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MAXIMAL LINEAR TOPOLOGIES AND THE COMPLEMENT OF LINEAR TOPOLOGIES

Dedicated to Professor Hisao Tominaga on his 60th birthday

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Introduction. The purpose of this paper is two-fold. First, we characterize the maximal left linear topology on a ring. Applying this, we again prove the equivalence of conditions on a ring, obtained by Nicholson and Sarath [6], to have a unique maximal left linear topology. Secondly, as in parallel with Meijer and Smith [7], we investigate the complement of left linear topologies on a ring R . Thus we consider the collection $C(R)$ of those left ideals of R which do not belong to any proper left linear topology on R . Two extremes when $C(R)$ consists of all proper left ideals of R and $C(R) = \{0\}$ are examined.

0. Preliminaries. Let R be a ring with identity. We denote by $\mathcal{L}(R)$ the set of all left ideals of R , and by $R\text{-mod}$ the category of all unital left R -modules. For $A \in \mathcal{L}(R)$ and a subset F of R , we set $AF^{-1} = \{x \in R \mid xF \subseteq A\}$. A nonempty subset \mathcal{L} of $\mathcal{L}(R)$ is called a *left linear topology* if the following conditions are satisfied :

- T1. If $I \in \mathcal{L}$, $J \in \mathcal{L}(R)$ and $I \leq J$, then $J \in \mathcal{L}$.
- T2. If I and J belong to \mathcal{L} , then $I \cap J \in \mathcal{L}$.
- T3. If $I \in \mathcal{L}$ and $a \in R$, then $Ia^{-1} \in \mathcal{L}$.

A left linear topology \mathcal{L} on R is called a *left Gabriel topology* if \mathcal{L} satisfies a further condition :

- T4. If $I \in \mathcal{L}(R)$ and there exists $J \in \mathcal{L}$ such that $Ij^{-1} \in \mathcal{L}$ for every $j \in J$, then $I \in \mathcal{L}$.

A left linear topology \mathcal{L} is called *proper* if $0 \notin \mathcal{L}$. If \mathcal{L}_1 and \mathcal{L}_2 are left linear topologies on R , we define $\mathcal{L}_1 \leq \mathcal{L}_2$ if every member of \mathcal{L}_1 is a member of \mathcal{L}_2 . A subclass \mathcal{S} of $R\text{-mod}$ is called a *hereditary pretorsion class* if \mathcal{S} is closed under isomorphisms, submodules, factor modules and direct sums. \mathcal{S} is called *proper* if $R \notin \mathcal{S}$. A preradical r for $R\text{-mod}$ is called *left exact* if $r(N) = r(M) \cap N$ for every $M \in R\text{-mod}$ and every submodule N of M . It is called *proper* if $r(R) \neq R$, and is called *cofaithful* if $r(Q) = Q$ for every injective $Q \in R\text{-mod}$. For preradicals r and s for

$R\text{-mod}$, we define a preradical $r+s$ by $(r+s)(M) = r(M)+s(M)$ for all $M \in R\text{-mod}$. For a module $Q \in R\text{-mod}$, we define a preradical t_Q for $R\text{-mod}$ by $t_Q(M) = \sum \text{Im}\alpha$, α ranging over $\text{Hom}_R(Q, M)$, for each $M \in R\text{-mod}$. We remark that t_Q is cofaithful if and only if Q is cofaithful, i.e., Q generates all injective left R -modules, or equivalently, R can be embedded in a finite direct sum of copies of Q ([1, Proposition 4.5.4]). We naturally define the ordering of hereditary pretorsion classes of $R\text{-mod}$ and that of left exact preradicals for $R\text{-mod}$. It is well known that there is an order preserving bijective correspondence between left linear topologies on R , hereditary pretorsion classes of $R\text{-mod}$ and left exact preradicals for $R\text{-mod}$ (see [9, p. 145]).

