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ExtA(Z2[y]/Z2,Z2), A being the mod 2 Steenrod algebra

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$\mathrm{Ext}_A(Z_2[y]/Z_2, Z_2)$, A BEING THE mod 2 STEENROD ALGEBRA

Dedicated to Professor Atuo Komatu on his 60th birthday

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§ 1. Introduction.

Let A be the mod 2 Steenrod algebra, Z be a ring of integers, $Z_m = Z/mZ$, (m: a positive integer). Let $\underline{M}^k$ be a polynomial algegra with one generator y of degree 2^k, (k: a non-negative integer) over Z_2. Let x be the generator of $\underline{M}^0$. Let M_i^k be a A-submodule of $\underline{M}^k$ generated by y^q, $q \geqq i$, and $M_{i,p}^k$ be a quotient module of M_i^k by an A-submodule generated by y^q, $p<q$. Particularly we denote $M^k = M_1^k$, $M_i = M_i^0$, $M_{i,p} = M_{i,p}^0$.

Let RP^i, CP^i, HP^i be i-dimensional real (complex, quaternion) projective space, respectively. Then reduced cohomology groups of them with coefficient group Z_2 are

$$\mathrm{H}^*(RP^\infty) = M^0, \ \mathrm{H}^*(CP^\infty) = M^1, \ \mathrm{H}^*(HP^\infty) = M^2.$$

There is no space such that $\mathrm{H}^*(X) = M^k$, for $k \geqq 3$. (see [10] Chapter 1, Theorem 4.5; [3] Theorem 4. 6. 1.) But we can naturally make M^k (and M_i^k) a left A-module algebraically such that the axiom [10] by Steenrod hold on M^k. This definition has no contradiction since in the case $k=0$ there is a space $X (=RP^\infty)$ such that $\mathrm{H}^*(X) = M^0$ and since we have Proposition 2. 3 in this paper.

The determination of $\mathrm{Ext}_A(M_i^k, Z_2)$, $k = 0, 1, 2$, is used to the determination of 2-primary components of stable homotopy of S^0 to RP^∞/PP^{i-1}, CP^∞/CP^{i-1}, HP^∞/HP^{i-1}, respectively by [1] and [2].

After the author determined $\mathrm{Ext}_A^{s,t}(M^0, Z_2)$, $t-s \leqq 27$, the author fined that M. Mahowald [6] had determined $\mathrm{Ext}_A^{s,t}(M_i^0, Z_2)$, $t-s \leqq 29$, by his own method different to my method. Since his representation of generators is different to mine and the relationship between generators of $\mathrm{Ext}_A(M^k, Z_2)$ and those of $\mathrm{Ext}_A(Z_2, Z_2)$ is more clear at a glance by my method, we will offer the table of $\mathrm{Ext}_A(M^0, Z_2)$ in the last of this parer, for reference. So the main purpose of this paper is the determination of generators and relations of $\mathrm{Ext}_A^{s,t}(M_i^k, Z_2)$ in general (without restriction on $t-s$).

We conjecture by our table of $\mathrm{Ext}_A(M^0, Z_2)$, $t-s \leqq 27$, that $\mathrm{Ext}_A(Z_2, Z_2)/Z_2[h_0]$ *is isomorphic to a direct summand of* $\mathrm{Ext}_A(M^0, Z_2)$ *by an appropriate correspondence.* But we have not finded the effective method to prove this.

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§ 2. $\mathrm{Ext}_A^0(M_i^k, Z_2)$.

If 2-adic expansion of a positive integer i is

(2.1) $\quad i=2^{i_1}+\cdots\cdots+2^{i_m},\ i_1>\cdots\cdots>i_m\geqq 0,$

then we define *2-th set* $[i]$ and *2-th number* $\sharp[i]$ of i in the following;

$$[i]=\{i_1, \cdots\cdots, i_m\},\ \sharp[i]=m.$$

If 2-adic expansion of another positive integer j is

(2.2) $\quad j=2^{j_1}+\cdots\cdots+2^{j_n},\ j_1>\cdots\cdots>j_n\geqq 0,$

then $[i]\geqq[j]$ means the condition that

$$m\geqq n,\ i_1\geqq j_{m-n+1}, \cdots\cdots, i_m\geqq j_n:$$

or

$$m\leqq n,\ i_{n-m+1}\geqq j_1, \cdots\cdots, i_m\geqq j_n.$$

The following lemma on binomial coeficients is an alternative representation of Lemma 2.6 Chapter 1 in [10] and plays an essential part of proving many propositions and lemmas in this paper.

Lemma 2.1

$$\binom{i}{j}\equiv\begin{cases}1, & [j]\subset[i]\\ 0, & [j]\not\subset[i]\end{cases} \qquad (\mathrm{mod}\ 2)$$

Remark. $[j]\subset[i]$ means that the set $[j]$ is contained in the set $[i]$.

Proposition 2.2

If $k\geqq 0$, $a<2b$, *then as operations on* M_i^k,

$$\mathrm{Sq}^{2^k a}\mathrm{Sq}^{2^k b}=\sum_{t=0}^{[\frac{a}{2}]}\binom{b-t-1}{a-2t}\mathrm{Sq}^{2^k(a+b-t)}\mathrm{Sq}^{2^k t}$$

where $\left[\frac{a}{2}\right]$ *stands for the maximal integer which does not exceed* i *only in this proposttion.*

Proof. By the following equality and congruence:

$$\mathrm{Sq}^{2^k a}\mathrm{Sq}^{2^k b}=\sum\binom{2^k b-2^k t-1}{2^k a-2^{k+1}t}\mathrm{Sq}^{2^k(a+b-t)}\,\mathrm{Sq}^{2^k t}$$

$$\binom{2^k b-2^k t-1}{2^k a-2^{k+1}t}=\binom{2^k(b-t-1)+2^k-1}{2^k(a-2t)}\equiv\binom{2^k(b-t-1)}{2^k(a-2t)}\equiv\binom{b-t-1}{a-2t} \quad (\mathrm{mod}\ 2)$$

Proposition 2.3

$$\mathrm{Sq}^j y^i=\begin{cases}y^{j'+i}, & j=2^k j' \textit{ and } [j']\subset[i]\\ 0, & \textit{otherwise.}\end{cases}$$

Proof. By Cartan formula. This proposition is a generalization of Lemma 2.4 Chapter 1 in [10].

Theorem 2.4

$A\cdot y^j = Z_2\{y^i; i\geqq j,\ [i]\geqq[j],\ \sharp[i]\leqq\sharp[j]\}$, where $Z_2\{a: \mathrm{C}(a)\}$ means a Z_2-module generated by a satisfying the condition $\mathrm{C}(a)$.

Proof. We denote by C the right hand side of the equality to prove. By Proposition 2.2 and 2.3, it is sufficient to prove this Theorem in the case $k=0$. Since $A\cdot x^j\subset C$ easily follows from Proposition 2.3, we will only prove $A\cdot x^j\supset C$.

If e is a positive integer such that

$$(2.3)\qquad e>j,\ [e]\geqq[j],\ \sharp[e]\leqq\sharp[i],$$

then we denote

$$B=Z_2\{x^i\in C;\ e>i\}.$$

We will show by induction on e that if $B\subset A\cdot x^j$, then

$$x^e=a\cdot x^i,\ \text{for some } a\in A,\ x^i\in B,\ a\neq 1.$$

If 2-adic expansion of e is

$$e=2^{e_1}+\cdots\cdots+2^{e_q},\ e_1>\cdots\cdots>e_q\geqq 0$$

and that of i, j is the same as in (2.1) and (2.2), then by $j<e$, there is an integer a satisfying conditions (2.4) and, either (2.5), (2.6), (2.7) or (2.8):

$$(2.4)\qquad j_1=e_1,\ \cdots\cdots,\ j_{a-1}=e_{a-1},\ j_a<e_a,\ 0\leqq a\leqq q;$$

$$(2.5)\qquad e_u=e_{u+1}+1,\ a\leqq u<b<q,\ e_b>e_{b+1}+1,\ \text{for some } b;$$

$$(2.6)\qquad e_u=e_{u+1}+1,\ a\leqq u<q;$$

$$(2.7)\qquad e_a=e_{a+1}+1;$$

$$(2.8)\qquad a=q.$$

In all cases, $[e]\geqq[j]$ implies $e_b\geqq j_{n-q+b}$. We have $e_b>j_{n-q+b}$. (If $e_b=j_{n-q+b}$, then by $e_a>j_a$

$$\sharp\left[\textstyle\sum_{u=a}^{a} 2^{e_u}\right]>\sharp\left[\textstyle\sum_{u=a}^{u-q+b} 2^{j_u}\right].$$

Therefore $\sharp[e]>\sharp[j]$. This is contrary to (2.3).)

In the case (2.5), set $e'=2^{e_b-1}$, then (2.4) implies $e-e'>j$, $e_b>j_{n-q+b}$ implies $[e-e']\geqq[j]$, and we have

$$\sharp[e-e']=\sharp[e]\leqq\sharp[j].$$

Therefore by the inductive hypothesis,

$$x^e=\mathrm{Sq}^{e'}x^{e-e'},\ x^{e-e'}\in B.$$

In the cases (2.6), (2.8), we have $e_q>0$.

(If otherwise, then we have not both $e>j$, and $\#[e]\leqq\#[j]$.)
The proof in the case is the same as in the case (2.5), after replacing b with q.

