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Sen-Yen Shaw*

*National Central University

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MEAN AND POINTWISE ERGODIC THEOREMS FOR COSINE OPERATOR FUNCTIONS

SEN-YEN SHAW

1. Introduction. The purpose of this paper is to present a mean ergodic theorem and two pointwise ergodic theorems for a strongly continuous cosine operator function.

Let X be a Banach space and $B(X)$ be the Banach algebra of all bounded linear operators on X. A one-parameter family $\{C(t)\ ;\ t \geq 0\}$ in $B(X)$ is called a strongly continuous cosine function if it satisfies the three conditions:

(1) $C(t+s)+C(t-s) = 2C(t)C(s)$ for all $t \geq s \geq 0$;

(2) $C(0) = I$ (the identity operator) ;

(3) $C(t)$ is strongly continuous in t on $[0,\infty)$.

The associated sine function $S(\cdot)$ is defined by $S(t)x = \int_0^t C(s)xds$ $(x \in X)$.

There exist constants $w > 0$ and $M_w > 0$ such that $\| C(t) \| \leq M_w e^{wt}$ for all $t \geq 0$. We shall denote by w_0 the infimum of the set of all such w and call it the type of $C(\cdot)$. Let A be the infinitesimal generator of $C(\cdot)$, defined as $Ax := \lim_{t \to 0+} 2t^{-2}(C(t)-I)x$ in its natural domain $D(A)$. Then A is a densely defined closed operator, the resolvent set $\rho(A)$ contains all λ^2 with $\lambda > w_0$, and for each such λ

$$\lambda(\lambda^2 I - A)^{-1} = \int_0^\infty e^{-\lambda t} C(t)dt.$$

We shall use $L(\lambda)$ to denote this operator. For these and other fundamental properties of $C(\cdot)$ the reader is refered to [3] and [11].

The operators $t^{-1}S(t)$, $t > 0$, and $\lambda L(\lambda)$, $\lambda > 0$, are the Cesaro averages and the Abel averages of $C(\cdot)$, respectively. In section 2 we shall relate the convergence of $\lim_{t\to\infty} t^{-1}S(t)x$, $\lim_{\lambda\to 0} \lambda L(\lambda)x$, and $\lim_{n\to\infty} n^{-1}\sum_{i=0}^{n-1} C(it)x$. In section 3, X is assumed to be a Lebesgue space $L_p(S, \Sigma, \mu\ ;\ Y)$, $1 \leq p < \infty$, with Y a reflexive space. Under suitable conditions the almost everywhere convergence of $t^{-1}S(t)f$ for $f \in L_p \cap L_\infty$ and of $\lambda L(\lambda)f$ for f in L_p will be justified.

2. Mean ergodic theorems. Suppose $C(\cdot)$ is a strongly continuous cosine function such that $\|C(t)\| \leq M$ for all $t \geq 0$. Then $C(\cdot)$ has type $w_0 = 0$. We denote by P_c[resp. P_a] the operator defined by

$$P_c x := \lim_{t\to\infty} t^{-1}S(t)x \text{ [resp. } P_a := \lim_{\lambda\to 0} \lambda L(\lambda)x],$$

with domain consisting of all those x for which the limit exists. Also we define for each $t > 0$ the operator P_t by

$$P_t\, x := \lim_{n\to\infty} n^{-1} \sum_{i=0}^{n-1} C(it)x.$$

The following theorem is proved in [10] ; it characterizes the range $R(P_c)$, the null space $N(P_c)$, and the domain $D(P_c)$ of P_c, and also those of P_t.

Theorem A. *Under the hypothesis* : $\|C(t)\| \leq M$ *for all* $t \geq 0$, *one has* :

(i) $P_c = P_a$ *and is a bounded linear projection with* $R(P_c) = N(A) = \bigcap_{s>0} N(C(s)-I)$, $N(P_c) = \overline{R(A)} = \overline{\bigcup_{s>0} R(C(s)-I)}$, *and* $D(P_c) = \bigcap_{s>0} N(C(s)-I) \oplus \overline{\bigcup_{s>0} R(C(s)-I)}$ $= \{x \in X ;\ \exists \{t_n\} \to \infty \ni \text{w}-\lim_{n\to\infty} t_n^{-1}S(t_n)x \text{ exists}\}$.

(ii) *For each* $t > 0$, P_t *is a bounded linear projection with* $R(P_t) = N(C(t)-I)$, $N(P_t) = \overline{R(C(t)-I)}$, *and* $D(P_t) = N(C(t)-I) \oplus \overline{R(C(t)-I)}$ $= \{x \in X ;\ \exists \{n_k\} \to \infty \ni \text{w}-\lim_{k\to\infty} n_k^{-1} \sum_{i=0}^{n_k-1} C(it)x \text{ exists}\}$.

