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# On the relation of real cobordism to KR-theory

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# ON THE RELATION OF REAL COBORDISM TO $KR$-THEORY

Dedicated to Professor Tominosuke Otsuki on his 60th birthday

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## Introduction

In the previous paper [6] we have introduced the cobordism theory with reality. The purpose of this paper is to give an analogue of the cobordism interpretation for $K$-theory of Conner-Floyd [4] for theories with reality.

Throughout this paper, by a *real space* and a *real map* we mean a Hausdorff space with involution and an equivariant map between real spaces, respectively(cf. [2], [6]). By a *real complex* we mean a $CW$-complex with nice involution (cf. [4], [6]). By *real vector bundles* over real spaces we mean real vector bundles in the sense of Atiyah [2].

Let $MR^{*,*}(\ )$ and $KR^{*,*}(\ )$ be the cobordism theory with reality [6] and the real $K$-theory of Atiyah [2], respectively. They are multiplicative generalized cohomology theories in some sense. By making use of Thom classes in $KR$-theory, we can get a natural transformation

$$\mu_R : MR^{*,*}(X) \longrightarrow KR^{*,*}(X)$$

of the cohomologies. Furthermore we can define a group homomorphism

$$c_0 : \ KR^{*,*}(X) \longrightarrow MR^{*,*}(X)$$

by using the first $MR^{*,*}$-Chern classes for real vector bundles. And then, it holds a relation

$$\mu_R\, c_0 = -\text{ id}.$$

Hence we obtain

**Theorem 1.** *For any pair $(X, A)$ of finite real complexes, $KR(X, A)$ is embedded additively as a direct summand of $MR^{0,0}(X, A)$.*

Since the transformation $\mu_R : MR^{*,*} \longrightarrow KR^{*,*}$ is a ring homomorphism, we can regard $KR^{*,*}$ as a left $MR^{*,*}$-module by defining $\omega\, a = \mu_R(\omega)\, a$ for $\omega \in MR^{*,*}$ and $a \in KR^{*,*}$. Then we have the following

**Theorem 2.** *For any pair $(X, A)$ of finite real complexes, we have an* *isomorphism*

$$\hat{\mu}_R : MR^{*,*}(X, A) \underset{MR^{*,*}}{\otimes} KR^{*,*} \cong KR^{*,*}(X, A).$$

In §1 we summarize some basic properties of $KR$-theory. In §2 we discuss on the relation between $MR^{*,*}$-theory and $KR^{*,*}$-theory by making use of the transformation $\mu_R$ of cohomology theories and prove Theorem 1. The proof of Theorem 2 is given in §3 by using $MR^{*,*}$- and $KR^{*,*}$-cohomology structures of the Grassmann manifold $G_k(C^n)$ which is a real space with the reality given by the conjugation.

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## 1. Preliminaries

In this section we summarize some basic properties on $KR$-theory which are needed in the later sections.

For any real pair $(X, A)$, we define

$$KR^{-p,-q}(X, A) = \widetilde{KR}(\Sigma^{p,q} \wedge (X/A))$$

for any integers $p, q \geqq 0$, where $\Sigma^{p,q}$ is the real space of [6], (2.1).[1] Then, there is the following Bott isomorphism.

**Proposition 1.1** (cf. [2], Theorem 2.3).

$$\beta : KR^{-p,-q}(X, A) \longrightarrow KR^{-p-1,-q-1}(X, A), \quad x \longmapsto bx,$$

*is an isomorphism, where* $b \in \widetilde{KR}(\Sigma^{1,1}) \cong Z$ *is a generator.*

By making use of this proposition we can define $KR^{p,q}(X, A)$ for any pair $(p, q)$ of integers. And we have

**Proposition 1.2.** *For any integer $p$, $KR^{p,*}(\ ,\ )$ is a generalized cohomology theory. The theory $KR^{*,*}(\ ,\ )$ is a multiplicative theory.*

Let $\xi$ be a real vector bundle over a real compact space $X$ and $T(\xi)$ the real Thom space (cf. [6]) of $\xi$. As in the usual way (cf. [3], Chap. II, §2.6 or [5], §3) the Thom class of $\xi$

$$\mathfrak{T}(\xi) \in \widetilde{KR}(T(\xi))$$

is defined by the exterior algebra of $\xi$. And we have

1) According to the definition of Atiyah [2] $KR^{p,q}(X, A) = \widetilde{KR}(\Sigma^{p,q} \wedge (X/A))$.