In the case (2.7) or (2.8), if $e_a>j_a+1$, then the proof is similar to that in the case (2.5), after replacing b with a.

In the case (2.5), set $e'=2^{e_b-1}$ then (2.4) implies $e-e'>j$, $e_b>j_{n-q+b}$ implies $[e-e']\geqq[j]$, and we have

$$\#[e-e']=\#[e]\leqq\#[j].$$

Therefore by the inductive hypothesis,

$$x^e=\mathrm{Sq}^{e'}x^{e-e'},\ x^{e-e'}\in B.$$

In the case (2.7), if

$$e_a=j_a+1\,;\ j_u=j_{u+1}+1,\ a\leqq u<n \ \text{ or } \ a=e=q,$$

then $\#[e]\leqq\#[j]$ implies $[j]-[e]\neq\phi$ and we take

$$c=min([j]-[e]),$$

where signature "—" means a subtraction of two tets $[j]$ and $[e]$. Take

$$e'=2^c+\textstyle\sum_{u=a+1}^{q}2^{e_u}.$$

Clearly $[e']\subset[j]$, and $x^e=\mathrm{Sq}^{e'}x^j$, $x^j\in B$.

In the case (2.7), let

$$e_a=j_a+1\,;\ j_u=j_{u+1}+1,\ a\leqq u<b,\ j_b>j_{b+1}+1, \ \text{ or } \ j_a>j_{a+1}+1$$

(If $j_a>j_{a+1}+1$, take $b=a$.)
If $e_{a+1}=j_b-1$, then we take

$$c=min\ (\{j_a,\ \cdots\cdots,\ j_b,\ j_b-1\}-[e]),$$

$$d=min\ \{u\,;\ e_u\geqq j_b-1\},$$

$$e'=2^c+\textstyle\sum_{u=a+1}^{d}2^{e_u},$$

$$j'=j+2^{j_b-1}-2^{j_{b+1}}.$$

$$j'>j,\ [j']\geqq[j],\ \#[j']=\#[j]$$

implies by the inductive hypothesis

$$x^{j'}=gx^u\in C,\ \text{for some } g\in A,\ x^u\in B. \tag{2.9}$$

Clearly $[e']\subset[j']$, and

$$x^e=\mathrm{Sq}^{e'}x^{j'}.$$

If $e_{a+1}<j_b-1$, then we have (2.9). Clearly $j_b-1\in[j']$, and

$$x^e=\mathrm{Sq}^{2^{j_b-1}}x^{j'}.$$

In the case (2.8), if

$$e_a=j_a+1\,;\ j_u=j_{u+1}+1,\ a\leqq u<n \ \text{ or } \ a=q=n,$$

then $j_n\in[j]$ implies

$$x^e = \mathrm{Sq}^{j'} x^j,\ x^j \in B.$$

(We mrite $j'=2^{j_n}$. In this case $j=e-j'$.)

In the case (2.8), if, for some b,

$$e_a = j_a + 1,\ j_u = j_{u+1}+1,\ a \leqq u < b < n,\ j_b > j_{b+1}+1,$$

and we take $j' = 2^{j_b - 1}$, then

$$j_b - 1 \in [e-j']$$

implies

$$x^e = \mathrm{Sq}^{j'} x^{e-j'},\ x^{e-j'},\ \in B$$

[Q. E. D. of Theorem 2.4]

For the next theorem we give the following notation:

If i is such an integer as (2.1), and u is such an integer that $i_v > u > i_{v+1}$, or $i_m > u \geqq 0$, we define (in the last case, we set $v=m$)

$$(i, u) = 2^{i_1} + \cdots\cdots + 2^{i_v} + 2^u - 1.$$

Theorem 2.5

(1) *If* $2^j - 1 < i \leqq 2^{j+1} - 1$, *then*

$$\mathrm{Ext}_A^0(M_{i,p}^k, Z_2) = Z_2\{\underline{h}_u,\ max[p] \geqq u > j+k\ ;$$
$$h_{i,u},\ j > u \geqq 0,\ u \notin [i],\ p \geqq (i, u)\}$$

$$\mathrm{Ext}_A^0(M_i^k, Z_2) = Z_2\{\underline{h}_u,\ u > j+k\ ;\ h_{i,u},\ j > u \geqq 0,\ u \notin [i]\}$$

(2) *In the particular case* $i = 2^j - 1$,

$$\mathrm{Ext}_A^0(M_i^k, Z_2) = Z_2\{\underline{h}_u,\ u \geqq j+k\}.$$

Remark. where $Z_2\{a; \mathrm{C}(a)\}$ stand for a Z_2-free module generated by a satisfying the condition $\mathrm{C}(a)$; $\underline{h}_u$, $h_{i,u}$ stands for the cohomology classes of $[\ ]\ y_{2^{u-k}-1}$, $[\ ]\ y_{(i,u)}$ of degrees $2^u - 2^k$, (i, u) in the cobar construction $\overline{\mathrm{F}}(A^*, M_{i,p}^{k*})$ of $M_{i,p}^{k*}$ over A^*, and y_u stands for the element in $M_{i,p}^{k*}$ dual to y^u in $M_{i,p}^k$.

Proof. By Proposition 2.3, it is sufficient to prove the proposition in the case $k=0$ and $p=\infty$.

First we show that

$$x^{2^u - 1},\ u > j\ ;\ x^{(i,u)},\ j > u \geqq 0,\ u \notin [i],$$

generate M_i as a left A-module. If $2^j - 1 < e < 2^{j+1} - 1$, then

$$\#[e] = j = \#[2^j - 1],\ [e] \geqq [2^j - 1],$$

so by Theorem 2.4,

$$x^e = a \cdot x^{2^j - 1},\ \text{for some } a \in A.$$

If $i < e < 2^{j+1} - 1$, and e cannot be expressed in the form of (i, u), we denote

$$u=max([e]-[i]).$$

Clearly $u\notin[i]$, $e>i, u$, and $e\neq(i, u)$ implies $\{u-1, \cdots\cdots, 1, 0\}-[e]\neq\phi$. Therefore

$$[e]\geqq[(i, u)],\ \sharp[e]\leqq\sharp([i, u)].$$

Then by Theorem 2.4,

$$x^e=a\cdot x^{(i,u)},\ \text{for some } a\in A.$$

Secondly we show that (2. 10) is a minimal generating set of M_i as a left A-module. If $u>v>j$, then

$$\sharp[2^u-1]=u>v=\sharp[2^v-1],\ 2^u-1>2^v-1$$

implies by Theorem 2.4 that x^{2^u-1} and x^{2^v-1} are linearly independent. If

$$j>u>v\geqq0,\ u\notin[i],\ v\notin[i],$$

then

$$\sharp[(i, u)]-\sharp[(i, v)]=\sharp[2^u-1]-\sharp[2^v-1]=v-v>0,$$
$$(i, u)-(i, v)=(2^u-1)-(2^v-1)>0,$$

implies by Theorem 2.4 that $x^{(i,u)}$ and $x^{(i,v)}$ are linearly independent. If

$$u>j>v\geqq0,\ v\notin[i],$$

then

$$2^u-1>2^{j+1}-1>(i, v)$$
$$\sharp[2^u-1]=u>j>\sharp[(i, v)]$$

implies by Theorem 2.4 That x^{2^u-1} and $x^{(i,u)}$ are linearly independent. Thus the proof is completed.

For the next alternative Proof of Theorem 2.5 (2), we give the following definition.

Definition 2.6

We define A-maps

$$f_k:\ \bar{A}\longrightarrow M^k,\ \bar{A}=A/Z_2,$$
$$\underline{f}_k:\ \bar{A}\longrightarrow \underline{M}^k$$

for an admissible monomial $\mathrm{Sq}^{i_1}\mathrm{Sq}^{i_2}\cdots\cdots\mathrm{Sq}^{i_n}$ *in the following*;

$$f_k(\mathrm{Sq}^{i_1}\cdots\cdots\mathrm{Sq}^{i_n})=\begin{cases} y^{i-1},\ i_1=2^k i,\ i\geqq2,\ n=1, \\ 0,\ \textit{otherwise.}\end{cases}$$

$$\underline{f}_k(\mathrm{Sq}^{i_1}\cdots\cdots\mathrm{Sq}^{i_n})=\begin{cases} y^{i-1},\ i_1=2^k i,\ i\geqq1,\ n=1, \\ 0,\ \textit{otherwise.}\end{cases}$$

We denote

$$f=f_0,\ \underline{f}=\underline{f}_0,\ L^k=\ker f_k,\ K=\ker \underline{f}.$$

Remark. Adem relations ensure that f_k and $\underline{f}_k$ are A-maps in the following; If $0<i<2j$, then

(We show only in the case $k=0$ by Proposition 2.2 and 2.3.)

$$\underline{f}(\mathrm{Sq}^i\mathrm{Sq}^j)=\sum\binom{j-t-1}{i-2t}\underline{f}(\mathrm{Sq}^{i+j-t}\mathrm{Sq}^t)$$
$$=\binom{j-1}{i}\underline{f}(\mathrm{Sq}^{i+j})=\binom{j-1}{i}x^{i+j-1},$$
$$\mathrm{Sq}^i\underline{f}(\mathrm{Sq}^j)=\mathrm{Sq}^i x^{j-1}=\binom{j-1}{i}x^{i+j-1}.$$

[Alternative proof of (2) in Proposision 2.5]

Since f_k is an A-map, and by Lemma 4.2, Chapter 1 in [10], and Sq^i is indecomposable if and only if i is a power of 2, if $y^i\notin A\cdot M^k$, then $i=2^u-1$, for some $u\geqq k$.

