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Throughout, A will represent an (associative) ring, A^+ the additive group of A, and D the maximal divisible ideal of A. We shall use the symbol $\boxplus$ (resp. $\oplus$) for a (ringtheoretic) direct sum (resp. grouptheoretic direct sum). Let Π be the set of prime numbers.

The purpose of this note is to prove the following theorem, which gives a weak answer of a problem raised by Szász [1, Problem 79].

Theorem 1. *The following are equivalent:*

1) *For each integer n, nA is a direct summand of A.*

2) *A contains an ideal I consisting of torsion elements with square-free order such that $D \cap I = 0$ and either $A = D \boxplus I$ or $A^+/(D \boxplus I)^+$ is divisible.*

In advance of proving our theorem, we state the next

Lemma 1. (1) *Let $p \in \Pi$, and U a subgroup of A^+ with $pU = U$. If u is a non-zero element of U with $pu = 0$, then there exists a quasicyclic subgroup of U containing u; $u \in D$.*

(2) *Let P be a subset of Π. Suppose that $A = pA \boxplus A_p$ or $A = pA$ according as $p \in P$ or $p \in \Pi \setminus P$. If $p_1, \cdots, p_k \in P$ are different, then $A = A_{p_1} \boxplus \cdots \boxplus A_{p_k} \boxplus p_1 p_2 \cdots p_k A$, where $q(p_1 \cdots p_k A) = p_1 \cdots p_k A$ for every $q \in \{p_1, \cdots, p_k\} \cup \Pi \setminus P$. In particular, if P is finite then $A = D \boxplus (\boxplus_{p \in P} A_p)$.*

Proof. (1) Straightforward.

(2) Obviously, $pA = p^2 A$ and $pA_p = 0$ for every $p \in P$. The rest of the proof is a routine.

Proof of Theorem 1. 1) $\Rightarrow$ 2). Let P be the set of all $p \in \Pi$ with $pA \neq A$; $A = pA \boxplus A_p$. If P is empty then $A = D$. Now, let P be non-empty, and $p \in P$. Then $pA_p = 0$ and $pA = p^2 A$. By Lemma 1 (2), we readily see that $B = D + \sum_{p \in P} A_p = D \boxplus (\boxplus_{p \in P} A_p)$. Furthermore, since $A = pA + B$ for every $p \in \Pi$, A^+/B^+ is divisible provided $A \neq B$.

2) $\Rightarrow$ 1). We set $B = D \boxplus I = D \boxplus (\boxplus_{p \in P} A_p)$, where P is a subset of

Π and A_p is an ideal of A with $pA_p=0$. Further, we may assume that $n=p_1^{m_1}\cdots p_r^{m_r}q_1^{n_1}\cdots q_s^{n_s}$ with different $p_i\in P$ and $q_j\in\Pi\backslash P$.

Let $p\in\Pi$. Since $A=B$ or A^+/B^+ is divisible, we have

$$A=pA+B=\begin{cases} pA+pB=pA & \text{if } p\notin P \\ pA+pB+A_p=pA+A_p & \text{if } p\in P. \end{cases}$$

Suppose that $p\in P$ and $pA\cap A_p$ contains a non-zero element a. Then, because $pA=p^2A$, Lemma 1 (1) forces a contradiction $a\in D\cap A_p=0$. Thus, we have shown that $A=pA$ or $A=pA\boxplus A_p$ according as $p\notin P$ or $p\in P$. Then, by Lemma 1 (2), $A=A_{p_1}\boxplus\cdots\boxplus A_{p_r}\boxplus p_1\cdots p_rA=A_{p_1}\boxplus\cdots\boxplus A_{p_r}\boxplus nA$.

Corollary 1. *Suppose that A^+ contains no quasicyclic subgroups. If nA is a grouptheoretic direct summand of A^+ for each integer n, then nA is a direct summand of A.*

Proof. Let P be the set of all $p\in\Pi$ with $pA\neq A$. Suppose now that P is non-empty, and let $p\in P$. Then $A^+=pA\oplus A_p$, and so $pA_p=0$ and $pA=p^2A$. Now, let $a, b\in A_p$, and $ab=c+c'$ with $c\in pA$, $c'\in A_p$. Then $pc=0$, and Lemma 1 (1) shows that $c=0$. This proves that A_p is a subring of A, and therefore $A=pA\boxplus A_p$. Hence, R has the property 1) (see the last part of the proof of Theorem 1).

Remark 1. Let P be an infinite subset of Π, and A_p rings with $pA_p=0$ $(p\in P)$. Then $A_0=\boxplus_{p\in P}A_p$ has the property 1). Also, the complete direct sum A of A_p $(p\in P)$ has the same and A^+/A_0^+ is divisible. Now, decompose P into the disjoint infinite subsets $P^{(i)}$ $(i=1, 2, \cdots)$, and denote by $A^{(i)}$ the complete direct sum of A_p $(p\in P^{(i)})$. Then $B=\boxplus_{i=1}^{\infty}A^{(i)}$ has the property 1) and $A\supsetneqq B\supsetneqq A_0$.

Remark 2. Let A be a ring with property 1). If the maximal torsion ideal T of A is a grouptheoretic direct summand of A^+ and contains no quasicyclic subgroups, then T is a direct summand of A.

Reference

[1] F. Szász: Radicals of Rings, John Wiley & Sons, Chichester-New York-Brisbane-Toronto, 1981

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