**Proposition 1. 3.** (i) *Let $h: \eta \longrightarrow \xi$ be a real bundle map and $T(h): T(\eta) \longrightarrow T(\xi)$ the real map of the real Thom spaces induced by $h$. Then*

$$\mathfrak{T}(\eta) = T(h)^* \, \mathfrak{T}(\xi).$$

(ii) *Under the identification $T(\xi \times \xi') = T(\xi) \wedge T(\xi')$* we have

$$\mathfrak{T}(\xi \times \xi') = \mathfrak{T}(\xi) \wedge \mathfrak{T}(\xi').$$

(iii) *If $\theta^n$ is the $n$-dimensional trivial real vector bundle over a point, then $b_n = \mathfrak{T}(\theta^n) \in \widetilde{KR}(\Sigma^{n,n})$ is a generator $(b = b_1)$.*

Let the sequence $\{MU(k), \ \varepsilon_k \mid k \in N\}$ be the real Thom spectrum of [6], (2. 4), and

$$\mu_{m,n}: MU(m) \wedge MU(n) \longrightarrow MU(m+n)$$

be the real map of [6], (2. 2).

**Proposition 1. 5.** *Let $\gamma_n = (E(\gamma_n), p, BU(n))$ be the $n$-dimensional universal real vector bundle* (cf. [6], §1) *and $i_n: \Sigma^{n,n} \subset MU(n)$ the natural real inclusion. Then, we have*

(i) $\mu_{m,n}{}^* (\mathfrak{T}(\gamma_{m+n})) = \mathfrak{T}(\gamma_m) \wedge \mathfrak{T}(\gamma_n)$,

(ii) $b_n = i_n{}^* \, (\mathfrak{T}(\gamma_n))$.

Let $CP_n$ be the $n$-dimensional complex projective space and $\eta_n$ the canonical complex line bundle over $CP_n$. The space $CP_n$ is a real space and the bundle $\eta_n$ is a real line bundle with the reality induced by the conjugation.

As in the usual case (cf. [5], Chap. I, §4) we have

**Proposition 1. 6.** (i) *$T(\eta_{n-1}) = CP_n$ as real spaces with base points.*

(ii) $\mathfrak{T}(\eta_{n-1}) = 1 - \eta_n$ in $KR(CP_n)$.

Let us consider that the Thom class $\mathfrak{T}(\xi)$ of the $n$-dimensinal real vector bundle $\xi$ belongs to $\widetilde{KR}^{n,n}(T(\xi))$, that is

$$\mathfrak{T}(\xi) \in \widetilde{KR}^{n,n}(T(\xi)),$$

by the Bott isomorphism $\beta^{-n}: \widetilde{KR}(T(\xi)) \cong \widetilde{KR}^{n,n}(T(\xi))$. It is convenient to think so for considerations of cohomology theories. Then

**Proposition 1.7** (Thom Isomorphism Theorem) (cf. [2], Theorem 2.4). *For any $n$-dimensional real vector bundle $\xi$ over a real compact space $X$, the homomorphism*

$$\Psi : KR^{p,q}(X) \longrightarrow \widetilde{KR}^{p+n,q+n}(T(\xi)),$$

*defined by $\Psi(x) = \mathfrak{T}(\xi) \cdot x$ for $x \in KR^{p,q}(X)$, is an isomorphism.*

Furthermore we have the followings.

**Proposition 1.8** (cf. [2], p. 374). *Let $u_n = \beta^{-1}(1-\eta_n) \in KR^{1,1}(CP_n)$. Then $KR^{*,*}(CP_n)$ is a free $KR^{*,*}$-module with basis $1, u_n, \cdots, (u_n)^n$, with the relation $(u_n)^{n+1} = 0$. In other words,*

$$KR^{*,*}(CP_n) = KR^{*,*}[u_n]/((u_n)^{n+1}).$$

**Proposition 1.9.** *It holds the splitting principle in the $KR^{*,*}$-theory.*

**Proposition 1.10.** *There exists a unique function assigning to each $n$-dimensional real vector bundle $\xi$ over a real compact space $X$ an element*

$$\sigma(\xi) = 1 + \sigma_1(\xi) + \cdots + \sigma_n(\xi)$$

*where $\sigma_i(\xi) \in KR^{i,i}(X)$, such that*

1) *if a real bundle map $f : \eta \longrightarrow \xi$ covers a real map $f : X \longrightarrow Y$ of base spaces, then $f^* \sigma(\xi) = \sigma(\eta)$,*

2) *if $\xi$ and $\eta$ are real vector bundles over $X$, then*
$$\sigma(\xi \oplus \eta) = \sigma(\xi)\, \sigma(\eta),$$

3) *if $\eta_n$ is the canonical real line bundle over the real space $CP_n$, then $\sigma(\eta_n) = 1 + u_n$ where $u_n$ is the element in Proposition* 1.8.