If $y^{2^u-1}=a\cdot y^j$, for some j and $a\in A$, then

$$f_k(\mathrm{Sq}^{2^{u+k}})=y^{2^u-1}=a\cdot y^j=a\cdot f_k(\mathrm{Sq}^{2^k(j+1)})=f_k(a\cdot\mathrm{Sq}^{2^k(j+1)})$$

Therefore

$$\mathrm{Sq}^{2^{u+k}}+a\cdot\mathrm{Sq}^{2^k(j+1)}=b,\ \text{for some } b\in\ker f_k$$

This is contrary to the fact that $\mathrm{Sq}^{2^{u+k}}$ is indecoposable.

§ 3. Relations in $\mathrm{Ext}_A(M_i^k, Z_2)$.

We determine some typical relations in $\mathrm{Ext}_A(M_i^k, Z_2)$ by using the cobar construction $\mathrm{F}(A^*, M_i^{k*})$ of M_i^{k*} over A^* in this section.

We denote by $\alpha\beta$ the image of $\alpha\otimes\beta$ by the composition map

$$\mathrm{Ext}_A^{s,t}(M_i^k, Z_2)\otimes\mathrm{Ext}_A^{u,v}(Z_2, Z_2)\longrightarrow\mathrm{Ext}_A^{s+u,t+v}(M_i^k, Z_2)$$

Let h_m be the generator in $\mathrm{Ext}_A^{1,2^m}(Z_2, Z_2)$ corresponding to Sq^{2^m}.

Theorem 3.1

If $n\geqq 0$, i *is such as* (2.1), *then in* $\mathrm{Ext}_A(M_i^k, Z_2)$,

$\underline{h}_{n+1}h_n=0,\ n+1>i_1+k,$

$\underline{h}_{n+2}h_n^2=\underline{h}_{n+1}h_{n+1}^2,\ n+1>i_1+k,$

$\underline{h}_{n+2}h_{n+2}h_n=0,\ n+2>i_1+k.$

Remark. Similar relations holds in $\mathrm{Ext}_A(Z_2, Z_2)$, but the following relations are not true;

$$\underline{h}_n h_{n+1}=0,\ \underline{h}_n h_{n+2}^2=0.$$

The remainder of this section is devoted to the proof of this theorem. The direct proof is remained in the last of this section.

Lemma 3.2

$a<2b,\ c\geqq 2d,$

$\mathrm{Sq}^a\mathrm{Sq}^b=\mathrm{Sq}^{2^n c}\mathrm{Sq}^{2^n d}+\cdots\cdots$(*Adem relation*) *implies* $a, b\equiv 0 \pmod{2^n}$.

Proof. If $a=a_1 2^n+a_2$, $0<a_2<2^n$, then

$$\binom{2^n-a_2-1}{a_2}\equiv 0 \pmod 2$$

Therefore

$$\binom{b-2^n d-1}{a-2^{n+1}d}=\binom{2^n(c-a_1)+2^n-a_2-1}{2^n(a_1-2d)+a_2}\equiv\binom{c-a_1}{a_1-2d}\binom{2^n-a_2-1}{a_2}\equiv 0 \pmod 2$$

Proposition 3.3

$r\geqq 2s$,

$$\{(a, b)\,;\, a<2b,\ \mathrm{Sq}^a\mathrm{Sq}^b=\mathrm{Sq}^r\mathrm{Sq}^s+\cdots\cdots(\textit{Adem relation})\}$$
$$\xrightarrow{g}\{(c, d)\,;\, c<2d,\ \mathrm{Sq}^c\mathrm{Sq}^d=\mathrm{Sq}^{2^n r}\mathrm{Sq}^{2^n s}+\cdots(\textit{Adem relation})\}$$

This map is a bijection by defining

$$g(a, b)=(2^n a, 2^n b)$$

Proof. The latter equality in the proof of Proposition 2.2 implies that g is a map and the definition of g implies that g is a monomorphism. If the latter Adem relation in this proposition holds, then by Lemma 3.2

$$c=2^n c',\ d=2^n d',\ c'<2d'.$$

Therefore by the latter equality in the proof of Proposition 2.2, g is an epimorphism.

Proposition 3.4

Let B be a module over a field R, $\{b_u, u\in U\}$ be a basis for B, b^u be the element in the dual R-module B^* dual to b_u.

(1) *If B is an algebra with product φ, and*

$$\varphi(b_u\otimes b_v)=\sum c_{u,v}^w b_w,\quad c_{u,v}^w\in R$$

then B^* is a coalgebra with coproduct φ^* such that

$$\varphi^*(b^w)=\sum(-1)^e c_{u,v}^w b^u\otimes b^v,\quad e=\deg b_u\times\deg b_v$$

(2) *If B is a coalgebra with coproduct ψ and*

$$\psi(b_w)=\sum c_w^{u,v} b_u\otimes b_v,\quad c_w^{u,v}\in R,$$

then B^* is an algebra with product ψ^* such that

$$\psi^*(b^u\otimes b^v)=\sum(-1)^e c_w^{u,v} b^w.$$

Proof. standard.

Lemma 3.5

If $\mathrm{Sq}^I\neq\mathrm{Sq}^{2^u}$, *then* $(\mathrm{Sq}^I)^*$ *has not* $\xi_1^{2^u}$ *as a summand.*

The proof is left to my paper to appear. This lemma is used only to prove Proposition 3.6 in the case of $\deg(I)=2^q$, $q\geqq 0$, and this is not necessary to prove Theorem 3.1.

Proposition 3.6

$$(\mathrm{Sq}^I)^* = \sum \xi^J$$

implies

$$(\mathrm{Sq}^{2^n \cdot I})^* = \sum \xi^{2^n \cdot J},$$

where J *runs over the same set in the two summations above and if* $I=(i_1, \cdots\cdots, i_m)$, *then we denote*

$$2^n \cdot I = (2^n i_1, \cdots\cdots, 2^n i_m).$$

Proof. By induction on $\deg(I)$.

$$\varphi^*(\sum \xi^J) = \varphi^*((\mathrm{Sq}^I)^*) = \sum (\mathrm{Sq}^{I_1})^* \otimes (\mathrm{Sq}^{I_2})^*,$$

where the last summation runs over all pairs I_1, I_2 such that

$$\mathrm{Sq}^{I_1}\mathrm{Sq}^{I_2} = \mathrm{Sq}^I + \cdots\cdots$$

By inductive hypothesis

$$\begin{aligned}\varphi^*(\sum \xi^{2^n \cdot J}) &= (\varphi^*(\sum \xi^J))^{2^n} \\ &= \sum ((\mathrm{Sq}^{I_1})^*)^{2^n} \otimes ((\mathrm{Sq}^{I_2})^*)^{2^n} \\ &= \sum_{I_1, I_2 \neq I} (\mathrm{Sq}^{2^n \cdot I_1})^* \otimes (\mathrm{Sq}^{2^n \cdot I_2})^* + ((\mathrm{Sq}^I)^*)^{2^n} \otimes 1 + 1 \otimes ((\mathrm{Sq}^I)^*)^{2^n},\end{aligned}$$

On the other hand, by Proposition 3.4 (1),

$$\varphi^*((\mathrm{Sq}^{2^n \cdot I})^*) = \sum (\mathrm{Sq}^{2^n \cdot I_1})^* \otimes (\mathrm{Sq}^{2^n \cdot I_2})^*.$$

Uuing Lemma 3.5, we have the conclusion.

For the next proposition we denote $\mathrm{Sq}^I = \mathrm{Sq}(i_1, i_2, \cdots\cdots, i_n)$, for convenience' sake, if $I=(i_1, i_2, \cdots\cdots, i_n)$ is complicated.

Proposition 3.8

(1) $\mathrm{Sq}(2^{n+j}, \cdots\cdots, 2^{n+1}, 2^n)^* = \xi_{j+1}^{2^n}$, $j \geqq 0$

(2) $\mathrm{Sq}(2^n(2^j + 2^{m+q}), \cdots\cdots, 2^n(2^{j-q} + 2^m), 2^{n+j-q-1}, \cdots\cdots, 2^{n+1}, 2^n)^*$
$= \xi_{q+1}^{2^{n+m}} \xi_{j+1}^{2^n}$, $0 \leqq m \leqq j$, $0 \leqq q \leqq j$.

(3) $\mathrm{Sq}(2^n(2^{m+q} + 2^j), \cdots\cdots, 2^n(2^{m+q-j+1} + 2), 2^n(2^{m+q-j} + 1), 2^{n+m+q-j-1},$
$\cdots\cdots, 2^{n+m+1}, 2^{n+m})^* = \xi_{q+1}^{2^{n+m}} \xi_{j+1}^{2^n}$, $0 \leqq m \leqq j \leqq q$.

(4) $\mathrm{Sq}(2^{n+j+m}, \cdots\cdots, 2^{n+j+2}, 2^{j+n}, \cdots\cdots, 2^{n+1}, 2^n)^*$
$= \xi_{m-1}^{2^{n+j+1}} \xi_{j+m}^{2^n}$, $m \geqq 2$, $j \geqq 0$.

