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HOPF GALOIS EXTENSIONS WITH HOPF ALGEBRAS OF DERIVATION TYPE

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Throughout the present paper, R will denote a commutative ring with identity of prime characteristic p . Let H be a Hopf algebra over R . A commutative ring extension A of R is called an *H -Hopf Galois extension* of R (or A/R is an *H -Hopf Galois extension*) if A is a finitely generated faithful projective R -module and an H -module algebra and the natural homomorphism (arising from the H -module structure of A) from the smash product algebra $A \# H$ to the endomorphism algebra $\text{End}_R(A)$ is an isomorphism. For details, we refer to [2], [6], [7] and [8]. Unadorned $\otimes$ and Hom etc. are taken over R and every map is R -linear. All the modules and R -algebra homomorphisms considered are unitary.

By the *Hopf algebra of derivation type of degree p^m* (cited below as $H(p^m)$) we mean a Hopf algebra over R defined as follows: $H(p^m)$ is an R -algebra freely generated by d with relation $d^{p^m} = 0$ and its Hopf algebra structure is given by

$$\Delta(d) = d \otimes 1 + 1 \otimes d, \quad \epsilon(d) = 0 \quad \text{and} \quad \lambda(d) = -d,$$

where Δ , ϵ and λ are the diagonalization, augmentation and antipode, respectively. (Hereafter, the letter " d " will always mean the above generator.) In [4, Corollary 1.4], the first named author shows that any $H(p^m)$ -Hopf Galois extension of R is of the form

$$R[X_1]/(X_1^p - \alpha_1) \otimes \cdots \otimes R[X_m]/(X_m^p - \alpha_m) \quad (\alpha_i \in R).$$

As is easily checked, such an extension is an $H(p)^m$ -Hopf Galois extension of R , where $H(p)^m = H(p) \otimes \cdots \otimes H(p)$. Conversely, in case $A_m = R[X_1]/(X_1^p - \alpha_1) \otimes \cdots \otimes R[X_m]/(X_m^p - \alpha_m)$ is a field, the existence of a non-integrable element in A_{m-1} enabled R. Baer [1] to show inductively that any derivation of A_{m-1} over R can be extended to that of A_m satisfying the same conditions as for d (especially its p^m -th power equals zero and the invariant subfield is R). This fact may be interpreted as A_m to be an $H(p^m)$ -Hopf Galois extension of R . But unfortunately the explicit form of the non-integrable element was not given.

In this paper we shall consider a typical non-integrable element and construct a derivation (compatible with the above characterization) on an $H(p)^m$ -Hopf Galois extension of R . Furthermore, we shall show that a

commutative R -algebra A is an $H(p^m)$ -Hopf Galois extension of R if and only if it is an $H(p)^m$ -Hopf Galois extension.

Now, we begin our study with stating two propositions quoted from [4].

Proposition 1 ([4, Corollary 1.6]). *Let A be a commutative R -algebra. Then A is an $H(p)$ -Hopf Galois extension of R if and only if A is isomorphic to $R[X]/(X^p - a)$ ($a \in R$) as H -module algebra, where the action of $d \in H(p)$ on $R[X]/(X^p - a)$ is defined by $d(x) = 1$, x the residue class of X .*

Proposition 2 ([4, Lemma 1.2 and Corollary 1.4]). *Let A/R be an $H(p^m)$ -Hopf Galois extension. Then A is isomorphic to $R[X_1]/(X_1^p - \alpha_1) \otimes \cdots \otimes R[X_m]/(X_m^p - \alpha_m)$ ($\alpha_i \in R$). When this is the case, $d(x_1) = 1$, $x_i = d^{p^{m-1}-p^{i-1}}(x_m)$ and $d(x_i) \in R[x_1, \dots, x_{i-1}]$ ($2 \leq i$), where x_i is the residue class of X_i .*

