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Roger Yue Chi Ming*

*Université Paris VII

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ROGER YUE CHI MING

In [5], it was shown that a ring without non-zero nilpotent elements is von Neumann regular if and only if every principal left ideal is the left annihilator of an element of A. This together with our result mentioned in [1] will be extended in the first theorem of this note. In [3] and [4], rings whose maximal left ideals are left annihilators are studied. A characterization of an arbitrary regular ring is here given in terms of finitely generated maximal and projective left annihilators (Theorem 2). Several characteristic properties of a semi-simple Artinian ring are obtained in terms of annihilators (Theorem 3).

Throughout, A denotes an associative ring with identity and Z is the left singular ideal of A. We recall that (1) A is called *reduced* if A contains no non-zero nilpotent elements; (2) A left ideal I is *closed* in A if I has no proper essential extensions in A; (3) A left (right) A-module M is *p-injective* if, for any principal left (right) ideal P of A and any left (right) A-homomorphism $g: P \longrightarrow M$, there exists $c \in M$ such that $g(b) = bc$ $(g(b) = cb)$ for all $b \in P$. Then, A is strongly regular if and only if A is regular reduced, and it is easy to see that any p-injective left (right) ideal of A is idempotent.

The next lemma is proved in [6] (see [4, Lemma 3] for the proof).

Lemma 1. *The following conditions are equivalent:*

1) $Z = 0$ *and every closed left ideal of* A *is two-sided;*

2) A *is a reduced ring and* $I + l(I)$ $(= I \oplus l(I))$ *is essential in* A *for every left ideal* I *of* A;

3) A *is a reduced ring and every closed left ideal of* A *is a left annihilator.*

The characterization of a strongly regular ring given in [5] and [6] (see [1]) is improved in the next theorem.

Theorem 1. *The following conditions are equivalent:*

1) A *is strongly regular;*

2) A *is a reduced ring and every principal left ideal of* A *is a left annihilator;*

3) A *is a reduced ring and every maximal left ideal of* A *is p-injective;*

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4) *A is a reduced ring and every maximal left ideal of A is either p-injective or a left annihilator.*

5) $Z = 0$ *and every finitely generated left ideal of A is the left annihilator of a left ideal*;

6) $Z = 0$ *and every principal left ideal of A is closed in A and two-sided.*

Proof. Obviously, 3) implies 4) and 1) implies 5) and 6). 1) implies 3) by [5, Lemma 2] (cf. also [2]), and each of 5) and 6) does 2) by Lemma 1.

2) $\Longrightarrow$ 1) Let $b \in A$. In the reduced ring A, as is well-known, $ab^2 = 0$ if and only if $ab = 0$, namely, $l(b) = l(b^2)$. Since Ab is a left annihilator, $Ab = l(r(Ab)) = l(r(b)) = l(l(b)) = l(l(b^2)) = Ab^2$, which proves 1).

4) $\Longrightarrow$ 1) Let $b \in A$. We claim first $Ab + l(b) = A$. If not, there exists a maximal left ideal L containing $Ab + l(b)$. In case L is p-injective, considering the canonical injection $i : Ab \longrightarrow L$, we can find $c \in L$ with $b = bc$. Then, $1 - c \in r(b) = l(b) \subseteq L$, whence it follows a contradiction $1 \in L$. On the other hand, in case $L = l(t)$ with some $0 \neq t \in A$, we have $t \in r(Ab + l(b)) \subseteq r(b) \subseteq L$. Then, $t^2 = 0$, a contradiction. Now, let $ab + d = 1$, $a \in A$, $d \in l(b)$. Then $ab^2 = b$, which proves 1).

A left ideal I of A is called a *maximal left annihilator* if $I = l(S)$ for some non-empty subset $S \neq 0$ of A and for any left annihilator J with $I \subseteq J$, either $J = I$ or $J = A$. In that case, $I = l(s)$ for any $0 \neq s \in S$.

Theorem 2. *The following conditions are equivalent*:

1) *A is regular*;

2) *A is a semi-prime ring such that every finitely generated left ideal is the projective left annihilator of an element of A.*

3) *A is a semi-prime ring such that every finitely generated left ideal is either a maximal left annihilator or the projective left annihilator of an element of A.*

Proof. If A is regular, every finitely generated left ideal of A is obviously the projective left annihilator of an element. Hence, 1) $\Longrightarrow$ 2) $\Longrightarrow$ 3). Assume 3). At any rate, every finitely generated left ideal of A is the left annihilator of an element. And so, it suffices to prove that any principal left ideal is projective. First, we claim $Z = 0$. Suppose there exists $0 \neq z \in Z$. Since Az can not be projective, Az is a maximal left annihilator. Accordingly, for any $w \notin Az$ we have $(Az \subset)\ Az + Aw = A$, which means that Az is a maximal left ideal and $Az = Z$. Moreover, recalling that Z contains no non-zero idempotents, we have $Az = l(t)$ with some $0 \neq t \in Z$. But then $(At)^2 \subseteq ZAt \subseteq Zt = Azt = 0$, which contradicts the semi-primeness of A. Thus $Z = 0$. Now, assume that a principal left ideal I is a maximal left annihilator. Since $Z=0$, there exists $0 \neq b \in A$ with $I \cap Ab = 0$. Recalling that $I \oplus Ab\ (\supset I)$ is projective, we see that I is projective, completing the proof.

The next result is motivated by [3, Corollary 2].

Theorem 3. *The following conditions are equivalent:*

1) *A is semi-simple, Artinian.*

2) *Every left ideal of A is the left annihilator of an idempotent right ideal;*

3) *The right annihilator of any maximal left ideal is a non-zero p-injective right ideal;*

4) *The right annihilator of any maximal left ideal contains a non-zero idempotent right ideal.*

Proof. Obviously 1)$\Longrightarrow$2)$\Longrightarrow$4), and 1)$\Longrightarrow$3)$\Longrightarrow$4) (cf. [5, Lemma 2] or [2]). In order to see that 4) implies 1), it suffices to prove that every maximal left ideal I of A is a direct summand. Let J be a nonzero idempotent right ideal contained in $r(I)$. Let $s,\ t \in J$ with $st \neq 0$. Then $I = l(J) = l(s) = l(t) = l(st)$. Suppose I is essential in A. Then there exists $b \in A$ such that $0 \neq bs \in I$, which implies $b \in l(st) = I$. This contradicts $bs \neq 0$.

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UNIVERSITÉ PARIS VII
U. E. R. DE MATHÉMATIQUES
2, PLACE JUSSIEU, 75005 PARIS, FRANCE

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