(5) $\mathrm{Sq}(2^{n+j}(2^m + 1), \cdots\cdots, 2^{n+j-m+1}(2^m + 1), 2^{n+j-m}, \cdots\cdots, 2^{n+1}, 2^n)^*$
$= \xi_m^{2^{n+j+1}} \xi_{j+1}^{2^n} + \xi_{j+m+1}^{2^n}$, $j \geqq m-1 \geqq 0$.

(6) $\mathrm{Sq}(2^{n+j}(2^m+1), \cdots\cdots, 2^{n+1}(2^m+1), 2^n(2^m+1), 2^{n+m-1}, \cdots\cdots, 2^{n+1},$
$2^n)^* = \xi_m^{2^{n+j+1}} \xi_{j+1}^{2^n} + \xi_{j+m+1}^{2^n}, \ m > j \geqq 0.$

(7) $\mathrm{Sq}(2^{n+j+m}+2^{n+j}, \cdots\cdots, 2^{n+j+2}+2^{n+j-m+2}, 2^{n+j-m+1}, \cdots\cdots, 2^{n+1},$
$2^n)^* = \xi_{m-1}^{2^{n+j+2}} \xi_{j+1}^{2^n} + \xi_{m-1}^{2^{n+j+1}} \xi_{j+m}^{2^n}, \ 2 \leqq m \leqq j+2.$

(8) $\mathrm{Sq}(2^{n+j+m}+2^{n+j}, \cdots\cdots, 2^{n+m}+2^n, 2^{n+m-1}, \cdots\cdots, 2^{n+j+3}, 2^{n+j+2})^*$
$= \xi_{m-1}^{2^{n+j+2}} \xi_{j+1}^{2^n} + \xi_{m-1}^{2^{n+j+1}} \xi_{j+m}^{2^n}, \ m \geqq j+2.$

Proof. It is sufficient by Proposition 3.6 to prove this proposition in the case $n=0$.

Proof of (5); If

$$\psi(\mathrm{Sq}^I) = \mathrm{Sq}(2^{j+m}, \cdots\cdots, 2^{j+2}, 2^{j+1}) \otimes \mathrm{Sq}(2^j, \cdots\cdots, 2, 1) + \cdots\cdots,$$

then I is either

$$I_1 = (2^{j+m}, \cdots\cdots, 2, 1)$$

or

$$I_2 = (2^{j+m}+2^j, \cdots\cdots, 2^{j+1}+2^{j-m+1}, 2^{j-m}, \cdots\cdots, 2, 1)$$

Applying Proposition 3.4 (2),

$$\begin{aligned} \xi_m^{2^{j+1}} \xi_{j+1} &= \mathrm{Sq}(2^{j+m}, \cdots\cdots, 2^{j+1})^* \mathrm{Sq}(2^j, \cdots\cdots, 2, 1)^* \\ &= (\mathrm{Sq}^{I_1})^* + (\mathrm{Sq}^{I_2})^* \\ &= \xi_{j+m+1} + (\mathrm{Sq}^{I_2})^*. \end{aligned}$$

Thus the proof of (5) is completed.

(4) is a special case of (2).

Proof of (7): If

$$\psi(\mathrm{Sq}^I) = \mathrm{Sq}(2^{j+m}, \cdots\cdots, 2^{j+2}) \otimes \mathrm{Sq}(2^j, \cdots\cdots, 2, 1) + \cdots\cdots$$

then I is either

$$I_1 = (2^{j+m}, \cdots\cdots, 2^{j+2}, 2^j, \cdots\cdots, 2, 1)$$

or

$$I_2 = (2^{j+m}+2^j, \cdots\cdots, 2^{j+2}+2^{j-m+2}, 2^{j-m+1}, \cdots\cdots, 2, 1)$$

Applying Proposition 3.4 (2) and the formula (4),

$$\begin{aligned} \xi_{m-1}^{2^{j+2}} \xi_{j+1} &= \mathrm{Sq}(2^{j+m}, \cdots\cdots, 2^{j+2})^* \mathrm{Sq}(2^j, \cdots\cdots, 2, 1)^* \\ &= (\mathrm{Sq}^{I_1})^* + (\mathrm{Sq}^{I_2})^* \\ &= \xi_{m-1}^{2^{g+1}} \xi_{j+m} + (\mathrm{Sq}^{I_2})^* \end{aligned}$$

Thus the proof is completed.

The proofs of other formulas are similar.

Remark. The following formula is expected to be true:

$$(\mathrm{Sq}^j)^* = \sum b(J)\,\xi^J,\ \deg(J)=j,$$

where if $J=(j_1, \cdots\cdots, j_n)$, then we denote

$$b(J)=\frac{(j_1+\cdots\cdots+j_n)!}{j_1!\cdots\cdots j_n!}$$

In particular

$$(\mathrm{Sq}^{2^n-1})^* = \sum \xi^I,\ \deg(I)=2^n-1.$$

For the proof of Theorem 3.1, we use only the following special cases of the formulas above:

$$\mathrm{Sq}(2^{n+2}, 2^n)^* = \xi_1^{2^{n+1}}\xi_2^{2^n}$$

$$\mathrm{Sq}(2^{n+1}+2^n)^* = \xi_1^{2^{n+1}+2^n}+\xi_2^{2^n}.$$

We denote the A^*-comodule map of M_i^{k*} by

$$\Delta: M_i^{k*} \longrightarrow A^*\otimes M_i^{k*}$$

There are some properties of this map.

Proposition 3.9

$$\Delta x_j = \sum \xi^I \otimes x_n$$

implies

$$\Delta y_j = \sum \xi^{2^k\cdot I}\otimes y_n.$$

Proof. If $\mathrm{Sq}^I x^u = x^j$, $j\geqq u\geqq i$, in M_i, then by Proposition 2.3, $\mathrm{Sq}^{2^k\cdot I}y^u = y^j$ in M_i^k. If $\mathrm{Sq}^J y^u = y^j$, $j\geqq u\geqq i$, then by Proposition 2.3, $J=2^k\cdot I$ for some I, and $\mathrm{Sq}^I x^u = x^j$. Therefore by Proposition 3.6, we have the proposition.

Lemma 3.10

$$\{(I,q);\ \mathrm{Sq}^I y^q = y^m\} \xrightarrow{g} \{(J,j);\ \mathrm{Sq}^J y^j = y^{2^n m+2^n-1}\}$$

This map g is a bijection by defining

$$g(I,q)=(2^n\cdot I, 2^n q+2^n-1)$$

Proof. By Proposition 2.3, g is a monomorphism. If

$$\mathrm{Sq}^J y^j = y^{2^n m+2^n-1},$$

then by Theorem 2.4, $[j]\leqq[2^n m+2^n-1]$, that is, $[j]\leqq[2^n-1]$. Therefore $j=2^n q+2^n-1$ for some q and Sq^J is such that

$$\mathrm{Sq}^J y^{2^n q} = y^{2^n m}.$$

By Proposition 2.3, $J=2^n\cdot I$, for some I. Thus g is an epimorphism and a bijection.

Proposition 3.11

$$\Delta y_m = \sum \xi^I \otimes y_j$$

implies

$$\Delta y_{2^n m+2^n-1} = \sum \xi^{2^n \cdot I} \otimes y_{2^n j+2^n-1}.$$

Proof. By Lemma 3.10.

Proposition 3.12

$$\Delta x_{2^n-1} = 1 \otimes x_{2^n-1}$$

$$\Delta x_{3\cdot 2^n-1} = 1 \otimes x_{3\cdot 2^n-1} + \xi_1^{2^n} \otimes x_{2^{n+1}-1}$$

$$\Delta x_{5\cdot 2^n-1} = 1 \otimes x_{5\cdot 2^n-1} + \xi_1^{2^n} \otimes x_{2^{n+2}-1} + \xi_1^{2^{n+1}} \otimes x_{3\cdot 2^n-1} + \xi_2^{2^n} \otimes x_{2^{n+1}-1}$$

$$\Delta y_{7\cdot 2^n-1} = 1 \otimes y_{7\cdot 2^n-1} + \xi_1^{2^n} \otimes x_{3\cdot 2^{n+1}-1} + (\xi_1^{3\cdot 2^n} + \xi_2^{2^n}) \otimes x_{2^{n+2}-1}$$

The formulas replaced x_j with y_j and ξ^I with $\xi^{2^k \cdot I}$ above are true.

Proof. It is sufficient by Proposition 3.9 to prove the formulas in the case $k=0$. We will prove only the second, for example. The proof of the second is reduced by Proposition 3.11 to that of

$$\Delta x_2 = 1 \otimes x_2 + \xi_1 \otimes x_1,$$

which is clearly true.

[The proof of Theorem 3.1]

Let δ be the coboundary map of the cobar construction $\bar{\mathrm{F}}(A^*, M_i^{k*})$. Then it is sufficient to calculate

$$\delta\left([\ ]x_{3\cdot 2^n-1}\right)$$

$$\delta\left([\xi_1^{2^n}]x_{5\cdot 2^n-1} + [\xi_1^{3\cdot 2^n} + \xi_2^{2^n}]x_{3\cdot 2^n-1} + [\xi_1^{2^n}\xi_2^{2^n}]x_{2^{n+1}-1}\right)$$

$$\delta\left([\xi_1^{2^{n+2}}]x_{5\cdot 2^n-1} + [\xi_2^{2^{n+1}}]x_{3\cdot 2^n-1} + [\xi_3^{2^n}]x_{2^{n+1}-1}\right)$$

§ 4. Minimal sets of generators.