Proposition 3. *Let H_1, H_2 be finite cocommutative Hopf algebras over R . If A/R is an $H_1 \otimes H_2$ -Hopf Galois extension, then there exist subalgebras A_1, A_2 of A such that A_i/R is an H_i -Hopf Galois extension ($i = 1, 2$) and $A = A_1 \otimes A_2$ as $H_1 \otimes H_2$ -module algebra. Conversely, if A_i/R is an H_i -Hopf Galois extension, then $A_1 \otimes A_2$ is an $H_1 \otimes H_2$ -Hopf Galois extension of R .*

Proof. The converse part has been proved in [2, Proposition 3.2] and [8, Lemma 4.2]. Let $\varepsilon_i: H_i \rightarrow R$ be an augmentation, $p_1 = 1 \otimes \varepsilon_2: H_1 \otimes H_2 \rightarrow H_1$ and $p_2 = \varepsilon_1 \otimes 1: H_1 \otimes H_2 \rightarrow H_2$. Then, by [8, Lemma 4.1], $A_i = \text{Hom}_{H_1 \otimes H_2}(H_i, A)$ is an H_i -Hopf Galois extension of R , where H_i is regarded as an $H_1 \otimes H_2$ -module via p_i . Thus $A_1 \otimes A_2$ is an $H_1 \otimes H_2$ -Hopf Galois extension of R . By taking the image of $1 \in H_i$, we may identify A_1 with $A^{H_2} = \{a \in A \mid h \cdot a = \varepsilon(h)a \text{ for all } h \in H_2\}$; and A_2 with A^{H_1} . Under these identifications we can define the homomorphism $\phi: A_1 \otimes A_2 \rightarrow A$ by the product in A . Since A is commutative, ϕ is a well-defined $H_1 \otimes H_2$ -module algebra homomorphism. Thus ϕ is an isomorphism by [2, Theorem 1.1.12].

By Propositions 1, 2 and 3 we get the following:

Corollary 4. *Let $A = R[X_1]/(X_1^p - \alpha_1) \otimes \cdots \otimes R[X_m]/(X_m^p - \alpha_m)$ be an $H(p^m)$ -Hopf Galois extension of R . Then A is an $H(p)^m$ -Hopf Galois extension of R , where $\partial_i = \partial/\partial x_i = 1 \otimes \cdots \otimes 1 \otimes d \otimes 1 \otimes \cdots \otimes 1 \in 1 \otimes \cdots \otimes 1 \otimes H(p) \otimes 1 \otimes \cdots \otimes 1$ (i -th position) acts on A as $\partial_i(x_j) = \delta_{ij}$ (Kronecker's delta).*

Conversely, if A is an $H(p)^m$ -Hopf Galois extension of R , then A is isomorphic to $R[X_1]/(X_1^p - \alpha'_1) \otimes \cdots \otimes R[X_m]/(X_m^p - \alpha'_m)$ as $H(p)^m$ -module algebra, ∂_i acts as $\partial_i(x_j) = \delta_{ij}$.

For further investigation, we need the following lemmas.

Lemma 5. $\binom{p^{n-1}-1}{k} \equiv (-1)^k \pmod{p}$, $n \geq 2$, $0 \leq k \leq p^{n-1}-1$.

Proof. Since $(1+X)^{p^{n-1}} \equiv 1+X^{p^{n-1}} \pmod{p}$, we have

$$\binom{p^{n-1}-1}{k} \equiv \begin{cases} 1 \pmod{p} & \text{for } k=0, \\ 0 \pmod{p} & \text{otherwise.} \end{cases}$$

Combining this with $\binom{p^{n-1}-1}{k} + \binom{p^{n-1}-1}{k+1} = \binom{p^{n-1}}{k+1}$, we can easily get the assertion by induction.

Lemma 6. Let $A = R[X_1]/(X_1^p - \alpha_1) \otimes \cdots \otimes R[X_m]/(X_m^p - \alpha_m)$ be an $H(p)^m$ -Hopf Galois extension of R . Then there exists a nilpotent R -derivation $\delta: A \rightarrow A$ of index p^m such that $\delta(x_1) = 1$ and $\delta^{p^{k-1}-p^{k-2}}(x_k) = x_{k-1}$ ($2 \leq k \leq m$).

