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The Prime Ideal Factorization of 2 in Pure Quartic Fields with Index 2

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Abstract

The prime ideal decomposition of 2 in a pure quartic field with field index 2 is determined explicitly.

KEYWORDS: pure quartic field, discriminant, prime decomposition

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ABSTRACT. The prime ideal decomposition of 2 in a pure quartic field with field index 2 is determined explicitly.

1. INTRODUCTION

Let K be an algebraic number field and O_K its ring of integers. When determining generators of the ideals in the prime ideal factorization of a (rational) prime p in O_K , the most difficult case occurs when p divides the field index $i(K)$ of K . In this paper we examine the case when K is a pure quartic field. Here $i(K) = 1$ or 2 , and we determine explicit generators of the prime ideals in the decomposition of 2 when $i(K) = 2$.

Let K be a pure quartic field. Then there exists a fourth power free integer m such that $K = \mathbb{Q}(m^{1/4})$. It follows from the work of Funakura [1, p. 36] that the field index $i(K)$ of K is given by

$$i(K) = \begin{cases} 2, & \text{if } m \equiv 1 \pmod{16}, \\ 1, & \text{if } m \not\equiv 1 \pmod{16}. \end{cases}$$

From now on we assume that $i(K) = 2$ so that $m \equiv 1 \pmod{16}$, say $m = 16k + 1$. In this case the prime ideal factorization of $\langle 2 \rangle$ in O_K is

$$\langle 2 \rangle = P_1^2 P_2 P_3,$$

where P_1, P_2, P_3 are distinct prime ideals, see [1, p. 36]. In this paper we determine explicit generators of P_1, P_2 and P_3 .

Theorem. *Let m be a fourth power free integer such that $K = \mathbb{Q}(m^{1/4})$ is a pure quartic field with $i(K) = 2$. Then $\langle 2 \rangle = P_1^2 P_2 P_3$, where the*

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distinct prime ideals P_1, P_2, P_3 of O_K are given by

$$P_1 = \langle 2, \frac{3}{2} + m^{1/4} + \frac{1}{2}m^{1/2} \rangle,$$

$$P_2 = \begin{cases} \langle 2, \frac{5}{4} + \frac{1}{4}m^{1/4} + \frac{1}{4}m^{1/2} + \frac{1}{4}m^{3/4} \rangle, & \text{if } m \equiv 1 \pmod{32}, \\ \langle 2, \frac{3}{4} + \frac{5}{4}m^{1/4} + \frac{3}{4}m^{1/2} + \frac{1}{4}m^{3/4} \rangle, & \text{if } m \equiv 17 \pmod{32}, \end{cases}$$

$$P_3 = \begin{cases} \langle 2, \frac{5}{4} - \frac{1}{4}m^{1/4} + \frac{1}{4}m^{1/2} - \frac{1}{4}m^{3/4} \rangle, & \text{if } m \equiv 1 \pmod{32}, \\ \langle 2, \frac{3}{4} - \frac{5}{4}m^{1/4} + \frac{3}{4}m^{1/2} - \frac{1}{4}m^{3/4} \rangle, & \text{if } m \equiv 17 \pmod{32}. \end{cases}$$

2. PROOF OF THEOREM

Let $L = \mathbb{Q}(m^{1/2})$ so that $\mathbb{Q} \subset L \subset K$ and $[L : \mathbb{Q}] = 2$. Set

$$Q_1 = \langle 2, \frac{1 + m^{1/2}}{2} \rangle, \quad Q_2 = \langle 2, \frac{1 - m^{1/2}}{2} \rangle.$$

Q_1 and Q_2 are distinct prime ideals of O_L such that $\langle 2 \rangle = Q_1 Q_2$. Let m_2 be the largest integer such that $m_2^2 \mid m$. Set $m_1 = m/m_2^2$ so that m_1 is a squarefree integer having the same sign as m . Clearly $m^{1/2} = m_2 m_1^{1/2}$. Then

$$Q_1 = \begin{cases} \langle 2, \frac{1 + m_1^{1/2}}{2} \rangle, & \text{if } m_2 \equiv 1 \pmod{4}, \\ \langle 2, \frac{1 - m_1^{1/2}}{2} \rangle, & \text{if } m_2 \equiv 3 \pmod{4}. \end{cases}$$