The elements $\sigma_i(\xi)$, $i = 1, \cdots, n$, will be called *$KR^{*,*}$-Chern classes* of the $n$-dimensional real vector bundle $\xi$.

## 2. The relation between $MR$-theory and $KR$-theory

In the previous paper [6] we have defined the real cobordism group for any finite real complex $X$ with base point as follows:

$$\widetilde{MR}^{p,q}(X) = \mathrm{Dir}_k \ \mathrm{Lim} \ [\Sigma^{k-p,\, k-q} \wedge X ; \ MU(k)]_R.$$

We now define a natural transformation

$$\mu_R : \widetilde{MR}^{*,*}(X) \longrightarrow \widetilde{KR}^{*,*}(X)$$

in the same way as the definition of the natural transformation

$$\mu_C : \widetilde{MU}^*(\cdot) \longrightarrow \widetilde{K}^*(\cdot)$$

in Conner-Floyd [5], Chap. I, §5 : Let $\alpha \in \widetilde{MR}^{p,q}(X)$ be represented by $f : \Sigma^{k-p,k-q} \wedge X \longrightarrow MU(k)$. Then, let $\mu_R(\alpha)$ be the image of $\mathfrak{T}(\gamma_k)$ in the composition

$$\widetilde{KR}(MU(k)) \xrightarrow{f^*} \widetilde{KR}(\Sigma^{k-p,k-q} \wedge X) = \widetilde{KR}^{p-k,\, p-k}(X) \overset{\beta^{-k}}{\cong} \widetilde{KR}^{p,q}(X)\,.$$

**Proposition 2.1.** (i) *The transformation*

$$\mu_R : \widetilde{MR}^{*,*}(\cdot) \longrightarrow \widetilde{KR}^{*,*}(\cdot)$$

*is a multiplicative transformation of cohomology theories.*

(ii) *If* $t(\xi) \in \widetilde{MR}^{n,n}(T(\xi))$ *is the Thom class of an n-dimensional real vector bundle* $\xi$ *over a real compact space* [6], §4, *then* $\mu_R(t(\xi))$ *is the Thom class of* $\xi$ *in the KR-theory.*

**Proposition 2.2.** *Let* $\xi$ *be an n-dimensional real vector bundle over a finite real complex* $X$, *and let* $c_i(\xi) \in MR^{i,i}(X)$ *and* $\sigma_i(\xi) \in KR^{i,i}(X)$, $i = 1, \cdots, n$, *be the* $MR^{*,*}$*-Chern classes* [6] *and* $KR^{*,*}$*-Chern classes, respectively. Then* $\mu_R c_i(\xi) = \sigma_i(\xi)$.

*Proof.* To prove this, it suffices to show that

$$\mu_R : MR^{1,1}(CP_n) \longrightarrow KR^{1,1}(CP_n)$$

maps $x_n$ into $u_n$, where $x_n$ is the element of [6], Theorem 6.2, and $u_n$ is the element of Proposition 1.8. Since the element $x_n$ is represented by the real inclusion $j_n : CP_n \subset MU(1) = CP(\infty)$,

$$\begin{aligned} \mu_R(x_n) &= \beta^{-1} j_n^* (\mathfrak{T}(\gamma_1)) \\ &= \beta^{-1} \mathfrak{T}(\eta_{n-1}) && \text{by Prop. 1.3, (i), and Prop. 1.6, (i)} \\ &= \beta^{-1}(1 - \eta_n) && \text{by Prop. 1.6, (ii)} \\ &= u_n\,. \end{aligned}$$

q. e. d.