Proof. Let $\partial_i: A \rightarrow A$ be a derivation defined by $\partial_i(x_j) = \delta_{ij}$. We put $\delta = P_0\partial_1 + P_1\partial_2 + \cdots + P_{m-1}\partial_m$, where for $p \neq 2$, $P_0 = 1$, $P_i = (-1)^i x_1^{p-1} \cdots x_i^{p-1}$ ($1 \leq i \leq m-1$) and for $p = 2$, $P_0 = 1$, $P_1 = x_1$, $P_i = P_{i-1}x_i + \cdots + x_1 \cdots x_{i-1}x_{i-1}$ ($2 \leq i \leq m-1$). Since the assertion is valid for $m = 1, 2$, we proceed by induction on m . Assume that $\delta(x_1) = 1$ and $\delta^{p^{k-1}-p^{k-2}}(x_k) = x_{k-1}$ ($2 \leq k \leq m-1$). Then, it is easy to see that $\delta^{p^{k-1}}(x_k) = 1$ ($1 \leq k \leq m-1$), and hence $\delta^{p^{k-1}}(P_{k-1}) = 0$. First, we consider the case $p \neq 2$. Noting that $\delta^{p^{m-2}}$ is a derivation with $\delta^{p^{m-2}}(P_{m-2}) = 0$, we have

$$\begin{aligned} \delta^{p^{m-1}-p^{m-2}}(x_m) &= \delta^{p^{m-1}-p^{m-2}-1}((-1)^{m-1}x_1^{p-1} \cdots x_{m-1}^{p-1}) \\ &= \delta^{p^{m-2}-1} \delta^{p^{m-2}(p-2)}((-1)^{m-1}x_1^{p-1} \cdots x_{m-1}^{p-1}) \\ &= \delta^{p^{m-2}-1}((-1)^{m-1}x_1^{p-1} \cdots x_{m-2}^{p-1}(\delta^{p^{m-2}})^{p-2}(x_{m-1}^{p-1})) \\ &= \delta^{p^{m-2}-1}((-1)^{m-1}x_1^{p-1} \cdots x_{m-2}^{p-1}(p-1)x_{m-1}) \\ &= \delta^{p^{m-2}-1}(\delta(x_{m-1})x_{m-1}) \\ &= \sum_{k=0}^{p^{m-2}-1} \binom{p^{m-2}-1}{k} \delta^{k+1}(x_{m-1}) \delta^{p^{m-2}-1-k}(x_{m-1}) \\ &= \delta(x_{m-1}) \delta^{p^{m-2}-1}(x_{m-1}) - \delta^2(x_{m-1}) \delta^{p^{m-2}-2}(x_{m-1}) + \cdots \\ &\quad + \delta^{p^{m-2}-2}(x_{m-1}) \delta^2(x_{m-1}) - \delta^{p^{m-2}-1}(x_{m-1}) \delta(x_{m-1}) \\ &\quad + \delta^{p^{m-2}}(x_{m-1})x_{m-1} \quad (\text{by Lemma 5}) \\ &= x_{m-1}. \end{aligned}$$

Next, we consider the case $p = 2$. Noting that $\delta^{2^{k-2}}(x_{k-1}) = 1$, we have

