Mathematical Journal of Okayama University

Volume 19, *Issue* 1 1976 *Article* 4

DECEMBER 1976

Elementary proofs of some theorems of special Fourier series

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ELEMENTARY PROOFS OF SOME THEOREMS ON SPECIAL FOURIER SERIES

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1. Introduction. A real sequence $\{a_n\}$ is said to be quasi-convex if

$$\sum_{n=1}^{\infty}(n+1)|\Delta^2 a_n|<\infty$$

where $\Delta^2 a_n=\Delta a_n-\Delta a_{n+1}$, $\Delta a_n=a_n-a_{n+1}$.

It is known that a bounded convex sequence is quasi-convex and a bounded quasi-convex sequence is of bounded variation, viz

$$\sum_{n=1}^{\infty}|\Delta a_n|<\infty.$$

We denote by $(Ta)_n$, the n-th arithmetic mean of $\{a_n\}$,

$$(Ta)_n=\frac{1}{n}(a_1+a_2+\cdots+a_n).$$

Hardy [4] proved that if

(1) $$\sum_{n=1}^{\infty} a_n \sin nx$$

is the Fourier series of a function $f(x)\in L^p(0, 2\pi)$ $(p\geq 1)$, then

(2) $$\sum_{n=1}^{\infty}(Ta)_n \sin nx$$

is the Fourier series of a function $\varphi(x)\in L^p(0, 2\pi)$.

G. and S. Goes [2] obtained for a special sequence $\{a_n\}$ the following

Theorem A. *Let $\{a_n\}$ be a real null-sequence of bounded variation. Then* (2) *is the Fourier series of an L^1-function if and only if*

(3) $$\sum_{n=1}^{\infty}\frac{|a_n|}{n}<\infty.$$

Hence Theorem A, when combined with the above result of Hardy, has the following interesting corollary.

Corollary 1.[1] *If $\{a_n\}$ is a real null-sequence of bounded variation, then* (3) *is necessary for* (1) *being a Fourier series.*

[1] Cf. [2; Proof of Theorem 5.3]

Mathematical Journal of Okayama University, Vol. 19 [1976], Iss. 1, Art. 4

If $\{a_n\}$ is a quasi-convex null-sequence, the following important theorem due to Teljakovskiĭ [7] is known.

Theorem B. *Let $\{a_n\}$ be a quasi-convex real null-sequence. Then* (1) *is the Fourier series of an L^1-function if and only if* (3) *holds.*

Thus we have

Corollary 2. *When $\{a_n\}$ is a quasi-convex null-sequence,* (1) *is a Fourier series if and only if* (2) *is a Fourier series.*

For the proof of Theorem A, the following theorem [2 ; Theorem 6. 1] is required.

Theorem C. *A bounded sequence $\{a_n\}$ is of bounded variation if and only if $\{(Ta)_n\}$ is a quasi-convex sequence.*

Proofs of Theorems A and C due to G. and S. Goes are by a theory of the so-called BK-space, and so they are not elementary. In the next section we shall first give an elementary proof of Theorem C and then prove Theorem A in an elementary way depending upon Theorems B and C.

2. We need the following lemmas to prove Theorem C.

Lemma 1. *Put* $S_n=\sum_{k=1}^{n} a_k,\ t_n=(Ta)_n=\frac{S_n}{n}$. *Then*

$$\sum_{k=1}^{n} |\Delta t_k| \leq \sum_{k=1}^{n} |\Delta a_k|.$$

Proof. Writing $d_k=\Delta a_k=a_k-a_{k+1}$, we have

$$\begin{aligned}\Delta t_k=t_k-t_{k+1}&=\frac{S_k}{k}-\frac{S_{k+1}}{k+1}\\&=\left(\frac{1}{k}-\frac{1}{k+1}\right)(S_k-ka_{k+1})\\&=\left(\frac{1}{k}-\frac{1}{k+1}\right)(d_1+2d_2+\cdots+kd_k).\end{aligned}$$

Hence

$$\begin{aligned}\sum_{k=1}^{n}|\Delta t_k|&\leq\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)(|d_1|+2|d_2|+\cdots+k|d_k|)\\&\leq|d_1|\left(1-\frac{1}{n+1}\right)+2|d_2|\left(\frac{1}{2}-\frac{1}{n+1}\right)+\cdots+n|d_n|\left(\frac{1}{n}-\frac{1}{n+1}\right)\\&\leq|d_1|+|d_2|+\cdots+|d_n|\\&=\sum_{k=1}^{n}|\Delta a_k|.\end{aligned}$$

Lemma 2. *Assume that $\{a_n\}$ is of bounded variation. Then $\{t_n\}$ is quasi-convex if and only if it is of bounded variation.*

Proof. It will suffice to prove that $\{t_n\}$ is quasi-convex when it is of bounded variation. A simple calculation shows that

$$\Delta t_n = \frac{S_n}{n} - \frac{S_{n+1}}{n+1} = \frac{1}{n(n+1)}(S_n - na_{n+1}),$$

$$\Delta^2 t_n = \Delta(\Delta t_n) = \frac{S_n}{n(n+1)} - \frac{S_{n+1}}{(n+1)(n+2)} + \frac{a_{n+2}}{n+2} - \frac{a_{n+1}}{n+1}$$

$$= \frac{2(S_n - na_{n+1})}{n(n+1)(n+2)} + \frac{1}{n+2}(a_{n+2} - a_{n+1}).$$

Hence

(4) $$(n+2)\Delta^2 t_n = 2\Delta t_n + \Delta a_{n+1}.$$

Thus we have by Lemma 1

$$\sum_{n=1}^{\infty}(n+2)|\Delta^2 t_n| \leq 2\sum_{n=1}^{\infty}|\Delta t_n| + \sum_{n=1}^{\infty}|\Delta a_n| < \infty,$$

which shows that $\{t_n\}$ is quasi-convex.