If $\xi$, $\eta$ are $m$, $n$-dimensional real vector bundles over a finite real complex $X$ respectively, then $c_1(\xi \oplus \eta) = c_1(\xi) + c_1(\eta)$. Hence there exists a unique additive homomorphism

$$c_1 : KR(X) \longrightarrow MR^{1,1}(X)$$

taking a class of a real vector bundle $\xi$ into $c_1(\xi)$.

**Proposition 2. 3.** *If $\xi$ is an n-dimensional trivial real vector bundle over a finite real complex, then* $c_i(\xi) = 0$ *for* $i \geqq 1$.

*Proof.* Since every trivial bundle is induced by a map into a point, it suffices by the naturality to prove $c_i(\theta^n) = 0$, $i \geqq 1$, for the trivial real bundle $\theta^n$ over a point. By [6], Theorem 6. 2, we have $(x_{n-1})^n = 0$ in $MR^{*,*}(CP_{n-1})$. Hence $c_i(\theta^n) = 0$, $i = 1, \cdots, n$, by the definition of $MR^{*,*}$-Chern classes. q. e. d.

**Proposition 2. 4.** *For any connected finite real complex $X$ with base point, the following diagram is commutative:*

$$\begin{array}{ccc}
\widetilde{KR}(X) & \xrightarrow{c_1} & \widetilde{MR}^{1,1}(X) \\
{\scriptstyle -1}\downarrow & & \downarrow{\scriptstyle \mu_R} \\
\widetilde{KR}(X) & \xrightarrow{\beta^{-1}} & \widetilde{KR}^{1,1}(X).
\end{array}$$

*Proof.* First we have

$$c_1(\eta_n - 1) = x_n \in MR^{*,*}(CP_n)$$

for the canonical real line bundle $\eta_n$ over the real space $CP_n$. Hence

$$\mu_R\, c_1(\eta_n - 1) = \mu_R(x_n) = \beta^{-1}(1 - \eta_n)$$

is just the computation of the proof of Proposition 2. 2. Therefore, by the naturality we have

$$\mu_R\, c_1(\xi - 1) = \beta^{-1}(1 - \xi)$$

for any real line bundle $\xi$ over $X$.

Every element of $\widetilde{KR}(X)$ is of the form $\xi - k$, where $\xi$ is a $k$-dimensional real vector bundle and $k$ is a $k$-dimensional trivial real vector bundle over $X$. In virtue of Proposition 1. 9, there is a real space $F$ and a real map $\pi : F \longrightarrow X$ such that

1) $\pi^* : KR^{*,*}(X) \longrightarrow KR^{*,*}(F)$ is a monomorphism, and
2) $\pi^*\xi$ splits as a sum of $k$ real line bundles $\xi_1, \cdots, \xi_k$.

Then

$$\begin{aligned}
\pi^*\, \mu_R\, c_1(\xi - k) &= \mu_R\, c_1((\xi_1 - 1) + \cdots + (\xi_k - 1)) \\
&= \beta^{-1}(1 - \xi_1) + \cdots + \beta^{-1}(1 - \xi_k) \\
&= \pi^*\, \beta^{-1}(k - \xi).
\end{aligned}$$

Hence

$$\mu_R c_1(\xi - k) = \beta^{-1}(k - \xi). \qquad \text{q. e. d.}$$

We now define, for a finite real complex $X$ with base point,

$$c_0 : \widetilde{KR}(X) \longrightarrow \widetilde{MR}^{0,0}(X)$$

as the composition

$$\begin{tikzcd}
\widetilde{KR}(X) \arrow[rr, dashed, "c_0"] \arrow[d, "\sigma^{1,1}"'] & & \widetilde{MR}^{0,0}(X) \arrow[d, "\sigma^{1,1}"] \\
\widetilde{KR}^{1,1}(\Sigma^{1,1} \wedge X) \arrow[dr, "\beta"'] & & \widetilde{MR}^{1,1}(\Sigma^{1,1} \wedge X), \\
& \widetilde{KR}(\Sigma^{1,1} \wedge X), \arrow[ur, "c_1"'] &
\end{tikzcd}$$

where $\sigma^{1,1}$ is the suspension isomorphism.