$$\begin{aligned}
 \delta^{2^{k-1}-1}(x_1 \cdots x_{k-1}) &= \sum_{i=0}^{2^{k-1}-1} \binom{2^{k-1}-1}{i} \delta^i(x_1 \cdots x_{k-2}) \delta^{2^{k-1}-1-i}(x_{k-1}) \\
 &= \sum_{i=0}^{2^{k-1}-1} \delta^i(x_1 \cdots x_{k-2}) \delta^{2^{k-1}-1-i}(x_{k-1}) \\
 &= (x_1 \cdots x_{k-2}) \delta^{2^{k-2}-1}(\delta^{2^{k-2}}(x_{k-1})) \\
 &\quad + \delta(x_1 \cdots x_{k-2}) \delta^{2^{k-2}-2}(\delta^{2^{k-2}}(x_{k-1})) + \cdots \\
 &\quad + \delta^{2^{k-2}-1}(x_1 \cdots x_{k-2}) \delta^{2^{k-2}}(x_{k-1}) \\
 &\quad + \delta(\delta^{2^{k-2}-1}(x_1 \cdots x_{k-2})) \delta^{2^{k-2}-1}(x_{k-1}) + \cdots \\
 &\quad + \delta^{2^{k-2}}(\delta^{2^{k-2}-1}(x_1 \cdots x_{k-2})) x_{k-1} \quad (\text{by Lemma 5}) \\
 &= \delta^{2^{k-2}-1}(x_1 \cdots x_{k-2}) \delta^{2^{k-2}}(x_{k-1}) \\
 &\quad + \delta(\delta^{2^{k-2}-1}(x_1 \cdots x_{k-2})) \delta^{2^{k-2}-1}(x_{k-1}) + \cdots \\
 &\quad + \delta^{2^{k-2}}(\delta^{2^{k-2}-1}(x_1 \cdots x_{k-2})) x_{k-1}.
 \end{aligned}$$

Hence, by induction method we get $\delta^{2^{k-1}-1}(x_1 \cdots x_{k-1}) = 1$ ($2 \leq k \leq m-1$).

Now, we see that

$$\begin{aligned}
 \delta^{2^{m-1}-2^{m-2}}(x_m) &= \delta^{2^{m-2}}(x_m) = \delta^{2^{m-2}-1}(P_{m-1}) \\
 &= \delta^{2^{m-2}-1}(P_{m-2}x_{m-1} + x_1 \cdots x_{m-2}\alpha_{m-2}) \\
 &= \delta^{2^{m-2}-1}(\delta(x_{m-1})x_{m-1} + \alpha_{m-2}\delta^{2^{m-2}-1}(x_1 \cdots x_{m-2})) \\
 &= \sum_{i=0}^{2^{m-2}-1} \delta^{1+i}(x_{m-1}) \delta^{2^{m-2}-1-i}(x_{m-1}) + \alpha_{m-2} \\
 &= \delta(x_{m-1}) \delta^{2^{m-2}-1}(x_{m-1}) + \delta^2(x_{m-1}) \delta^{2^{m-2}-2}(x_{m-1}) + \cdots \\
 &\quad + \delta^{2^{m-3}}(x_{m-1}) \delta^{2^{m-3}}(x_{m-1}) + \cdots \\
 &\quad + \delta^{2^{m-2}-2}(x_{m-1}) \delta^2(x_{m-1}) + \delta^{2^{m-2}-1}(x_{m-1}) \delta(x_{m-1}) \\
 &\quad + \delta^{2^{m-2}}(x_{m-1}) x_{m-1} + \alpha_{m-2} \\
 &= (\delta^{2^{m-3}}(x_{m-1}))^2 + x_{m-1} + (x_{m-2})^2 = x_{m-1}.
 \end{aligned}$$

It is easy to see that $\delta^{p^{m-1}} \neq 0$ and $\delta^{p^m} = 0$.

We now return back to the investigation of an $H(p)^m$ -Hopf Galois extension A/R . Let δ be such a derivation as in Lemma 6. Then δ acts naturally on A and makes A an $H(p^m)$ -module algebra. Since $\delta^0 = 1$, δ^1 , δ^2 , $\dots$, δ^{p^m-1} are left A -linearly independent, the usual argument of passing to the residue class fields and counting the ranks shows that A/R is an $H(p^m)$ -Hopf Galois extension; especially $A^{(\delta)} = \{a \in A \mid \delta(a) = 0\} = R$ by [7, Proposition 1.2]. Conversely, by Corollary 4, every $H(p^m)$ -Hopf Galois extension is an $H(p)^m$ -Hopf Galois extension. Summarizing the above, we get