1. Maximal linear topologies. It is not assured that, for a given proper left Gabriel topology on R , there exists a maximal left Gabriel topology containing given one. In [7, Theorem 3.4], Meijer and Smith proved that the above property on a ring R holds if and only if every nonzero injective left R -module has a nonzero submodule whose annihilator is an M -ideal. If R satisfies the maximum condition for ideals, then R has the above property ([3, Proposition 3.2]). But we can prove the next

Proposition 1.1. *For every proper left linear topology \mathcal{L} on R , there exists a maximal left linear topology containing \mathcal{L} .*

Proof. This is done by Zorn's lemma.

Lemma 1.2. *For every left linear topology \mathcal{L} on R and left ideal A of R , there exists a unique minimal left linear topology \mathcal{L}^* containing \mathcal{L} and A . For $J \in \mathcal{L}(R)$, J belongs to \mathcal{L}^* if and only if there exist $I \in \mathcal{L}$ and a finite subset F of R such that $J \geq I \cap AF^{-1}$.*

Proof. Let \mathcal{L}^* be the set of left ideals J of R such that there exist $I \in \mathcal{L}$ and a finite subset F of R satisfying $J \geq I \cap AF^{-1}$. It is sufficient to show that \mathcal{L}^* is in fact a left linear topology. Clearly \mathcal{L}^* satisfies T1. Assume J_1 and J_2 belong to \mathcal{L}^* . Then there exist left ideals I_1 and I_2 and finite subsets F_1 and F_2 of R such that $J_i \geq I_i \cap AF_i^{-1}$ ($i = 1, 2$). Since $I_1 \cap I_2 \in \mathcal{L}$ and $AF_1^{-1} \cap AF_2^{-1} = A(F_1 \cup F_2)^{-1}$, we have $J_1 \cap J_2 \geq (I_1 \cap I_2) \cap A(F_1 \cup F_2)^{-1}$, proving \mathcal{L}^* satisfies T2. Now assume $J \in \mathcal{L}^*$ and $a \in R$. Then there exist $I \in \mathcal{L}$ and a finite subset F of R such that $J \geq I \cap AF^{-1}$. Now we have $Ja^{-1} \geq (I \cap AF^{-1})a^{-1} = Ia^{-1} \cap (AF^{-1})a^{-1}$

$= Ia^{-1} \cap A(aF)^{-1}$. Since $Ia^{-1} \in \mathcal{L}$, we obtain $Ja^{-1} \in \mathcal{L}^*$, proving \mathcal{L}^* satisfies T3.

Now we have a criterion of the maximality of left linear topologies.

Theorem 1.3. *The following conditions are equivalent for a proper left linear topology \mathcal{L} on R :*

- (1) \mathcal{L} is maximal.
- (2) For each left ideal $A \notin \mathcal{L}$, there exist $I \in \mathcal{L}$ and a finite subset F of R such that $I \cap AF^{-1} = 0$.
- (3) For each left ideal $A \notin \mathcal{L}$, there exist $I \in \mathcal{L}$ and a natural number n such that R can be embedded in $R/I \oplus (R/A)^{(n)}$.
- (4) For each left ideal $A \notin \mathcal{L}$, there exists $I \in \mathcal{L}$ such that $R/I \oplus R/A$ is cofaithful.

Proof. \mathcal{L} is maximal if and only if, for each left ideal $A \notin \mathcal{L}$, 0 belongs to the unique minimal left linear topology containing \mathcal{L} and A . Hence by using Lemma 1.2, we have (1) \Leftrightarrow (2). Now assume, for each left ideal $A \notin \mathcal{L}$, there exist $I \in \mathcal{L}$ and a finite subset $\{r_1, \dots, r_n\}$ of R such that $I \cap Ar_1^{-1} \cap \dots \cap Ar_n^{-1} = 0$. Then R is embedded in $R/I \oplus R/Ar_1^{-1} \oplus \dots \oplus R/Ar_n^{-1}$. But $R/Ar_i^{-1} \cong (Rr_i + A)/A \leq R/A$ for each $i = 1, \dots, n$. Hence we have (2) \Rightarrow (3). The implication (3) \Rightarrow (4) is trivial. Finally, assume for each left ideal $A \notin \mathcal{L}$, there exist $I \in \mathcal{L}$ and a natural number n with a monomorphism $f: R \rightarrow (R/I)^{(n)} \oplus (R/A)^{(n)}$. Put $f(1) = (\bar{s}_1, \dots, \bar{s}_n, \bar{r}_1, \dots, \bar{r}_n)$, where $s_i, r_i \in R$ and $\bar{s}_i = s_i + I$ and $\bar{r}_i = r_i + A$ for $i = 1, \dots, n$. Since $f(x) = (x\bar{s}_1, \dots, x\bar{s}_n, x\bar{r}_1, \dots, x\bar{r}_n) = 0$ implies $x = 0$, we have $Is_1^{-1} \cap \dots \cap Is_n^{-1} \cap Ar_1^{-1} \cap \dots \cap Ar_n^{-1} = 0$. Thus we have proved (4) \Rightarrow (2), because $Is_1^{-1} \cap \dots \cap Is_n^{-1} \in \mathcal{L}$.