We shall use the above theorem to prove the following theorem, which gives a sufficient condition for P_t to coincide with P_c. It is known that the same assertion holds for semigroups (cf. Sato [8]).

Theorem 1. *Let* $C(\cdot)$ *be a strongly continuous cosine function of uniformly bounded operators. Suppose there exists a* $\delta > 0$ *such that* $C(t)+I$ *is invertible (particularly,* $\|C(t)-I\| < 2$*) for all* $t \in (0, \delta)$. *Then* $P_t = P_c$ *for all* $t \in (0, 2\delta)$.

Proof. Since by Theorem A one has that $R(P_c) \subset R(P_t)$ and $N(P_t) \subset N(P_c)$, it remains for us to show $R(P_t) \subset D(P_c)$ and $N(P_c) \subset N(P_t)$ for all t in $(0, 2\delta)$.

Using (1) we can easily show by induction that each $C(it)-I$ is a polynomial of $C(t)$ and is divisible by $C(t)-I$. Also we can write

$$\begin{aligned}
&\left(\left(n-\frac{1}{2}\right)t\right)^{-1} S\left(\left(n-\frac{1}{2}\right)t\right)\\
&\quad=\left(\left(n-\frac{1}{2}\right)t\right)^{-1}\left\{\int_0^{\frac{1}{2}t}+\sum_{i=1}^{n-1}\left[\int_{(i-\frac{1}{2})t}^{it}+\int_{it}^{(i+\frac{1}{2})t}\right]\right\}C(s)bs\\
&\quad=\left(\left(n-\frac{1}{2}\right)t\right)^{-1}\left\{S\left(\frac{1}{2}t\right)+\sum_{i=0}^{n-1}\int_0^{\frac{1}{2}t}[C(it-s)+C(it+s]ds\right\}\\
&\quad=\left(\left(n-\frac{1}{2}\right)t\right)^{-1}\left\{S\left(\frac{1}{2}t\right)+\sum_{i=1}^{n-1}\int_0^{\frac{1}{2}t}2C(s)C(it)ds\right\}\\
&\quad=\frac{2n}{2n-1}(t/2)^{-1}S(t/2)\left[n^{-1}\sum_{i=0}^{n-1}C(it)\right].
\end{aligned}$$

Hence, if $x\in R(P_t)=N(C(t)-I)$, then $x\in N(C(it)-I)$ so that

$$\left(\left(n-\frac{1}{2}\right)t\right)^{-1} S\left(\left(n-\frac{1}{2}\right)t\right)x=\frac{2n}{2n-1}(t/2)^{-1}S(t/2)x,$$

which converges to $(t/2)^{-1}S(t/2)x$ as $n\to\infty$. So, Theorem A(i) implies that x belongs to $D(P_c)$.

Next, let E be the set of all $s>0$ such that $R(C(s)-I)$ is contained in $\overline{R(C(t)-I)}$. Then to show $N(P_c)\subset N(P_t)$ is equivalent to showing that $E=(0,\infty)$. We first prove $t/2\in E$. If $x\in R(C(t/2)-I)$, then we have

$$\begin{aligned}
&\frac{1}{2}\left[I+C\left(\frac{t}{2}\right)\right]\left[\frac{1}{n}\sum_{i=0}^{n-1}C(it)\right]x\\
&\quad=\frac{1}{2n}\left\{\sum_{i=0}^{n-1}C(it)+C\left(\frac{t}{2}\right)+\frac{1}{2}\sum_{i=1}^{n-1}\left[C\left((2i+1)\frac{t}{2}\right)+C\left((2i-1)\frac{t}{2}\right)\right]\right\}x\\
&\quad=\frac{1}{2n}\sum_{j=0}^{2n-1}C\left(j\frac{t}{2}\right)x+\frac{1}{2n}\left[C\left(\frac{t}{2}\right)-C\left((2n-1)\frac{t}{2}\right)\right]x,
\end{aligned}$$

which converges to $P_{t/2}x=0$ as $n\to\infty$. Since $I+C(t/2)$ is invertible, we must have that $P_tx=\lim_{n\to\infty}n^{-1}\sum_{i=0}^{n-1}C(it)x=0$, i. e. $x\in\overline{R(C(t)-I)}$. Repeating the same process and noting that $C(ms)-I$ is divisible by $C(s)-I$, we see that E contains all numbers of the form $(m/2^n)t$, m, $n=1,2,\ldots$, which form a dense subset of $(0,\infty)$. Then the strong continuity of $C(\cdot)$ shows that the whole set $(0,\infty)$ is contained in E. Hence the theorem is proved.

