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NOTE ON RIGHT S-IDEMPOTENT IDEALS

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Let R be a ring, and $M(\neq 0)$ a right R-module. If $u \in uR$ for every $u \in M$, M is said to be *s-unital*. In particular, if R_R is s-unital, R is called a *right s-unital ring*. Given a finite subset U of an s-unital module M_R, there exists an element e in R such that $ue = u$ for all $u \in U$ (see [7, Theorem 1]). Following Lanski [6], a right ideal I of R is called *right s-idempotent* if $TI = T$ for every right ideal T of R contained in I, or equivalently, if $a \in |a)I$ for each $a \in I$, where $|a)$ is the principal right ideal generated by a. Finally, following [4], R is called *almost right Noetherian* if for each infinite ascending chain $I_1 \subseteq I_2 \subseteq \cdots$ of right ideals of R there exists a positive integer k such that $I_n R^k \subseteq I_k$ for all n.

The purpose of this note is to give the following theorem which includes [6, Theorems 2 and 3] and leads also to [6, Theorem 4].

Theorem 1. (1) *Let I be a non-zero right ideal of a ring R. Then the following are equivalent* :

1) *I is right s-idempotent.*

2) *$I^2 = I$ and RI is a right s-unital ring.*

(2) *Let R be either* i) *a right Goldie ring,* ii) *an almost right Noetherian ring,* iii) *a ring satisfying the minimum condition on right annihilators (equivalently, the maximum condition on left annihilators), or* iv) *a ring satisfying the maximum condition on principal left ideals. Let I be a non-zero right ideal of R. Then the following are equivalent* :

1) *I is right s-idempotent.*

2) *$I^2 = I$ and RI has a right identity.*

If, furthermore, R is semiprime (resp. prime), then 1) *is equivalent to*

2)′ *RI (resp. $RI = R$) has an identity.*

In preparation for proving Theorem 1, we state the next lemma.

Lemma 1. (1) *Let R be as in Theorem* 1 (2). *Then every right s-unital subring A of R has a right identity.*

(2) *Let I be an ideal of a semiprime ring R. If I has a right identity e, then e is the identity of I.*

Proof. (1) In view of [5, Theorems 3, 4 and Corollary 6], it suffices to prove the case iv). Choose $a \in A$ such that the principal left ideal $(a|$ generated by a is maximal in $\{(x| \mid x \in A\}$, and take $e \in A$ with $ae = a$. Then we can easily see that $(e| = (a|$ and $e^2 = e$. Suppose $Ae \neq A$, and choose a non-zero $b \in A(1-e)$. Take $c \in A$ such that $ec = e$ and $bc = b$. Then $Rc \supseteq Re \oplus (b| \supsetneq Re = (a|$. This contradiction proves that e is a right identity of A.

(2) Since $\{(1-e)I\}^2 \subseteq I(1-e)I = 0$, we have $(1-e)I = 0$, which proves that e is a left identity of I.

The next improves [1, Theorems 1 and 2].

Corollary 1. *Let R be a ring in which every element is a product of idempotents. If R satisfies the minimum condition on right annihilators or the maximum condition on principal left ideals, then R is a Boolean ring.*

Proof. By Lemma 1 (1), R has a right identity e. Now, for any $x \in R$ we have $R(ex-x) = R(e-1)x = 0$, whence it follows that $ex = x$. This proves that e is the identity 1 of R. Hence, R is a Boolean ring by [3, Lemma (2)].

Proof of Theorem 1. (1) 1) ⇒ 2). It is easy to see that $I^2 = I$ and I_{RI} is s-unital. Now, let $x = \sum_{i=1}^{n} x_i a_i$ $(x_i \in R,\ a_i \in I)$ be an arbitrary element of RI, and choose $e \in RI$ such that $a_i e = a_i$ $(i = 1, \cdots, n)$. Then $x = xe$.

2) ⇒ 1). Obviously, $I = I^2 \subseteq RI$. Hence, for any $a \in I$ we get $a \in aRI \subseteq \{a)I$.

(2) In view of (1) and Lemma 1 (1), it is immediate that 1) and 2) are equivalent. We assume henceforth that R is semiprime. Then, by Lemma 1 (2), we see that 1) implies 2)′. In order to see the converse, let e be an identity of RI. Then $RI(1-e) = 0$ shows that $I(1-e) = 0$, and therefore $I = Ie \subseteq I^2$. Hence, I is right s-idempotent by (1).

Corollary 2. (1) *Let I be a non-zero ideal of a ring R. Then the following are equivalent* :

1) *I is right s-idempotent.*

2) *I is a right s-unital ring.*

(2) *Let R be as in Theorem* 1 (2), *and I a non-zero ideal of R. Then the following are equivalent* :

1) *I is right s-idempotent.*

2) *I has a right identity.*

The next includes [6, Theorem 4] (see also [2, Corollaries 4 and 5]).

Corollary 3. *The following are equivalent :*

1) *Every right ideal of R is right s-idempotent.*

2) *Every ideal of R is right s-idempotent.*

3) *R is fully right idempotent, namely every right ideal of R is idempotent.*

If, furthermore, R is as in Theorem 1 (2) *then* 1) *is equivalent to*

4) *R is a finite direct sum of simple rings with identity.*

Proof. Obviously, 4) implies 3). In view of Theorem 1 (1), it is easy to see that 1) and 2) are equivalent and imply 3). Now, suppose 3). Then R is semiprime and every non-zero ideal of R is a right s-unital ring. Hence, by Corollary 2 (1), R satisfies 2). If, furthermore, R is as in Theorem 1 (2) then Corollary 2 (2) and Lemma 1 (2) show that every non-zero ideal of R has an identity and is a direct summand of R, and therefore R satisfies 4).

Remark. A ring R is said to have the *finite intersection property on right annihilators* provided that whenever $r(I) = 0$ for a right ideal I of R there exists a finite subset F of I such that $r(F) = 0$. On the other hand, R is called a *right strongly semiprime ring* provided if I is an ideal of R and is essential as a right ideal then there exists a finite subset F of I such that $r(F) = 0$. In [2, Theorem 2], it has been proved that the following are equivalent :

1) R is a right strongly semiprime, fully right idempotent ring.

2) R is a fully right idempotent ring and possesses the finite intersection property on right annihilators.

3) R is a finite direct sum of simple rings with identity.

As a matter of fact, [2, Corollary 4] was obtained as a corollary to the theorem.

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