Corollary 1.4. *The following conditions are equivalent for a proper hereditary pretorsion class \mathcal{S} of $R\text{-mod}$:*

- (1) \mathcal{S} is maximal.
- (2) For each (cyclic) left R -module $M \notin \mathcal{S}$, there exist a cyclic left R -module $C \in \mathcal{S}$ and a natural number n such that R can be embedded in $C \oplus M^{(n)}$.
- (3) For each (cyclic) left R -module $M \notin \mathcal{S}$, there exists a cyclic left R -module $C \in \mathcal{S}$ such that $C \oplus M$ is cofaithful.

Corollary 1.5. *The following conditions are equivalent for a proper*

left exact preradical r for R-mod :

- (1) *r is maximal.*
- (2) *For each (cyclic) left R-module M with r(M) ≠ M, there exists a cyclic left R-module C with r(C) = C such that t_C + t_M is cofaithful.*

In [8] Rubin called a left ideal A of R *weakly essential* if $AF^{-1} \neq 0$ for every finite subset F of R. Note that, if a left ideal A is weakly essential, then AX^{-1} is also weakly essential for every finite subset X of R. We remark that every member of a proper left linear topology on R is weakly essential. Now we shall consider the case when a left linear topology is unique maximal.

Proposition 1.6. *The following conditions are equivalent for a proper left linear topology \mathcal{L} on R :*

- (1) \mathcal{L} is unique maximal.
- (2) \mathcal{L} coincides with the set of all weakly essential left ideals of R.
- (3) \mathcal{L} contains all weakly essential left ideals of R.

Proof. For a left ideal A of R, we put \mathcal{L}_A a unique minimal left linear topology containing A. Then \mathcal{L}_A consists of those left ideals B such that $B \geq AF^{-1}$ for some finite subset F of R.

(1) \Rightarrow (2). If $A \in \mathcal{L}$, then $AF^{-1} \in \mathcal{L}$ for every finite subset F of R. Since \mathcal{L} is proper, we see that A is weakly essential. Conversely assume A is a weakly essential left ideal of R. Then \mathcal{L}_A is proper and so $\mathcal{L}_A \subseteq \mathcal{L}$. Hence we have $A \in \mathcal{L}$.

(2) \Rightarrow (3). Clear.

(3) \Rightarrow (1). Let \mathcal{L}' be a proper left linear topology on R. For each $A \in \mathcal{L}'$, we have $AF^{-1} \in \mathcal{L}'$ for every finite subset F of R. Since \mathcal{L}' is proper, we see A is weakly essential, and so $A \in \mathcal{L}$ by (3). Therefore we have proved \mathcal{L} is unique maximal.

The following corollary was proved by Nicholson and Sarath by using the notion of α -weak essentiality. But we can prove this directly.

Corollary 1.7 (Nicholson and Sarath [6, Theorem 1]). *The following conditions are equivalent for a ring R with the set \mathcal{L} of all weakly essential left ideals of R :*

- (1) *R has a unique maximal left linear topology.*
- (2) \mathcal{L} forms a left linear topology.

(3) *If A and B belong to \mathcal{L} , then $A \cap B \neq 0$.*

Proof. (1) \Leftrightarrow (2). This is clear by using Proposition 1.6.