3. Pointwise ergodic theorems. Throughout this section, (S,Σ,μ) is a σ-finite measure space, $(Y, |\cdot|)$ is a refiexive Banach space, and $C(\cdot)$ is a strongly continuous cosine function of linear operators on $L_1 = L_1(S, \Sigma, \mu; Y)$. In addition, we assume that $\|C(t)\|_1 \leq 1$ for all $t \geq 0$, and that for some constant $K \geq 1$ $\sup_{t\geq 0} \|C(t)f\|_\infty \leq K \|f\|_\infty$ for all $f \in L_1 \cap L_\infty$.

It is known that each $C(t)$ can be extended so that it is defined on each $L_p = L_p(S, \Sigma, \mu; Y)$, $1 \leq p < \infty$ (see [1]), and the extended operator $C(t)$ has norm $\|C(t)\|_p \leq K$, by the Riesz convexity theorem (see [2, VI. 10. 11]). Thus for each $1 \leq p < \infty$ $C(\cdot)$ is a cosine function of operators on L_p. Moreover, it is strongly continuous on $(0,\infty)$. To see this let $f \in L_1 \cap L_p$ so that the function $C(\cdot)f$ is continuous in L_1 and hence $(C(t)f)(s)$ is [t, s]-measurable on $(0, \infty)\times S$ (cf. [2, III. 11. 16-(a)]). It follows from part (b) of the same lemma that $C(\cdot)f$ as a L_p-valued function is Lebesgue measurable on $(0, \infty)$. Since $L_1 \cap L_p$, $1 \leq p< \infty$, is dense in L_p, $C(\cdot)$ is strongly measurable on $(0,\infty)$ when regarded as operators on L_p. It follows that $C(\cdot)$ is strongly continuous on $(0,\infty)$([3],[7]) and hence is also right continuous at 0, by (1).

By Theorem III.11.17 of [2] there is for each $f \in L_p$ a Y-valued function $g(t,s)$, defined on $(0, \infty)\times S$ and strongly measurable with respect to the product of Lebesgue measure and μ, such that for each fixed $t > 0$ $g(t, s)$ as a function of s belongs to the equivalence class of $C(t)f \in L_p$. We shall denote this function $g(t, s)$ by the notation $(C(t)f)(s)$. The same theorem also shows the existence of a μ-null set $N(f)$, dependent on f but independent of t, such that for every s not in $N(f)$ $(C(\cdot)f)(s)$ is Bochner integrable on every finite interval $[0, t]$ with respect to Lebesgue measure, and the function : $s \to (S(t)f)(s) := \int_0^t (C(u)f)(s)du$ belongs to the equivalence class of $S(t)f \in L_p$. Similarly, there exists a μ-null set $N'(f)$, dependent on f but independent of t, such that for every s not in $N'(f)$ the function : $t \to e^{-\lambda t}(C(t)f)(s)$ is Bochner integrable on $(0, \infty)$, and the function $s \to (L(\lambda)f)(s) := \int_0^\infty e^{-\lambda t}(C(t)f)(s)dt$ belongs to the equivalence class of $L(\lambda)f \in L_p$.

The pointwise ergodic theorems are concerned with μ-almost everywhere convergence of $t^{-1}(S(t)f)(s)$ and $\lambda(L(\lambda)f)(s)$ as $t \to \infty$ and $\lambda \to 0^+$, or as $t \to 0^+$ and $\lambda \to \infty$. They are stated as follows.

Theorem 2. *Let Y be a reflexive Banach space, (S, Σ, μ) a σ-finite measure space, and let $C(\cdot)$ be a strongly continuous cosine function of* *linear contractions on* $L_1(S, \Sigma, \mu; Y)$ *such that, for some constant* $K \geq 1$, $\sup_{t>0} \| C(t)f\|_\infty \leq K \|f\|_\infty$ *for all* $f \in L_1 \cap L_\infty$. *Then the following statements hold for all* $1 \leq p < \infty$:

(i) *For every* $f \in L_p$ *the Abel ergodic limit* $f_1(s) := \lim_{\lambda \to 0^+} \lambda(L(\lambda)f)(s)$ *exists almost everywhere on* S.

(ii) *For every* $f \in L_p \cap L_\infty$ *the Cesàro ergodic limit* $\lim_{t\to\infty} t^{-1}(S(t)f)(s)$ *exists and equals* $f_1(s)$ *for almost all* s *in* S.

Theorem 3. *Let* Y *and* $C(\cdot)$ *be as assumed in Theorem 2. Then the following statements hold for all* $1 \leq p < \infty$:

(i) *For every* $f \in L_p$ *the Abel ergodic limit* $f_2(s) := \lim_{\lambda\to\infty} \lambda(L(\lambda)f)(s)$ *exists almost everywhere on* S.

(ii) *For every* $f \in L_p \cap L_\infty$ *the local Cesàro ergodic limit* $\lim_{t\to 0^+} t^{-1}(S(t)f)(s)$ *exists and equals* $f_2(s)$ *for almost all* s *in* S.