For the next proposition we denote by K, L^0, $\bar{K}$, $\bar{L}^0$ a Z_2-module generated by the following admissible monomials, respectively:

K: $\mathrm{Sq}^{a_1}\mathrm{Sq}^{a_2}\cdots\cdots\mathrm{Sq}^{a_n}$, $n \geqq 2$

L^0: Sq^1, $\mathrm{Sq}^{a_1}\mathrm{Sq}^{a_2}\cdots\cdots\mathrm{Sq}^{a_n}$, $n \geqq 2$

$\bar{K}$: $\mathrm{Sq}^{2^a}\mathrm{Sq}^{2^b}$, $a > b \geqq 0$

$\bar{L}^0$: Sq^1, $\mathrm{Sq}^{2^a}\mathrm{Sq}^{2^b}$, $a > b > 0$.

Since $K = \ker \underline{f}$, $L^k = \ker f_k$, and $\underline{f}$ and f_k are A-maps, K and L^k are left A-modules.

We finally prove in Proposition 5.3 that

$K=\bar{K}+\bar{A}\cdot K$ (direct sum)

Proposition 4.1

$K=\bar{K}+\bar{A}\cdot K$ (*not direct sum*)

Proof. It is sufficient to prove that

$\mathrm{Sq}^a\mathrm{Sq}^b\in\bar{A}\cdot K$, if $a\geqq 2b$, $b>0$ and unless

$a=2^{a'}$, $b=2^{b'}$, for any a', b'.

Let 2-adic expansions of a and b are

$a=2^{a_1}+\cdots\cdots+2^{a_q}$, $a_1>\cdots\cdots>a_q\geqq 0$,

$b=2^{b_1}+\cdots\cdots+2^{b_r}$, $b_1>\cdots\cdots>b_r\geqq 0$.

The set of all cases not satisfying

$a=2^{a'}$, $b=2^{b'}$, for any a' and b'

are classified into following four cases (with no intersection to each other):

(4.1) $r\geqq 2$, $a_q\geqq b_r+2$,

(4.2) $r\geqq 2$, $a_q=b_r+1$, $q\geqq 2$,

(4.3) $a_q\leqq b_r$, $q\geqq 2$,

(4.4) $r=1$, $a_q>b_r$, $q\geqq 2$.

Proof of the case (4.1): Let

$a=a'\,2^{n+2}$, $b=b'\,2^{n+1}+2^n$, $a'>b'>0$.

Then

$$\mathrm{Sq}(2^{n+1}, a-2^n, b'\,2^{n+1})=(\mathrm{Sq}^a\mathrm{Sq}^{2^n}+\mathrm{Sq}^{a+2^n})\mathrm{Sq}^{b'2^{n+1}}$$
$$=\mathrm{Sq}^a\mathrm{Sq}^b+\sum_{t=0}^{n-1}\mathrm{Sq}(a, b-2^t, 2^t)+\mathrm{Sq}(a+2^n, b'\,2^{n+1}).$$

The last summand is reduced to the case (4.3).

Proof of the case (4.2): Let

$a=a'2^{n+2}+2^{n+1}$, $b=b'2^{n+1}+2^n$, $a'\geqq b'>0$.

Then

$$\mathrm{Sq}(2^{n+1}, a'2^{n+2}+2^n, b'2^{n+1})=\mathrm{Sq}(a, 2^n, b'2^{n+1})$$
$$=\mathrm{Sq}^a\mathrm{Sq}^b+\textstyle\sum_{t=0}^{n-1}\mathrm{Sq}(a, b-2^t, 2^t).$$

Proof of the case (4.3): Let

$a=a'2^{n+1}+2^n$, $b=b'2^n$, $a'\geqq b'>0$.

Then we prove it by induction on n. If $n=0$, then $\mathrm{Sq}^a\mathrm{Sq}^b=\mathrm{Sq}^1\mathrm{Sq}^{a-1}\mathrm{Sq}^b$. Therefore $\mathrm{Sq}^a\mathrm{Sq}^b\in\bar{A}\cdot K$. If $n>0$, then

$$(4.5)\quad \mathrm{Sq}(2^n, a'2^{n+1}, b)=\mathrm{Sq}^a\mathrm{Sq}^b+\textstyle\sum_{t=0}^{n-1}\mathrm{Sq}(a-2^t, 2^t, b)$$
$$=\mathrm{Sq}^a\mathrm{Sq}^b+\textstyle\sum_{t=0}^{n-1}\mathrm{Sq}(a-2^t, b+2^t)$$
$$+\textstyle\sum_{t=0}^{n-1}\sum_{s=0}^{t-1}\mathrm{Sq}(a-2^t, b+2^t-2^s, 2^s)$$

$\mathrm{Sq}(a-2^t, b+2^t)$ is not admissible only in the case $a'=b'$, $t=n-1$, but if $n\geqq 2$, then

$$\mathrm{Sq}(a-2^{n-1}, b+2^{n-1})=\mathrm{Sq}(a'2^{n+1}+2^{n-1}, a'2^n+2^{n-1})$$
$$=\textstyle\sum_{t=0}^{n-2}\mathrm{Sq}(2^{n+1}a'+2^n-2^t, 2^na'+2^t).$$

Transform the summands of $n-2>t\geqq 0$ in the form as (4.5), and we know that they are contained in $\bar{A}\cdot K$ by inductive hypothesis. the summand of $t=n-2$ is

$$\mathrm{Sq}(2^{n-1}c+2^{n-2}, 2^{n-2}c),\quad c=4a'+1.$$

Apply the same method above to this summand, and we know that there remains only one summand

$$\mathrm{Sq}(2^{n-3}d+2^{n-4}, 2^{n-4}d),\quad d=4c+1,$$

which is unknown to be contained in $\bar{A}\cdot K$, if $n\geqq 4$.

But by applying this method repeatedly the problem is reduced to either $\mathrm{Sq}(4e+1, 2e+1)$ or $\mathrm{Sq}(8e+2, 4e+2)$ according that n is odd or even. We have

$$\mathrm{Sq}(4e+1, 2e+1)=0$$
$$\mathrm{Sq}(8e+2, 4e+2)=\mathrm{Sq}(8e+3, 4e+1)$$
$$=\mathrm{Sq}(1, 8e+2, 4e+1)$$

Then in the cases (4.1), (4.2), and (4.3), by inductive hypothesis, $\mathrm{Sq}^a2\mathrm{q}^b\in\bar{A}\cdot K$.

Proof of the case (4.4): Let $b=2^n$. Using Proposition 3.2 we can decompose Sq^a in the form of

$$\mathrm{Sq}^a=\textstyle\sum_{u>n}c_u\mathrm{Sq}^{2^u},\ c_u\in A.$$

Therefore $\mathrm{Sq}^a\mathrm{Sq}^b\in\bar{A}\cdot K$.

Thus the proposition has been proved.

In this proof, we use the following formulas.

Lemma 4.2

$$\mathrm{Sq}(2^n, 2^{n+1}a+2^n)=\textstyle\sum_{t=0}^{n-1}\mathrm{Sq}(2^{n+1}a+2^{n+1}-2^t, 2^t)$$
$$\mathrm{Sq}(2^n, 2^{n+1}a)=\textstyle\sum_{t=0}^{n-1}\mathrm{Sq}(2^{n+1}a+2^n-2^t, 2^t)+\mathrm{Sq}(2^{n+1}a+2^n)$$
$$\mathrm{Sq}(2^{n+1}, 2^{n+2}a+2^n)=\mathrm{Sq}(2^{n+2}a+2^{n+1}, 2^n)$$
$$\mathrm{Sq}(a2^{n+2}+2^n, a2^{n+1}+2^n)=\textstyle\sum_{t=0}^{n-1}\mathrm{Sq}(2^{n+2}a+2^{n+1}-2^t, 2^{n+1}a+2^t)\ .$$

We define an A-map

$$\bar{f}: \bar{A}^2 \longrightarrow N$$
$$N=\bar{A}\cdot M^0=Z_2\{x^j;\, j>0,\, j\neq 2^n-1, \text{ for any } n\}$$

to be the restriction of $f: \bar{A} \longrightarrow M^0$. then

$$(4.6) \qquad 0 \longrightarrow K \longrightarrow \bar{A}^2 \xrightarrow{\bar{f}} N \longrightarrow 0$$

is an exact sequence of left A-modules.

Lemma 4.3

$$N=\bar{A}\cdot N+Z_2\{x^{2^{n+1}+2^n-1},\, n\geqq 0\}. \quad (\textit{direct sum})$$

Proof. If

$$a=2^{m+1}a'+2^m+2^n-1,\, m\geqq n+2,\, a'\geqq 0,$$
$$\text{or } \ m=n+1,\, a'>0,$$

then

$$\mathrm{Sq}^{2^{m-1}}x^{a-2^{m-1}}=x^a,\; x^{a-2^{m-1}}\in N.$$

If $m>n$, set

$$m'=2^{m+1}+2^m-1,\;\; n'=2^{n+1}+2^n-1,$$

then

$$m'>n',\; \sharp[m']=m+1>n+1=\sharp[n'].$$

Therefore $x^{m'}$ and $x^{n'}$ are linearly independent by Theorem 2.4. Thus the proof is completed.