Next, by [2, Table D, cases D1, D2, p. 92], we see that

$$Q_1 = P_1^2$$

for some prime ideal P_1 of O_K . We claim that

$$P_1 = \langle 2, \frac{3}{2} + m^{1/4} + \frac{1}{2}m^{1/2} \rangle.$$

First we show that P_1 is a prime ideal of O_K . The minimal polynomial of $\theta = \frac{3}{2} + m^{1/4} + \frac{1}{2}m^{1/2}$ over \mathbb{Q} is

$$g(x) = x^4 - 6x^3 + (13 - 8k)x^2 + (-14 - 8k)x + (6 + 16k + 16k^2).$$

Hence $N(\theta) = \pm(6 + 16k + 16k^2) \equiv 2 \pmod{4}$. Let $\langle \theta \rangle = S_1 S_2 \cdots S_r$ be the prime ideal factorization of $\langle \theta \rangle$ in O_K . Hence $N(\langle \theta \rangle) =$

$N(S_1)N(S_2)\cdots N(S_r)$. As $2 \parallel N(<\theta>)$ there exists a unique $S = S_i$ such that $2 \parallel N(S)$, that is $N(S) = 2$. Thus $<\theta>$ has exactly one prime ideal to exponent 1 in its prime factorization lying above 2. As $P_1 = <2, \theta>$ we deduce that $P_1 = S$ so that P_1 is a prime ideal of O_K . Next we show that $P_1 \mid Q_1$. We set $\phi = \frac{3}{2} - m^{1/4} + \frac{1}{2}m^{1/2}$. An easy calculation shows that

$$\frac{1 + m^{1/2}}{2} = \theta\phi - (2k + 1)2.$$

Hence, as $2 \in P_1$ and $\theta \in P_1$, we deduce that $\frac{1 + m^{1/2}}{2} \in P_1$. Thus we have $Q_1 = <2, \frac{1 + m^{1/2}}{2}> \subseteq P_1$, and so $P_1 \mid Q_1$. As Q_1 is the square of a prime ideal in O_K , we deduce that $Q_1 = P_1^2$ as asserted.

Let

$$k = \begin{cases} 2g, & \text{if } m \equiv 1 \pmod{32}, \\ 2g + 1, & \text{if } m \equiv 17 \pmod{32}. \end{cases}$$

For $\epsilon = \pm 1$, the minimal polynomial of

$$\alpha(\epsilon) = \begin{cases} \frac{5}{4} + \frac{\epsilon}{4}m^{1/4} + \frac{1}{4}m^{1/2} + \frac{\epsilon}{4}m^{3/4}, & \text{if } m \equiv 1 \pmod{32}, \\ \frac{3}{4} + \frac{5\epsilon}{4}m^{1/4} + \frac{3}{4}m^{1/2} + \frac{\epsilon}{4}m^{3/4}, & \text{if } m \equiv 17 \pmod{32}, \end{cases}$$

is

$$x^4 - 5x^3 + (9 - 12g)x^2 + (-7 + 24g - 64g^2)x + (2 - 12g + 64g^2 - 128g^3),$$

if $m \equiv 1 \pmod{32}$, and

$$\begin{aligned} x^4 - 3x^3 + (-37 - 76g)x^2 + (-75 - 240g - 192g^2)x \\ + (-38 - 172g - 256g^2 - 128g^3), \end{aligned}$$

if $m \equiv 17 \pmod{32}$. Clearly $N(\alpha(\epsilon)) \equiv 2 \pmod{4}$ in both cases, and similarly to the argument above, we deduce that $I_+ = <2, \alpha(1)>$ and $I_- = <2, \alpha(-1)>$ are conjugate prime ideals of O_K lying above 2. If $m \equiv 1 \pmod{32}$ we have

$$\frac{1 - m^{1/2}}{2} = 2(1 - g - gm^{1/2}) - \alpha(1)\alpha(-1) \in I_+ \cap I_-$$

and if $m \equiv 17 \pmod{32}$

$$\frac{1 - m^{1/2}}{2} = 2(-g - (1 + g)m^{1/2}) - \alpha(1)\alpha(-1) \in I_+ \cap I_-.$$

Hence $\frac{1-m^{1/2}}{2} \in I_+ \cap I_-$. Thus I_+ and I_- are conjugate prime ideals of O_K lying above the prime ideal Q_2 of O_L . As $\langle 2 \rangle = P_1^2 P_2 P_3 = Q_1 Q_2$ and $Q_1 = P_1^2$, we see that $Q_2 = P_2 P_3$ and that we can take

$$P_2 = I_+ = \langle 2, \alpha(1) \rangle$$

and

$$P_3 = I_- = \langle 2, \alpha(-1) \rangle.$$

This completes the proof.

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