Proof of Theorem C. If a bounded sequence $\{a_n\}$ is of bounded variation, then $(Ta)_n$ is also of bounded variation by Lemma 1, so it is quasi-convex by Lemma 2. Conversely, if $(Ta)_n$ is quasi-convex ($(Ta)_n$ is bounded since a_n is bounded), then it is necessarily of bounded variation. Therefore, from (4) we obtain

$$\sum_{n=2}^{\infty}|\Delta a_n| \leq \sum_{n=1}^{\infty}(n+2)|\Delta^2 t_n| + 2\sum_{n=1}^{\infty}|\Delta t_n| < \infty,$$

which proves that a_n is of bounded variation.

Proof of Theorem A. If $\{a_n\}$ is a null-sequence of bounded variation, then $t_n = (Ta)_n$ is quasi-convex according to Theorem C. Thus, by Theorem B, (2) is the Fourier series of an L^1-function if and only if

(5) $$\sum_{n=1}^{\infty}\frac{|t_n|}{n} < \infty.$$

On the other hand, since

$$\frac{a_{n+1}}{n+1} = t_{n+1} - t_n + \frac{t_n}{n+1},$$

we have

Mathematical Journal of Okayama University, Vol. 19 [1976], Iss. 1, Art. 4

$$\left| \sum_{n=2}^{\infty} \frac{|a_n|}{n} - \sum_{n=1}^{\infty} \frac{|t_n|}{n+1} \right| \leq \sum_{n=1}^{\infty} |\Delta t_n|,$$

which implies that (5) holds if and only if (3) holds, whenever $\{a_n\}$ is a null-sequence of bounded variation. Thus our proof is complete.

3. It has been stated without proof by Szidon [6] that if $\{a_n \log n\}$ is a real sequence of bounded variation, i. e.

$$\sum_{n=1}^{\infty} |\Delta(a_n \log n)| < \infty, \tag{6}$$

then

$$\sum_{n=1}^{\infty} a_n \cos nx \tag{7}$$

is the Fourier series of an L^1-function. A proof of this fact seems to have been first published by T. Kano [5 ; Theorem C], and an independent one by G. Goes [3 ; Theorem 5. 1]. Their proofs are of a different character.

On the other hand, it is elementary to prove that if

$$\sum_{n=1}^{\infty} | \Delta a_n | \log n < \infty \quad \text{and} \quad a_n \to 0, \tag{8}$$

then both of (1) and (7) converge in the metric of L^1, and hence they are Fourier series (cf. [1 ; p. 26]). Note that, in the case of sine series (1), (6) ceases to be a sufficient condition for (1) being a Fourier series, as is easily seen from the example $a_n = 1/\log (n+1)$. That condition (6) is weaker than condition (8) has been proved by G. Goes [3 ; Theorem 4. 3] in the following form.

Theorem D. *Condition* (8) *holds if and only if both of* (3) *and* (6) *hold.*

Goes applied a theory of *BK*-space to prove this theorem, however, we shall give an entirely simple and elementary proof of this theorem.

Since

$$\Delta(a_n \log n) = \Delta a_n \cdot \log n - a_{n+1} \log \left(1 + \frac{1}{n}\right),$$

we have inequalities

$$| \Delta a_n | \log n \leq | \Delta(a_n \log n) | + \frac{|a_{n+1}|}{n}, \tag{9}$$

$$| \Delta(a_n \log n) | \leq | \Delta a_n | \log n + \frac{|a_{n+1}|}{n}. \tag{10}$$

Therefore, if $\{a_n \log n\}$ is of bounded variation and in addition (3) holds, then (8) follows from (9). Conversely, if (8) holds, then

$$\sum_{n=1}^{\infty}\frac{|a_n|}{n}=\sum_{n=1}^{\infty}\left\{\frac{1}{n}\left|\sum_{k=n}^{\infty}\Delta a_k\right|\right\}\leq\sum_{n=1}^{\infty}\left\{\frac{1}{n}\sum_{k=n}^{\infty}|\Delta a_k|\right\}$$

$$=\sum_{N=1}^{\infty}\left\{|\Delta a_N|\sum_{n=1}^{N}\frac{1}{n}\right\}\ll\sum_{N=1}^{\infty}|\Delta a_N|\log N<\infty,$$

i. e. (3) holds. Thus we conclude from (10) that $\{a_n \log n\}$ is of bounded variation. This completes our proof of Theorem D.

Acknowledgement: The author wishes to thank Professor G. Goes for some helpful advice made during the preparation of this paper.

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(*Received October 21, 1976*)