(2) \Rightarrow (3). Clear.

(3) \Rightarrow (2). Clearly \mathcal{L} satisfies T1. As noted above, \mathcal{L} also satisfies T3. Now assume A and B belong to \mathcal{L} . For every finite subset F of R , we see that AF^{-1} and BF^{-1} belong to \mathcal{L} . Hence $(A \cap B)F^{-1} = AF^{-1} \cap BF^{-1} \neq 0$ by (3). Thus $A \cap B$ belongs to \mathcal{L} . Therefore we showed that \mathcal{L} satisfies T2.

Example 1.8. Let R be a ring and \mathcal{L} the set of all essential left ideals of R . It is well known that \mathcal{L} is a proper left linear topology on R . By using Theorem 1.3, we notice that \mathcal{L} is maximal if and only if every weakly essential left ideal of R is essential. In this case, \mathcal{L} is unique maximal by Proposition 1.6. In case R is commutative, we remark that \mathcal{L} is maximal if and only if every nonzero ideal of R is essential. Thus we conclude that, if R is a commutative semiprime ring, \mathcal{L} is maximal if and only if R is prime.

2. The complement of linear topologies. In [7] Meijer and Smith concerned with the collection $N(R)$ of those left ideals of R which do not belong to any proper left Gabriel topology on R . As mentioned in [7, Lemma 2.1], a left ideal I belongs to $N(R)$ if and only if $\text{Hom}_R(R/I, E) \neq 0$ for every nonzero injective left R -module E . Now we shall consider the set

$$C(R) = \{I \in \mathcal{L}(R) \mid I \notin \mathcal{L} \text{ for every proper left linear topology } \mathcal{L} \text{ on } R\}.$$

Clearly $0 \in C(R)$ and $R \notin C(R)$. If $A \in \mathcal{L}(R)$ and $A \leq B$ for some $B \in C(R)$, then $A \in C(R)$. Remark that $C(R) \subseteq N(R)$.

Theorem 2.1. *The following statements are equivalent for a left ideal A of a ring R :*

- (1) $A \in C(R)$.
- (2) A is not weakly essential, i.e., $AF^{-1} = 0$ for some finite subset F of R .
- (3) R/A is cofaithful.

Proof. (1) \Leftrightarrow (2). For a left ideal A , $A \in C(R)$ if and only if $0 \in \mathcal{L}$ for every left linear topology \mathcal{L} containing A , or equivalently

0 belongs to the unique minimal left linear topology containing A . As noted in the proof of Proposition 1.6, $0 \in \mathcal{L}_A$ if and only if $AF^{-1} = 0$ for some finite subset F of R .

(2) \Leftrightarrow (3). This is proved by the same method as is used in the proof of Theorem 1.3. (See [1, Proposition 4.5.4]).

A left linear topology \mathcal{L} is called *super* if \mathcal{L} contains a unique minimal member. Such a member is in fact a two-sided ideal. We denote by $C_s(R)$ the set of those left ideals which do not belong to any proper super left linear topology on R . Clearly $C_s(R) \supseteq C(R)$. If R is left artinian, then every left linear topology on R is super, and so $C_s(R) = C(R)$. For a ring R with Jacobson radical J , it was proved in [7, Proposition 2.9] that $N(R)$ consists of all proper left ideals of R if and only if J is right T-nilpotent and R/J is a simple artinian ring. By the definition, $C_s(R)$ consists of all proper left ideals of R if and only if R is a two-sided simple ring.

Theorem 2.2. *The following statements are equivalent for a ring R :*

- (1) $C(R)$ contains all maximal left ideals of R .
- (2) $C(R)$ consists of all proper left ideals of R .
- (3) R is a simple artinian ring.

Proof. (1) \Rightarrow (2). Clear.

(2) \Rightarrow (3). Assume (2). Then every nonzero cyclic left R -module is cofaithful by Theorem 2.1. Thus every nonzero left R -module is also cofaithful. In particular every nonzero left ideal of R is cofaithful, and so R is left strongly prime (see [5, Proposition 2.5]). Also since every faithful left R -module is cofaithful, $\text{soc}({}_R R) \neq 0$ by [2, Proposition 1]. Hence R must be simple artinian by [5, Theorem 4.3].

