, A_0/B is (Galois and so) h-Galois (Prop. 8) and $\mathfrak{H}=\mathfrak{G}(A_0/B)$.

(d) Since A/S is q-Galois and locally finite, (d) is an easy consequence of (c).

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