Theorem 7. *Let A be a commutative R -algebra. Then A is an $H(p^m)$ -Hopf Galois extension of R if and only if it is an $H(p)^m$ -Hopf Galois extension.*

Remark. It should be noted that if $j < i$ then $H(p^i) \otimes H(p)^{m-i}$ is not isomorphic to $H(p^j) \otimes H(p)^{m-j}$. In fact, $H(p^i) \otimes H(p)^{m-i}$ contains the element $d \otimes (1 \otimes \cdots \otimes 1)$ ($d \in H(p^i)$) with the property $(d \otimes 1 \otimes \cdots \otimes 1)^{p^{i-1}} \notin R$ but $c^{p^{i-1}} \in R$ for any $c \in H(p^j) \otimes H(p)^{m-j}$. It should also be noted that if A/R is an $H(p^m)$ -Hopf Galois extension, then it is an $H(p^i) \otimes H(p)^{m-i}$ -Hopf Galois extension ($m \neq 1$), so there exist many non-isomorphic Hopf algebras which make A/R an Hopf Galois extension. Furthermore, A. Hattori [3] has pointed out that $R[X]/(X^p - a)$ ($a \in R$) is a $\text{Hom}(RG, R)$ Hopf Galois extension of R , where G is a cyclic group of order p (cf. also [9]). Therefore there exist much more non-isomorphic Hopf algebras which make A/R an Hopf Galois extension.

Finally, we are going to make a precision of Proposition 2. To this end, we define the (*weighted*) *degree* of a monomial in an $H(p^m)$ -Hopf (or $H(p)^m$ -Hopf) Galois extension $R[X_1]/(X_1^p - \alpha_1) \otimes \cdots \otimes R[X_m]/(X_m^p - \alpha_m) = R[x_1, \dots, x_m]$ as follows:

$$\text{degree}(rx_1^{e_1} \cdots x_m^{e_m}) = \begin{cases} \sum_{i=1}^m e_i p^{i-1} & \text{if } r \neq 0 (\in R), 0 \leq e_i \leq p-1 \\ -1 & \text{if } r = 0. \end{cases}$$

Since $\{x_1^{e_1} \cdots x_m^{e_m}\}_{0 \leq e_i \leq p-1}$ is an R -free basis of A [4, Theorem 1.3], the above definition is well-defined. Further, considering the p -adic expansion, we see that a monic monomial of given degree is uniquely determined.

Lemma 8. Let $A = R[X_1]/(X_1^p - \alpha_1) \otimes \cdots \otimes R[X_m]/(X_m^p - \alpha_m) = R[x_1, \dots, x_m]$ be an $H(p^m)$ -Hopf Galois extension of R , and

$$d(x_1) = 1 \text{ and } d(x_i) = P_{i-1} \quad (i \geq 2),$$

where P_{i-1} is given in the proof of Lemma 6.

(1) For a monic monomial f of degree ℓ , $d(f)$ is the sum of a monomial of degree $\ell-1$ with unit coefficient and a sum of monomials of degree less than $\ell-1$.

(2) Every monomial g of degree less than p^m-1 is integrable, that is there exists an element G in A such that $d(G) = g$; every sum of monomials of degree less than p^m-1 is integrable.

Proof. (1) This can be shown by direct computation.

(2) We shall proceed by induction on degree ℓ . For $\ell = -1, 0$, the assertion is valid. We assume that there holds the assertion for all $k \leq \ell-1$ ($< p^m-2$). Let G_1 be the monic monomial of degree $\ell+1$ ($< p^m$), and $g = rx_1^{e_1} \cdots x_m^{e_m}$ ($0 \leq e_i \leq p-1$, $r \in R$) an arbitrary monomial of degree ℓ . Then by (1), we have

$$d(G_1) = ux_1^{p^1} \cdots x_m^{p^m} + h,$$

where u is a unit of R and h is a sum of monomials of degree less than $\ell-1$. We set $G_2 = ru^{-1}G_1$. Then $d(G_2) = g + ru^{-1}h$, and by induction hypothesis, $ru^{-1}h$ is integrable, say $d(G_3) = ru^{-1}h$. Thus, we get $d(G_2 - G_3) = g$.