Proposition 4.4

$$L^0=\bar{L}^0+\bar{A}\cdot L^0 \;\; (\textit{not direct sum}).$$

Proof. By Proposition 4.1,

$$\begin{aligned} L^0 &= K+Z_2\{\mathrm{Sq}^1\} \\ &= \bar{K}+\bar{A}\cdot K+Z_2\{\mathrm{Sq}^1\} \\ &= \bar{L}^0+\bar{A}\cdot L^0+Z_2\{\mathrm{Sq}^{2^j}\mathrm{Sq}^1,\, j>0\}+Z_2\{\mathrm{Sq}^1\} \\ &= \bar{L}^0+\bar{A}\cdot L^0 \;\; (\text{not direct sum}). \end{aligned}$$

§ 5. Exact sequences for Ext.

The author imagines that somebody has ever proved the following proposition.

Proposition 5.1

Let R be a commutative ring with unit and B be an algebra over R.

(1) *Then an short exact sequence of left B-modules*

$$0 \longrightarrow L \xrightarrow{i} N \xrightarrow{f} M \longrightarrow 0$$

and a left B-module G induce an exact seqnence of right $\mathrm{Ext}^r_A(G, G)$*-modules*, $r \geqq 0$,

$$\cdots\cdots \longrightarrow \mathrm{Ext}^s_B(M, G) \xrightarrow{F_s} \mathrm{Ext}^s_B(N, G) \xrightarrow{I_s} \mathrm{Ext}^s_B(L, G) \xrightarrow{\partial_s} \mathrm{Ext}^{s+1}_B(M, G) \longrightarrow \cdots\cdots$$

(2) F_s, I_s *and* ∂_s *are compatible with Massey products*; *in detail, if* $m \in \mathrm{Ext}_B(M, G)$, $l \in \mathrm{Ext}_B(L, G)$, $n \in \mathrm{Ext}_B(N, G)$, $a, b \in \mathrm{Ext}_B(G, G)$, *then*

$$F\langle m, a, b\rangle \subset \langle F(m), a, b\rangle, \text{ if } ma = 0 = ab,$$

$$I\langle n, a, b\rangle \subset \langle I(n), a, b\rangle, \quad \text{if } na = 0 = ab,$$

$$\partial\langle l, a, b\rangle \subset -\langle \partial(l), a, b\rangle, \text{ if } la = 0 = ab.$$

These properties holds for iterated Massey products. For example,

$$F\langle\langle m, a, b\rangle, a', b'\rangle \subset \langle\langle F(m), a, b\rangle, a', b'\rangle,$$

if $ma = 0 = ab$, $\langle m, a, b\rangle a' \ni 0$, *and* $a'b' = 0$,
where F, I *and* ∂ *stand for* F_s, I_s *and* ∂_s *for an appropriate s.*

We apply this proposition to the following short exact sequence of left A-modules:

$$0 \longrightarrow L^k \longrightarrow \overline{A} \xrightarrow{J_k} M^k \longrightarrow 0$$

Then the following exact sequence is induced:

$$(5.1) \quad \begin{array}{l} \cdots\cdots \longrightarrow \mathrm{Ext}^{s-1,t}_A(L^k, Z_2) \xrightarrow{\partial_s} \mathrm{Ext}^{s,t-2^k}_A(M^k, Z_2) \\ \xrightarrow{F_s} \mathrm{Ext}^{s,t}_A(\overline{A}, Z_2) \xrightarrow{I_s} \mathrm{Ext}^{s,t}_A(L^k, Z_2) \longrightarrow \cdots\cdots \\ \qquad \wr\wr \\ \mathrm{Ext}^{s+1,t}_A(Z_2, Z_2) \end{array}$$

By comparing the dimensions of generators,

$F_0(\underline{h}_n) = h_n$, $n > k$.

Proposition 5.2

$\mathrm{Ext}^1_A(M^0, Z_2) = Z_2\{\underline{h}_a h_b;\ a \neq b+1,\ a > 0,\ b \geqq 0\}$.

$L^0 = \overline{A} \cdot L^0 + \overline{L}^0$. (*direct sum*)

Remark. By Theorem 3.1, we have

$\underline{h}_a h_{a-1} = 0$, $a > 0$.

Proof.

$\mathrm{Sq}^{2^a} x^{2^b-1} = 0$, $a \geqq b$.

$\mathrm{Sq}^{2^a} x^{2^b-1} = \mathrm{Sq}^{2^{b-1}} \mathrm{Sq}^{2^a} x^{2^{b-1}-1}$, $a+2 \leqq b$,

implies

$\underline{h}_b h_a \neq 0,\ b>0,\ a \geqq 0,\ b \neq a+1,$

$\underline{h}_b h_a \neq \underline{h}_a h_b,\ a \geqq b+2,\ b>0.$

Since

$F_1(\underline{h}_b h_{b+1}) = h_b h_{b+1} = 0,\ b>0$

$F_1(\underline{h}_b h_a) = h_b h_a = h_a h_b = F_1(\underline{h}_a h_b),\ a-2 \geqq b>0$

$I_0(h_0) = h_0',$

(where h_0' is the cohomology class of $[\ ]\xi_1$ in the cobar construction $\overline{\mathrm{F}}(A^*, L^{0*})$.), we have

$Z_2\{\underline{h}_b h_{b+1},\ b>0;\ \underline{h}_b h_a + \underline{h}_a h_b,\ a-2 \geqq b>0\}$

$\subset \ker F_1 = \mathrm{im}\ \partial_0 = \mathrm{coker}\ I_0.$

coker I_0 is a Z_2-module generated by $g_{a,b}$, which is the cohomology class of $[\ ](\mathrm{Sq}^{2^a}\mathrm{Sq}^{2^b})^*$ in the cobar construction $\overline{\mathrm{F}}(A^*, L^{0*})$, for a, b such that $a>b>0$ and $\mathrm{Sq}^{2^a}\mathrm{Sq}^{2^b} \notin \overline{A}\cdot L^0$. (Therefore $g_{a,b} \neq 0$, if exists.) By comparing the dimensions,

$\mathrm{coker}\ I_0 = Z_2\{g_{a,b},\ a>b>0\},$

$\partial_0(g_{a,b}) = \underline{h}_a h_b + \underline{h}_b h_a,\ a-2 \geqq b>0$

$\partial_0(g_{a+1,a}) = \underline{h}_a h_{a+1},\ a>0,$

and two sets of generators in the left hand side and right hand side correspond bijectively to each other. Thus the proof is completed.

Proposition 5.3

$K = \overline{A}\cdot K + \overline{K}$ (*direct sum*).

Proof. Let $S = Z_2\{\mathrm{Sq}^1\}$. Then the short exact sequence of left A-modules:

$$0 \longrightarrow K \longrightarrow L^0 \longrightarrow S \longrightarrow 0$$

induces the long exact sequence:

$$\begin{array}{l} \cdots\cdots \longrightarrow \mathrm{Ext}_A^{s,t}(L^0, Z_2) \xrightarrow{I_s} \mathrm{Ext}_A^{s,t}(K, Z_2) \\ \xrightarrow{\partial_s} \mathrm{Ext}_A^{s+1,t}(S, Z_2) \xrightarrow{F_{s+1}} \mathrm{Ext}_A^{s+1,t}(L^0, Z_2) \longrightarrow \cdots\cdots \\ \qquad\quad \wr\!\wr \\ \quad \mathrm{Ext}_A^{s+1,t-1}(Z_2, Z_2) \end{array}$$

By Proposition 5.2 and 4.4, $\mathrm{Ext}_A^0(K, Z_2)$ is a Z_2-free module generated by $g_{a,b}$, which is the cohomology class in the cobar construction $\overline{\mathrm{F}}(A^*, K)$ for a, b, such that $a>b \geqq 0$ and $\mathrm{Sq}^{2^a}\mathrm{Sq}^{2^b} \notin \overline{A}\cdot K$. In $\mathrm{Ext}_A(L^0, Z_2)$, $h_0' h_0 \neq 0$, $h_0' h_u = 0,\ u>0.$

By comparing the dimensions of generators, $g_{u,v}$, $u>v \geqq 0$, are generators in $\mathrm{Ext}_A(K, Z_2)$ and

$F_0(1) = h_0',\ F_1(h_0) = h_0' h_0,$

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$I_0(g_{u,v})=g_{u,v},\ u>v>0,$

$\partial_0(g_{u,0})=h_u,\ u>0$ Q. E. D.

Corollary 5.4

$$\overline{A}^2=\overline{A}^3+Z_2\{\mathrm{Sq}^{2^a}\mathrm{Sq}^{2^b}\,;\,a>b\geqq 0,\ a+1=b>0\}. \quad (\textit{direct sum})$$

Proof. Apply Proposition 5.3 to the exact sequence (4.6) and we have $I_0(g_{a,b})=g_{a,b},\ a>b\geqq 0$, and $g_{a,b},\ a>b\geqq 0$, are generators of $\mathrm{Ekt}_A^0(\overline{A}^2, Z_2)$.