Passing to pairs $(X, A)$, we get an additive homomorphism

$$c_0 : KR(X, A) \longrightarrow MR^{0,0}(X, A).$$

**Proposition 2. 5.** *For any pair $(X, A)$ of finite real complexes, the homomorphisms*

$$KR(X, A) \xrightarrow{c_0} MR^{0,0}(X, A) \xrightarrow{\mu_R} KR(X, A)$$

*have $\mu_R c_0(\alpha) = -\alpha$ for every $\alpha \in KR(X, A)$.*

Now, as a corollary of this proposition we obtain the following

**Theorem 1.** *For any pair $(X, A)$ of finite real complexes, $KR(X, A)$ is embedded additively in $MR^{0,0}(X, A)$ as a direct summand.*

### 3. A real cobordism interpretation for $KR^{*,*}(X)$

Let $MR^{*,*} = \widetilde{MR}^{*,*}(\Sigma^{0,0})$ and $KR^{*,*} = \widetilde{KR}^{*,*}(\Sigma^{0,0})$. Then, in virtue of Proposition 2. 5,

$$\mu_R : MR^{*,*} \longrightarrow KR^{*,*}$$

is a ring epimorphism. We thus can regard $KR^{*,*}$ as a left $MR^{*,*}$-module by defining $\omega a = \mu_R(\omega) a$ for $\omega \in MR^{*,*}$ and $a \in KR^{*,*}$.

For a pair $(X, A)$ of finite real complexes, define

$$\Lambda^{*,*}(X, A) = MR^{*,*}(X, A) \underset{MR^{*,*}}{\otimes} KR^{*,*}.$$

Then, there is a natural epimorphism

$$h: MR^{*,*}(X, A) \longrightarrow \Lambda^{*,*}(X, A)$$

defined by $h(x) = x \otimes 1$ for $x \in MR^{*,*}(X, A)$. And it is easily seen that the epimorphism induces an isomorphism

$$\bar{h}: MR^{*,*}(X, A)/R \cong \Lambda^{*,*}(X, A),$$

where $R$ is the least subgroup of $MR^{*,*}(X, A)$ generated by all $x\omega - x\omega'$ for $x \in MR^{*,*}(X, A)$ and $\omega, \omega' \in MR^{*,*}$ such that $\mu_R(\omega) = \mu_R(\omega')$.

Since $\mu_R$ is multiplicative, there is a unique homomorphism

$$\hat{\mu}_R: \Lambda^{*,*}(X, A) \longrightarrow KR^{*,*}(X, A)$$

satisfying the following conditions:

1) $$\hat{\mu}_R(x \otimes a) = \mu_R(x)\, a$$

2) commutativity holds in

$$\begin{tikzcd} MR^{*,*}(X, A) \arrow[rr, "h"] \arrow[dr, "\mu_R"'] & & \Lambda^{*,*}(X, A) \arrow[dl, "\hat{\mu}_R"] \\ & KR^{*,*}(X, A). & \end{tikzcd}$$

Let define

$$\hat{c}_0: KR^{*,*}(X, A) \longrightarrow \Lambda^{*,*}(X, A)$$

by the composition $KR^{*,*}(X, A) \xrightarrow{c_0} MR^{*,*}(X, A) \xrightarrow{h} \Lambda^{*,*}(X, A)$. Then we have

**Proposition 3.1.** $\hat{\mu}_R \hat{c}_0 = -1.$

Now we can state the main theorem of this paper.

**Theorem 2.** *For any pair $(X, A)$ of finite real complexes,*

$$\hat{\mu}_R: MR^{*,*}(X, A) \underset{MR^{*,*}}{\otimes} KR^{*,*} \longrightarrow KR^{*,*}(X, A)$$

*is an isomorphism.*

For the proof of Theorem 2 we shall need the cohomology structures of the real Thom space of the classifying bundle.

Let $G_k(C^n)$ be the Grassmann manifold of $k$-planes in the $n$-dimensional complex space $C^n$, which is a real space with the reality given by the conjugation. We have real vector bundles $\gamma_k^n$($k$-plane, point in it) and $\bar{\gamma}_k^n$ ($k$-plane, point in the orthogonal ($n$-$k$)-plane) over $G_k(C^n)$ with $\gamma_k^n \oplus \bar{\gamma}_k^n$ trvial. We then have the Chern classes

$$c_i = c_i(\gamma_k^n),\ \bar{c}_i = c_i(\bar{\gamma}_k^n) \in MR^{i,i}(G_k(C^n)),$$
$$\sigma_i = \sigma_i(\gamma_k^n),\ \bar{\sigma}_i = \sigma_i(\bar{\gamma}_k^n) \in KR^{i,i}(G_k(C^n)),$$

related by the equations $c\bar{c} = 1$ and $\sigma\bar{\sigma} = 1$. Therefore $\bar{c}_j$ and $\bar{\sigma}_j$ are the polynomials of degree $j$ in the $c_i$ and $\sigma_i$ given by the formal inversions of $c$ and $\sigma$, respectively. Then we can obtain the following proposition in the same way as R. E. Stong [7].