Theorem 9. *Let A be a commutative R -algebra. Then A is an $H(p^m)$ -Hopf Galois extension of R if and only if A is isomorphic to $R[X_1]/(X_1^p - \alpha_1) \otimes \cdots \otimes R[X_m]/(X_m^p - \alpha_m) = R[x_1, \cdots, x_m]$, $x_i^p \in R$, as H -module algebra, where x_i is the residue class of X_i and the action of $d \in H(p^m)$ on $R[x_1, \cdots, x_m]$ is defined by $d(x_i) = P_{i-1}$ (see the proof of Lemma 6).*

Proof. "If" part is already proved in the paragraph preceding Theorem 7. "Only if" part will be proved by induction on m . By Proposition 2, we can choose $y_1, \cdots, y_m \in A$ as follows: $A = R[y_1, \cdots, y_m] \cong R[Y_1]/(Y_1^p - \beta_1) \otimes \cdots \otimes R[Y_m]/(Y_m^p - \beta_m)$, $y_i^p = \beta_i \in R$, $d(y_1) = 1$, $y_i = d^{p^{m-1}-p^{i-1}}(y_m)$ and $d(y_i) \in R[y_1, \cdots, y_{i-1}]$. Obviously, the assertion is valid for $m=1$. We assume that there holds the assertion for $m-1$, that is there exist $x_1, \cdots, x_{m-1}$ such that $R[x_1, \cdots, x_{m-1}] = R[y_1, \cdots, y_{m-1}]$ and the restriction of d to $R[x_1, \cdots, x_{m-1}]$ is of the form $P_0\partial_1 + \cdots + P_{m-2}\partial_{m-1}$. Since $d(y_m) \in R[y_1, \cdots, y_{m-1}] = R[x_1, \cdots, x_{m-1}]$ and $d^{p^{m-1}} = 1$, we have

$$d(y_m) = P_{m-1} + g(x_1, \cdots, x_{m-1}),$$

where $g(x_1, \cdots, x_{m-1})$ is a sum of monomials of degree less than $p^{m-1}-1$. By Lemma 8, $g(x_1, \cdots, x_{m-1})$ is integrable, say $dG(x_1, \cdots, x_{m-1}) = g(x_1, \cdots, x_{m-1})$. Setting $x_m = y_m - G(x_1, \cdots, x_{m-1})$, we get $d(x_m) = P_{m-1}$ and $R[x_1, \cdots, x_m] = R[y_1, \cdots, y_m]$. Since d is a derivation and $R = \{a \in A \mid d(a) = 0\}$, it follows that x_m^p is in R . This completes the proof.

Remark. Let $d = P_0\partial_1 + \cdots + P_{m-1}\partial_m$ be another derivation on $A = R[X_1]/(X_1^p - \alpha_1) \otimes \cdots \otimes R[X_m]/(X_m^p - \alpha_m)$ satisfying the condition in Lemma 6. Then, for such d , there holds an analogue of Theorem 9.

Corollary 10. *Let A/R be an $H(p^m)$ -Hopf Galois extension, and let $x_1, \cdots, x_m$ be elements of A such that $A = R[x_1, \cdots, x_m]$, $x_i^p = \alpha_i \in R$, $d(x_1) = 1$ and $x_i = d^{p^{m-1}-p^{i-1}}(x_m)$. Then, concerning the action of $d \in H(p^m)$ restricted to $A_i = R[x_1, \cdots, x_i]$, A_i/R is an $H(p^i)$ -Hopf Galois extension.*

Proof. From the proof of Theorem 9, we may assume that $x_1, \cdots, x_m$ are chosen as in the proof of the only if part of Theorem 9. Then the assertion follows immediately.

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