(3) \Rightarrow (1). Assume I is a maximal left ideal of R . Then ${}_R(R/I)$ is cofaithful and so I belongs to $C(R)$ by Theorem 2.1.

In [7] the other extreme when $N(R) = \{0\}$ was considered. It was shown in [7, Theorem 6.4] that $N(R) = \{0\}$ if and only if R is a reduced ring and $Ra + 0a^{-1}$ is essential left ideal of R for all $a \in R$. We remark that $C_s(R) = \{0\}$ if and only if every nonzero left ideal of R contains a nonzero ideal of R .

Theorem 2.3. *The following statements are equivalent for a ring R :*

- (1) $C(R) = \{0\}$.

- (2) Every nonzero left ideal of R is weakly essential.
- (3) Every nonzero cyclic left ideal of R is weakly essential.
- (4) For every nonzero element a of R and elements r_1, \dots, r_n of R , there exist a nonzero element a' of R and elements r'_i in R such that $a'r_i = r'_i a$ ($i = 1, \dots, n$).

Proof. (1) \Leftrightarrow (2). This is clear by Theorem 2.1.

(2) \Leftrightarrow (3). Clear.

(3) \Leftrightarrow (4). Let $A = Ra$ be a nonzero cyclic left ideal of R . Then A is weakly essential if and only if, for every elements r_1, \dots, r_n of R , $Ar_1^{-1} \cap \dots \cap Ar_n^{-1} \neq 0$ holds. This occurs if and only if, for every elements r_1, \dots, r_n of R , there exists a nonzero element a' of R such that $a'r_i \in A = Ra$ ($i = 1, \dots, n$).

Corollary 2.4 (cf. [7, Corollaries 5.2 and 6.5]). *If R is a domain, then $C(R) = \{0\}$ if and only if R satisfies the left Ore condition.*

Proof. Assume R is a left Ore domain with a classical left quotient ring $Q_{cl}'(R)$. For every nonzero element a of R and elements r_1, \dots, r_n of R , there exist nonzero elements a'_i of R and elements s_i of R such that $r_i a^{-1} = a'^{-1} s_i$ ($i = 1, \dots, n$). As is well known (see [4, p. 392]), there exist a nonzero element a' of R and elements t_i of R such that $a'^{-1} = a'^{-1} t_i$ ($i = 1, \dots, n$). Put $r'_i = t_i s_i$ ($i = 1, \dots, n$). Thus we have $a'r_i = r'_i a$ ($i = 1, \dots, n$), and so $C(R) = \{0\}$. We can also show this fact by using [7, Corollary 5.2] with $C(R) \subseteq N(R)$. The reverse implication is obvious.

Remark 2.5. The property that $C(R) = \{0\}$ of rings R is not a Morita invariant. To see this, let K be a field. By Theorem 2.3, we see $C(K) = \{0\}$. But consider the ring R of $n \times n$ matrices over K for some $n > 1$. As shown in Theorem 2.2, we have $C(R) \neq \{0\}$. On the other hand, the property that R has a unique maximal left linear topology is a Morita invariant ([6, Corollary to Theorem 2]). Hence we conclude that the above two properties on R are not equivalent.

By using Theorem 2.3, we shall prove the next two propositions.

Proposition 2.6. *If R is a left order in a ring Q , then $C(Q) = \{0\}$ implies $C(R) = \{0\}$. Furthermore, if R is a domain, then $C(Q) = \{0\}$.*

Proof. Suppose there are given elements $r (\neq 0), r_1, \dots, r_n$ in R .

By $\mathbf{C}(Q) = \{0\}$, there exist elements $q(\neq 0), q_1, \dots, q_n$ in Q such that $qr_i = q_i r$ ($i = 1, \dots, n$). We can find a regular element r' in R with $r'q(\neq 0), r'q_1, \dots, r'q_n \in R$. Thus we have $(r'q)r_i = (r'q_i)r$ ($i = 1, \dots, n$), and so $\mathbf{C}(R) = \{0\}$.