**Proposition** 3.2 (cf. [7], p. 69). *Let $h^{*,*}$ denote the cohomology functor $MR^{*,*}$ or $KR^{*,*}$ and $d_i$ the $h^{*,*}$-Chern class $c_i$ or $\sigma_i$. Then $h^{*,*}(G_k(C^n))$ is the quotient of the polynomial algebra over $h^{*,*}$ on $d_1, \cdots, d_k$, by the relations imposed by $\bar{d}_j = 0$ for $j > n-k$.*

*Proof.* The proof is by induction on $k$. Since $G_1(C^n) = CP_{n-1}$, the proposition being obvious for $k=1$ by Proposition 1.8 and [6], Theorem 6.2.

Suppose the result holds for all $G_t(C^n)$ with $t<k$, and consider $G_k(C^n)$. Let $(P(\gamma_k^n), \pi, G_k(C^n))$ be the associated real projective bundle of $\gamma_k^n$ and $(P(\bar{\gamma}_{k-1}^n), \bar{\pi}, G_{k-1}(C^n))$ the one of $\bar{\gamma}_{k-1}^n$. A point in $P(\gamma_k^n)$ is a pair $(V, [x])$ of a $k$-plane $V$ in $C^n$ and a line $[x]$ in $V$. Let $[x]^\perp$ be the orthogonal complement of $[x]$ in $V$. Then we can identify $P(\gamma_k^n)$ with $P(\bar{\gamma}_{k-1}^n)$ by means of the real homeomorphism defined by $(V, [x]) \longrightarrow ([x]^\perp, [x])$. Let $l = l(\gamma_k^n) = l(\bar{\gamma}_{k-1}^n)$ denote the canonical real line bundle over $P = P(\gamma_k^n) \equiv P(\bar{\gamma}_{k-1}^n)$ and $\xi = \bar{\pi}^*\gamma_{k-1}^n$, $\eta = \pi^*\bar{\gamma}_k^n$. Then we have

$$\text{(i)}\quad \pi^*\gamma_k^n = l \oplus \xi$$
$$\text{(ii)}\quad \bar{\pi}^*\bar{\gamma}_{k-1}^n = l \oplus \eta$$
$$\text{(iii)}\quad \xi \oplus l \oplus \eta = \theta^n$$

where $\theta^n$ is the trivial real bundle over $P$.

In virtue of Proposition 1.9 and [6], Theorem 6.6, $h^{*,*}(P)$ is a free $h^{*,*}(G_{k-1}(C^n))$-module with basis $1, c, \cdots, c^r$, with the relation $\sum_{i=0}^{r+1}(-1)^i c^{r+1-i} d_i(\bar{\gamma}_{k-1}^n) = 0$, where $c$ is the first $h^{*,*}$-Chern class $d_1(l)$ of $l$ and $r = n-k$. By making use of the inductive assumption and the above relations (i), (ii) and (iii) $h^{*,*}(P)$ is the quotient of the polynomial algebra over $h^{*,*}$ on $d_1(\xi), \cdots, d_{k-1}(\xi), c, d_1(\eta), \cdots, d_r(\eta)$ by the relations imposed by $d(\xi \oplus l \oplus \eta) = d(\xi)\, d(l)\, d(\eta) = 1$. Furthermore $h^{*,*}(P)$ is a free $A$-module with basis $1, c, \cdots, c^{k-1}$, with the relation $\sum_{i=0}^{k}(-1)^i c^{k-i} d_i = 0$, where $A$ is the quotient of the polynomial algebra over $h^{*,*}$ on $d_1, \cdots, d_k$ by the relations imposed by $\bar{d}_j = 0$ for $j > r$.

On the other hand, looking at $P$ as a bundle over $G_k(C^n)$, $h^{*,*}(P)$ is a free $h^{*,*}(G_k(C^n))$-module with basis $1, c. \cdots, c^{k-1}$, with the relation $\sum_{i=0}^{k}(-1)^i c^{k-i} d_i = 0$. Besides $h^{*,*}(G_k(C^n)) \supset A$. This completes the induction.

q. e. d.