Since $\| C(t)\|_1 \leq 1$ for all $t > 0$, $C(\cdot)$ has type $w_0 = 0$ so that the resolvent $R_\lambda = (\lambda I - A)^{-1} = \lambda^{-\frac{1}{2}} L(\lambda^{\frac{1}{2}})$ exists for all $\lambda > 0$. Moreover, we have $\|\lambda R_\lambda\|_1 \leq 1$ and for all $f \in L_1 \cap L_\infty$ and almost all $s \in S$

$$|\lambda(R_\lambda f)(s)| = |\lambda^{\frac{1}{2}} \int_0^\infty e^{-\lambda^{\frac{1}{2}}t}(C(t)f)(s)dt| \leq \lambda^{\frac{1}{2}} \int_0^\infty e^{-\lambda^{\frac{1}{2}}t} |(C(t)f)(s)| dt$$

$$\leq \lambda^{\frac{1}{2}} \int_0^\infty e^{-\lambda^{\frac{1}{2}}t} \|C(t)f\|_\infty dt \leq K \|f\|_\infty,$$

i. e. $\|\lambda R_\lambda f\|_\infty \leq K \|f\|_\infty$. Hence $\{R_\lambda ; 0 < \lambda < \infty\}$ satisfies the conditions in the following pointwise ergodic theorem of Sato [9] for pseudo-resolvents, and consequently the Abel averages $\lambda(L(\lambda)f)(s)$ converge almost everywhere for all $f \in L_p$, as either $\lambda \to 0^+$ or $\lambda \to \infty$.

Theorem B. *Let* $\{J_\lambda ; 0 < \lambda < \infty\}$ *be a pseudo-resolvent of linear contractions on* $L_1(S, \Sigma, \mu; Y)$ *such that, for some constant* $K \geq 1$, $\sup_{\lambda>0} \|\lambda J_\lambda f\|_\infty \leq K \|f\|_\infty$ *for all* $f \in L_1 \cap L_\infty$. *Then for every* $1 \leq p < \infty$ *and every* $f \in L_p$ *the limits*

$$\lim_{\lambda \to 0^+} \lambda(J_\lambda f)(s) \quad \textit{and} \quad \lim_{\lambda\to\infty} \lambda(J_\lambda f)(s)$$

exist almost everywhere on S.

Finally, the validity of the assertions in Theorems 2 and 3 about the Cesàro limits is guaranteed by the following theorem, which is contained as a special case in Theorems 18.2.1 and 18.3.3 (and a remark following it) of [6].

Theorem C. *Let g be a bounded and Lebesgue measurable Y-valued function on* $(0, \infty)$. *Then*

$$\lim_{t\to\infty} t^{-1}\int_0^t g(s)ds = \lim_{\lambda\to 0}+\lambda\int_0^\infty e^{-\lambda t}g(t)dt$$

provided one of the limits exists. The same assertion still holds when "$t\to\infty$" and "$\lambda\to 0^+$" are replaced by "$t\to 0^+$" and "$\lambda\to\infty$", respectively.

Remark. Since $L_P\cap L_\infty$ is dense in L_p, the conclusion (ii) of Theorems 2 and 3 might be extended to include all f in L_p provided that one could prove such a maximal ergodic inequality :

$$(*)\quad \mu(\{s\ ;\ \sup_{t>0}|t^{-1}(S(t)f)(s)|>a\})\le Ca^{-p}\|f\|_p^p\quad (a>0,\ f\in L_p).$$

(cf. [4, Theorem 1.1]). A key to $(*)$ would be the following cosine version of Chacon maximal ergodic inequality :

$$(**)\qquad \int_{e^*(ka)}(a-|f^{a-}(s)|)\,d\mu\le\int_S|f^{a+}(s)|\,d\mu\quad (a>0,\ f\in L_p),$$

where $e^*(ka):=\left\{s\ ;\ \sup_{n\ge 1}\left|\frac{1}{n}\sum_{i=0}^{n-1}(C(i)f)(s)\right|>ka\right\}$, $f^{a-}(s):=\frac{f(s)}{|f(s)|}\min(a,|f(s)|)$ and $f^{a+}(s):=f(s)-f^{a-}(s)$.

if $(**)$ is true, then one can use the same arguments in Theroem 2 of [5] to derive a continuous version of $(**)$, from which then follows a dominated ergodic theorem (like Theorem 3 of [5]) and in particular $(*)$. Moreover, this would enable one to directly prove the completed Theorems 2 & 3, without using Theorems B and C. At present, the author has not found a proof of $(*)$ or $(**)$ yet.

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DAPARTMENT OF MATHEMATICS
NATIONAL CENTRAL UNIVERSITY
CHUNG-LI, TAIWAN, R. O. C.

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