Let N^k be a Z_2-module generated by

$x^n\,;\,n\geqq 0$ and $n\equiv -1 \pmod{2^k}$ or $n=2^k-1$

Lemma 5.5

$$N^k=\overline{A}\cdot N^k+Z_2\{x^{2^j-1},\ 0\leqq j\leqq k : x^{2^j-2^{k-1}-1},\ j\geqq k+2\}. \quad (\textit{direct sum})$$

Proof. By Theorem 2.4. We denote by $\underline{h}_j$ and $b''_{k,j}$ the cohomology classes of $[\ ]x_{2^j-1}$ and $[\ ]x_{2^j-2^{k-1}-1}$ in $\overline{\mathrm{F}}(A^*, N^{k*})$, respectively. deg $\underline{h}_j=2^j-1$, deg $b''_{k,j}=2^j-2^{k-1}-1$.

Proposition 5.6

$$\mathrm{Ext}_A^0(L^k, Z_2)=Z_2\{h_u,\ u\leqq k\,;\,g_{u,v},\ u>v>k\,;\,b'_{k,j},\ j\geqq k+2\}.$$

deg $h_u=2^u$, deg $b''_{k,j}=2^j-2^{k-1}$, deg $g_{u,v}=2^u+2^v$.

Proof. There is a morphism of short exact sequences of left A-modules:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L^0 & \longrightarrow & \overline{A} & \longrightarrow & M^0 & \longrightarrow & 0\\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & K & \longrightarrow & L^k & \longrightarrow & N^k & \longrightarrow & 0
\end{array}$$

This induces a morphism of long exact sequences for Ext:

$$\begin{array}{ccccc}
 & & \mathrm{Ext}_A^{s+1,t}(Z_2, Z_2) & & \\
 & & \wr\wr & & \\
\cdots\cdots\longrightarrow \mathrm{Ext}_A^{s,t-1}(M^0, Z_2) & \xrightarrow{F_s} & \mathrm{Ext}_A^{s,t}(\overline{A}, Z_2) & \xrightarrow{I_s} & \mathrm{Ext}_A^{s,t}(L^0, Z_2)\\
\downarrow p_s & & \downarrow q_s & & \downarrow r_s\\
\cdots\cdots\longrightarrow \mathrm{Ext}_A^{s,t-1}(N^k, Z_2) & \xrightarrow{F'_s} & \mathrm{Ext}_A^{s,t}(L^k, Z_2) & \xrightarrow{I'_s} & \mathrm{Ext}_A^{s,t}(K, Z_2)
\end{array}$$

$$\begin{array}{l}
\xrightarrow{\partial_s} \mathrm{Ext}_A^{s+1,t-1}(M^0, Z_2) \longrightarrow\cdots\cdots\\
\qquad\qquad \downarrow p_{s+1}\\
\xrightarrow{\partial'_s} \mathrm{Ext}_A^{s+1,t-1}(N^k, Z_2) \longrightarrow\cdots\cdots
\end{array}$$

We denote $F'_0(\underline{h}_j)=h'_j,\ k\geqq j\geqq 0,\ F'_0(b''_{k,j})=b'_{k,j}$. Then $h'_j=F'_0(\underline{h}_j)=q_0F_0(\underline{h}_j)=q_0(h_j)$.

$$\partial_0(g_{u,v})=\begin{cases}0, & u>v>k\\ \underline{h}_v h_u, & u-2\geqq v,\ u>k\geqq v\\ \underline{h}_u h_v+\underline{h}_v h_u, & k\geqq u\geqq v+2\\ \underline{h}_v h_{v+1}, & k\geqq v=u-1\geqq 0.\end{cases}$$

Therefore $\ker \partial_0=Z_2\{g_{u,v},\ u>v>k\}$. Thus the proof is completed.

Theorem 5.7

If $k>0$, *then*

$\mathrm{Ext}_A^1(M^k, Z_2)=Z_2\{\underline{h}_u h_v,\ k<u\neq v+1\ ;\ b_{k,j},\ j\geqq k+2$.

$\deg(\underline{h}_u h_v)=2^u-2^k+2^v$, $\deg b_{k,j}=2^j-3\cdot 2^{k-1}$.

Proof. By comparing the dimensions in the exact sequence (5.1),

$F_0(\underline{h}_u)=h_u,\ u>k\ ;\ I_0(h_u)=h'_u,\ 0\leqq u\leqq k,$

$\partial_0(g_{u+1,u})=\underline{h}_u h_{u+1},\ u>k\ ;\ \partial_0(b'_{k,j})=b_{k,j},\ j\geqq k+2.$

$\partial_0(g_{u,v})=\underline{h}_u h_v+\underline{h}_v h_u,\ u-2\geqq v>k,$

and $F_1(\underline{h}_u h_v)=h_u h_v=F_1(\underline{h}_v h_u)$ implies $\underline{h}_u h_v\neq\underline{h}_v h_u$.

(also we can show this directly by the method similar to the proof of Proposition 5.2) Thus the proof is completed.

Proposition 5.8

In $\mathrm{Ext}_A(L^k, Z_2)$, $g_{a,b}h_u\neq 0$, $g_{a,b}h_{a-1}=0$, $g_{a,b}h_{b-1}=0$,

$g_{a,b}h_{a+1}=g_{a+1,a}h_b$, $g_{a,b}h_{b+1}=g_{b+1,b}h_a$, $a>b>k$:

$g_{a,b}h_c+g_{a,c}h_b+g_{b,c}h_a=0$, $a-4\geqq b-2\geqq c>k$.

In $\mathrm{Ext}_A(L^0, Z_2)$, $a\geqq 0$,

$g_{a+3,a+2}h_a^2=g_{a+3,a+1}h_{a+1}^2=g_{a+2,a+1}h_{a+2}^2$.

Theorem 5.9

(1) *If* α *and* β *are non-zero elements of* $\mathrm{Ext}_A(Z_2, Z_2)$, *and* $\alpha\beta\neq 0$, *then* $\underline{\alpha}\beta\neq 0$. *In particular* $\underline{h}_u\beta\neq 0$, $u>k$, *in* $\mathrm{Ext}_A(M^k, Z_2)$, *if* $h_u\beta\neq 0$.

(2) *If* α, β_u *and* γ_u *are in* $\mathrm{Ext}_A(Z_2, Z_2)$, *then we denote an iterated Massey product by*

$$\mathrm{M}(\alpha)=\langle\langle\cdots\cdots\langle\langle\alpha, \beta_1, \gamma_1\rangle, \beta_2, \gamma_2\rangle, \cdots\cdots\rangle, \beta_n, \gamma_n\rangle.$$

If $\mathrm{M}(\alpha)$ *and* $\mathrm{M}(\underline{\alpha})$ *are defined and* $\mathrm{M}(\alpha)\not\ni 0$, *then* $\mathrm{M}(\underline{\alpha})\not\ni 0$ *in* $\mathrm{Ext}_A(M^k, Z_2)$.

Proof. By Proposition 5.1.

Corollary 5.10

(1) $\underline{h}_u h_v h_w$, $u>k$, $u\neq v\pm 1$, $v\neq w\pm 1$, $u\neq w\pm 1$;

$\underline{h}_u h_0^{2^{u-1}-1}$, $u>k$

are non-zero in $\mathrm{Ext}_A(M^k, Z_2)$.

(2) $\underline{c}_0$, $\underline{h}_1 c_0=\underline{c}_0 h_1$, $\underline{d}_0$, $P^i\underline{h}_2$, $P^i\underline{h}_1$, $P^i\underline{c}_0$, $P^i\underline{d}_0$

are non-zero in $\mathrm{Ext}_A(M^0, Z_2)$.

Remark. In theorem 5.8 and Corollary 5.9, if tere is an element in $\mathrm{Ext}_A(M^k, Z_2)$ which is mapped to α by F_n, for an appropriate n, then we denote this element by $\underline{\alpha}$. The representation of generators of $\mathrm{Ext}_A(Z_2, Z_2)$ is due to [9] and [7].

§ 6. Tables.

We offer the tables of $\mathrm{Ext}_A^{s,t}(L^0, Z_2)$, $t-s\leqq 29$, *and* $\mathrm{Ext}_A^{s,t}(M^0, Z_2)$, $t-s\leqq 27$, in this section.

We first determine the former by determining the partial minimel resolution of L^0 over A. Secondly we determine the latter by the former and the table of $\mathrm{Ext}_A(Z_2, Z_2)$ in [9], [7] and the exact sepuence (5.1) in the case $k=0$, We only remark the fact that I_s is trivial for all generators in $\mathrm{Ext}_A^s(Z_2, Z_2)$, except for h_0^{s+1}, when $F_s(h_0^{s+1})=h_0' h_0^s$, $s\geqq 0$, in that range of s, t.

Since $\alpha_2=\langle g_{2,1}, h_1, h_3^2\rangle$, $\alpha_4=\langle g_{4,1}, h_0, h_2^2\rangle$,

$$F(\alpha_2)=\langle F(g_{2,1}), h_1, h_3^2\rangle=\langle \underline{h}_1 h_2, h_1, h_3^2\rangle=\underline{h}_1\langle h_2, h_1, h_3^2\rangle=\underline{h}_1 c_1$$

$$F(\alpha_4)=\langle F(g_{4,1}), h_0, h_2^2\rangle=\langle \underline{h}_4 h_1+\underline{h}_1 h_4, h_0, h_2^2\rangle$$

$$=\underline{h}_4\langle h_1, h_0, h_2^2\rangle+\langle \underline{h}_1, h_0, h_2^2\rangle h_4=\underline{h}_4 c_0+\underline{c}_0 h_4.$$

By exactness $\underline{h}_4 c_0\neq \underline{c}_0 h_4$. By constructing a minimal resolution $\underline{h}_1 c_0=\underline{c}_0 h_1(\neq 0)$ and by Theorem 5.8 $\underline{h}_1 c_0 h_4\neq 0$. By $F(\alpha_4 h_1)=\underline{h}_4 c_0 h_1+\underline{c}_0 h_4 h_1$, $\underline{h}_4 c_0 h_1\neq \underline{c}_0 h_4 h_1=\underline{h}_1 c_0 h_4$.