As a corollary of this proposition we obtain the following

**Proposition 3.3.** *Let $h^{*,*}$ and $d_i$ be as in the previous proposition. Then $h^{*,*}(G_k(C^n))$ is a free $h^{*,*}$-module with basis $e_1, \cdots, e_r$ $(r = \binom{n}{k})$, where $e_i$ is a polynomial of the Chern classes $d_1, \cdots, d_k$.*

Hence, by making use of the Thom isomorphism $\psi$ we have

**Proposition 3.4.** *Let $h^{*,*}$ and $e_i$ be as in the above proposition. Then $\tilde{h}^{*,*}(T(\gamma_k^n))$ is a free $h^{*,*}$-module with basis $\alpha_1, \cdots, \alpha_r$ $(r = \binom{n}{k})$, where $\alpha_i = \psi(e_i)$.*

*Proof of Theorem* 2. 1) The case of $X = T(\gamma_k^n)$: Let

$$\tilde{\Lambda}^{*,*}(X) = \widetilde{MR}^{*,*}(X) \underset{MR^{*,*}}{\otimes} KR^{*,*}.$$

We need to compute the kernel of $\mu_R : \widetilde{MR}^{*,*}(X) \longrightarrow \widetilde{KR}^{*,*}(X)$. An element is in this kernel if and only if the coefficients from $MR^{*,*}$ used in expressing this element in terms of the $\alpha_i$ all lie in the kernel of $\mu_R : MR^{*,*} \longrightarrow KR^{*,*}$. Hence Ker $\mu_R \subset$ Ker $h$, hence $\hat{\mu}_R$ is an isomorphism in the diagram

$$\begin{tikzcd}
\widetilde{MR}^{*,*}(X) \arrow[rr, "h"] \arrow[dr, "\mu_R"'] & & \tilde{\Lambda}^{*,*}(X) \arrow[dl, "\hat{\mu}_R"] \\
& \widetilde{KR}^{*,*}(X). &
\end{tikzcd}$$

2) The general case: Suppose $\hat{\mu}_R(\alpha) = 0$ for $\alpha \in \Lambda^{*,*}(X, A)$. Then, there exists $x \in MR^{*,*}(X, A)$ such that $\alpha = h(x)$ and $\mu_R(x) = 0$ in

$KR^{*,*}(X, A)$. Say $x = x_{i_1,j_1} + \cdots + x_{i_r,j_r}$ where $x_{i_k,j_k} \in MR^{i_k,j_k}(X, A)$. Let put $p = i_k$, $q = j_k$ for simplisity. Then $\mu_R(x_{p,q}) = 0$ in $KR^{p,q}(X, A)$. Let $x_{p,q}$ be represented by a real map

$$f: \Sigma^{n-p,n-q} \wedge (X/A) \longrightarrow MU(n).$$

Since $\Sigma^{n-p,n-q} \wedge (X/A)$ is compact, we have for sufficiently large $m$

$$f(\Sigma^{n-p,n-q} \wedge (X/A)) \subset T(\gamma_n^m)\,.$$

Then the suspension $\sigma^{n-p,n-q}(x_{p,q}) \in \widetilde{MR}^{n,n}(\Sigma^{n-p,n-q} \wedge (X/A))$ is in the image of

$$f^*: \widetilde{MR}^{n,n}(T(\gamma_n^m)) \longrightarrow \widetilde{MR}^{n,n}(\Sigma^{n-p,n-q} \wedge (X/A)).$$

Hence $\sigma^{n-p,n-q} h(x_{p,q})$ is in the image of

$$f^*: \widetilde{\Lambda}^{*,*}(T(\gamma_n^m)) \longrightarrow \widetilde{\Lambda}^{*,*}(\Sigma^{n-p,n-q} \wedge (X/A)).$$