Now assume R is a domain. For every elements $q(\neq 0), q_1, \dots, q_n$ of Q , there exist a regular element r in R with $rq(\neq 0), rq_1, \dots, rq_n \in R$. Since $\mathbf{C}(R) = \{0\}$ by Corollary 2.4, there exist $r'(\neq 0), r'_1, \dots, r'_n$ in R such that $r'(rq_i) = r'_i(rq)$ ($i = 1, \dots, n$). Noting that $r'r(\neq 0)$ and $r'_i r$ ($i = 1, \dots, n$) belong to Q , we obtain $\mathbf{C}(Q) = \{0\}$.

Proposition 2.7. *Suppose $R = R_1 \times \dots \times R_n$ is a direct sum of rings R_i ($i = 1, \dots, n$). Then $\mathbf{C}(R) = \{0\}$ if and only if $\mathbf{C}(R_i) = \{0\}$ for all $i = 1, \dots, n$.*

Proof. We may assume $n = 2$. Let S and T be rings. Assume $\mathbf{C}(S) = \{0\}$ and $\mathbf{C}(T) = \{0\}$. Let $(s, t)(\neq 0), (s_1, t_1), \dots, (s_n, t_n)$ be elements of $S \times T$. We may assume that $s \neq 0$. By $\mathbf{C}(S) = \{0\}$, there exist $s'(\neq 0), s'_1, \dots, s'_n$ in S such that $s's_i = s'_i s$ ($i = 1, \dots, n$). If $t = 0$, then we have $(s', t)(s_i, t_i) = (s'_i, t_i)(s, t)$ ($i = 1, \dots, n$). If $t \neq 0$, by $\mathbf{C}(T) = \{0\}$, there exist $t'(\neq 0), t'_1, \dots, t'_n$ in T such that $t't_i = t'_i t$ ($i = 1, \dots, n$), and so we have $(s', t')(s_i, t_i) = (s'_i, t'_i)(s, t)$ ($i = 1, \dots, n$). Therefore we have $\mathbf{C}(S \times T) = \{0\}$.

Conversely assume $\mathbf{C}(S \times T) = \{0\}$. To show $\mathbf{C}(S) = \{0\}$, let $s(\neq 0), s_1, \dots, s_n$ be elements of S . For the elements $(s, 0), (s_1, 1), \dots, (s_n, 1)$ in $S \times T$, there exist elements $(s', t')(\neq 0), (s'_1, t'_1), \dots, (s'_n, t'_n)$ in $S \times T$ such that $(s', t')(s_i, 1) = (s'_i, t'_i)(s, 0)$ ($i = 1, \dots, n$). Then we have $s's_i = s'_i s$ ($i = 1, \dots, n$) and $s' \neq 0$ because $t' = 0$. Therefore we showed $\mathbf{C}(S) = \{0\}$.

Example 2.8. There may be many rings R such that $\mathbf{C}(R)$ are not extreme. To give such an example, we shall calculate $\mathbf{C}(R)$ where R is the 2×2 upper triangular matrix ring over a field K . There are three types of minimal left ideals of R , namely $A = \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}$ and $B = \left\{ \begin{pmatrix} xa & xb \\ 0 & 0 \end{pmatrix} \mid x \in K \right\}$ for some fixed nonzero elements a and b of K . Let e_{11} , e_{12} and e_{22} be matrix units in R . Since $Ae_{12}^{-1} \cap Ae_{22}^{-1} = 0$, A belongs to $\mathbf{C}(R)$. Also since $Be_{11}^{-1} \cap Be_{22}^{-1} = 0$, B belongs to $\mathbf{C}(R)$. But since C is an ideal of R , it is weakly essential and so C does not belong to $\mathbf{C}(R)$. Now

let I be a left ideal of R which contains A or some B strictly. Then I also contains C and so I does not belong to $\mathbf{C}(R)$. Thus we conclude that $\mathbf{C}(R)$ consists precisely of A and those left ideals B .

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