As above $\underline{h}_1 c_1\neq 0$, but by constructing a minimal resolution $\underline{c}_1 h_1=0$.

In Table 6.1 and 6.2, "bar" means "multiplied by h_0, h_1 or h_2".

We imagine that bideg $({}_7h_1^2d_0)=(6, 23)$, bideg $({}_0i)=(7, 23)$, $d_2({}_7h_1^2d_0)={}_0P_1d_0$, $\pi_*(P_k)=Z_4+Z_{16}$ in Table 8.3, and $({}_6h_0h_3^2)h_0^2\neq 0$ in Table 8.2 in [6] are misprints and we think that they must be corrected in the following: bideg $({}_7h_1^2d_0)=(7, 23)$, bideg $({}_0i)=(6, 23)$, $d_2({}_0i)={}_0P_1d_0$, $\pi_*(P_k)=Z_2^2+Z_{16}$, and $({}_6h_0h_3^2)h_0^2=0$, $({}_2f_0)h_1\neq 0$.

In Table 8.3, $({}_2P_1h_1c_0)h_1=({}_4P_1c_0)h_0^2$, $({}_2P_1h_1c_0)h_2=({}_0P_1d_0)h_0$, $({}_6h_0^2h_2)h_2=({}_4c_0)h_0$, in Table 8.4, $({}_6P_1h_1^2)h_1={}_1P_1c_0$, $({}_6P_1h_1^2)h_2={}_3P_1c_0$, $({}_1P_1c_0)h_1^2=({}_3P_1c_0)h_0^2$.

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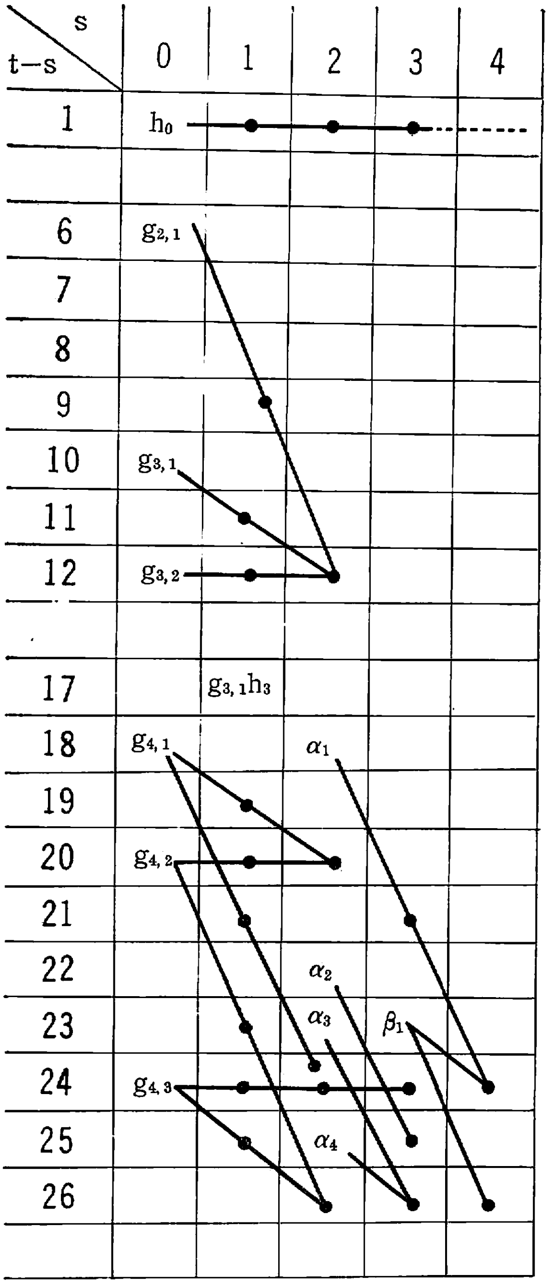

Table 6·1

$\mathrm{Ext}_A(L^0, Z_2)$

$\alpha_2 = \langle g_{2,1}, h_1, h_3^2 \rangle$

$\alpha_4 = \langle g_{4,1}, h_0, h_2^2 \rangle$

$\alpha_2 h_2 = \alpha_1 h_3$

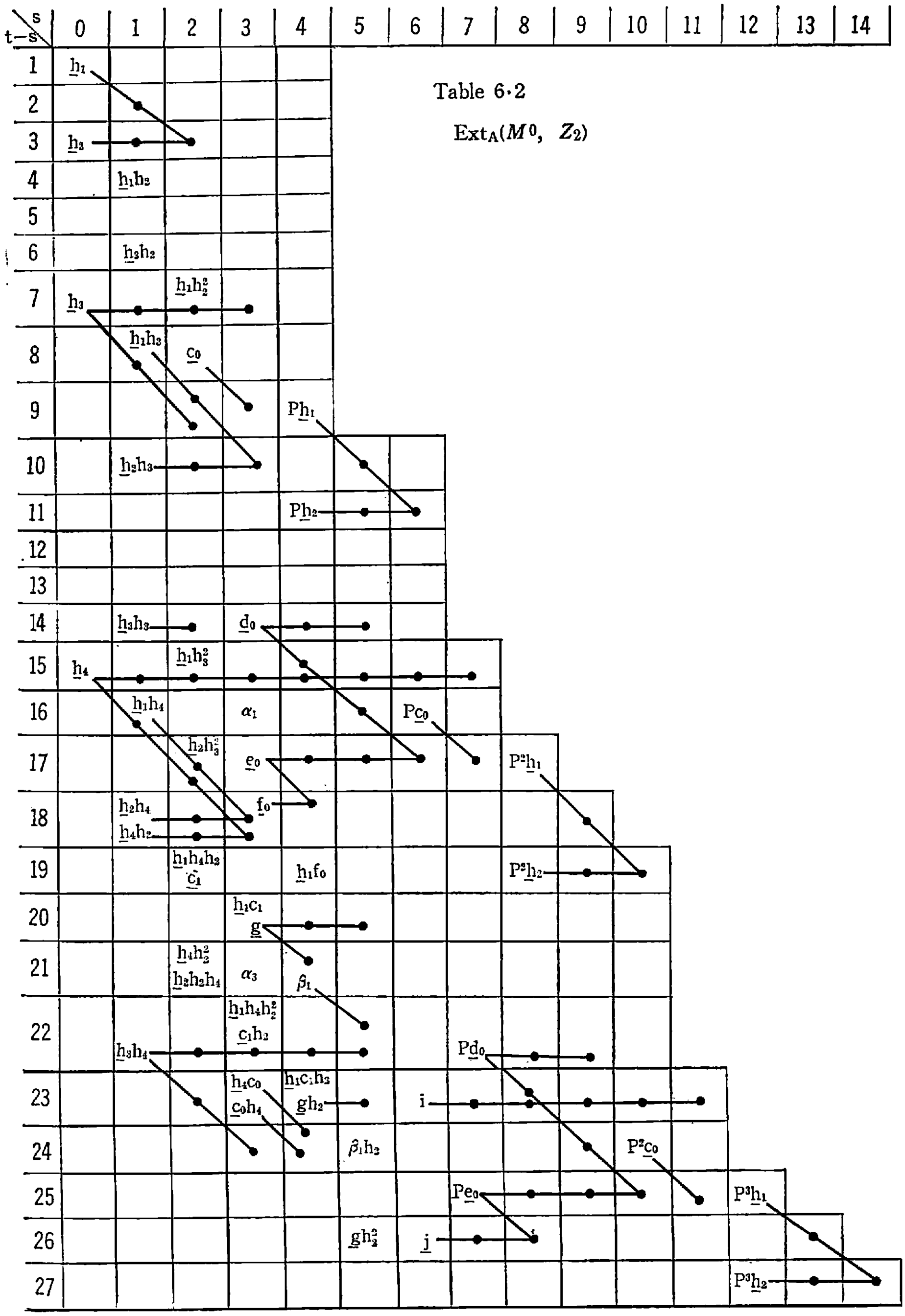
Table 6·2
$\mathrm{Ext}_A(M^0, Z_2)$
s
t−s
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27
h_1
h_2
h_1h_2
h_2h_2
$h_1h_2^2$
h_3
h_1h_3
c_0
Ph_1
h_2h_3
Ph_2
h_3h_3
d_0
$h_1h_3^2$
h_4
h_1h_4
α_1
Pc_0
$h_2h_3^2$
e_0
P^2h_1
h_2h_4
f_0
h_4h_2
$h_1h_4h_3$
c_1
h_1f_0
P^2h_2
h_1c_1
g
$h_4h_3^2$
$h_2h_2h_4$
α_3
β_1
$h_1h_4h_2^2$
c_1h_2
h_3h_4
Pd_0
h_4c_0
$h_1c_1h_3$
c_0h_4
gh_2
i
β_1h_2
P^2c_0
Pe_0
P^3h_1
gh_2^2
j
P^3h_2