Since $\hat{\mu}_R: \widetilde{\Lambda}^{*,*}(X/A) \longrightarrow \widetilde{KR}^{*,*}(X/A)$ maps $h(x_{p,q})$ into zero, so does

$$\hat{\mu}_R: \widetilde{\Lambda}^{*,*}(\Sigma^{n-p,n-q} \wedge (X/A)) \longrightarrow \widetilde{KR}^{*,*}(\Sigma^{n-p,n-q} \wedge (X/A))$$

map $\sigma^{n-p,n-q} h(x_{p,q})$ into zero. Consider the commutative diagram

$$\begin{tikzcd}
\widetilde{\Lambda}^{*,*}(T(\gamma_n^m)) \arrow[r, "f^*"] \arrow[d, shift right, "\hat{\mu}_R'"'] & \widetilde{\Lambda}^{*,*}(\Sigma^{n-p,n-q} \wedge (X/A)) \arrow[d, shift right, "\hat{\mu}_R"'] \\
\widetilde{KR}^{*,*}(T(\gamma_n^m)) \arrow[r, "f^*"] \arrow[u, shift right, "\hat{c}_0'"'] & \widetilde{KR}^{*,*}(\Sigma^{n-p,n-q} \wedge (X/A)). \arrow[u, shift right, "\hat{c}_0"']
\end{tikzcd}$$

Since $\hat{\mu}_R'$ is an isomorphism by the cass 1) and $\hat{\mu}_R'\hat{c}_0' = -1$, $\hat{c}_0'$ is an isomorphim. Therefore there exists $\varepsilon \in \widetilde{KR}^{*,*}(T(\gamma_n^m))$ with $\sigma^{n-p,n-q} h(x_{p,q}) = f^* \hat{c}_0'(\varepsilon)$. Then

$$-f^*(\varepsilon) = \hat{\mu}_R \hat{c}_0 f^*(\varepsilon) = \hat{\mu}_R f^* \hat{c}_0'(\varepsilon) = \hat{\mu}_R \sigma^{n-p,n-q} h(x_{p,q}) = 0.$$

Thus we have $\sigma^{n-p,n-q} h(x_{p,q}) = 0$ and $h(x_{p,q}) = 0$ in $\widetilde{\Lambda}^{*,*}(X/A)$. Hence $\alpha = h(x) = 0$ in $\Lambda^{*,*}(X, A)$. That is

$$\hat{\mu}_R: \Lambda^{*,*}(X, A) \longrightarrow KR^{*,*}(X, A)$$

is a monomorphism and the theorem follows. q. e. d.

Recently, S. Araki [1] has discussed on the structure of $MR^{*,*}$, in which he has introduced notations $MR^*$, $MR^{*+k}$ and $MR^{*-k}$. Now, by using these notations, let put

$$MR^* = \sum_p MR^{p,p}, \quad MR^{*+k}(X, A) = \sum_p MR^{p+k,p}(X, A),$$
$$KR^* = \sum_p KR^{p,p}, \quad KR^{*+k}(X, A) = \sum_p KR^{p+k,p}(X, A).$$

Then $MR^{*+k}(X, A)$ and $KR^{*+k}(X, A)$ are $MR^*$- and $KR^*$-modules, respectively. Furthermore, $MR^{*,*}(X, A)$ is a graded $MR^*$-module with grading $MR^{*+k}(X, A)$, $k \in Z$, and $KR^{*,*}(X, A)$ is a graded $KR^*$-module with grading $KR^{*+k}(X, A)$, $k \in Z$.

By a suggestion of P. S. Landweber we obtain the following

**Proposition 3.5.** *For any pair* $(X, A)$ *of finite real complexes, we have isomorphisms*

(i) $\hat{\mu}_R : MR^{*+k}(X, A) \underset{MR^*}{\otimes} KR^* \cong KR^{*+k}(X, A)$ *for any integer* $k$,

(ii) $\hat{\mu}_R : MR^{*,*}(X, A) \underset{MR^*}{\otimes} KR^* \cong KR^{*,*}(X, A)$.

*Proof.* Let $h^*$ denote $MR^*$ or $KR^*$. Then $\tilde{h}^*(T(\gamma_k^n))$ is a free $h^*$-module with basis $\alpha_1, \cdots, \alpha_r$, where $\alpha_i$ is as in Proposition 3.4. Therefore, the proof of (i) for $k = 0$ is quite similar to the proof of Theorem 2.

For a non-zero interger $k$, we have

$$MR^{*+k}(X, A) = \widetilde{MR}^*(\Sigma^{n-k,n} \wedge (X/A))$$
$$KR^{*+k}(X, A) = \widetilde{KR}^*(\Sigma^{n-k,n} \wedge (X/A)),$$

and the proposition follows from the case of $k = 0$. q. e. d.

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