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ON AN EXPONENTIAL SAM INVOLVING THE ARITHMETIC FUNCTION $\sigma_a(n)$

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1. Introduction. Let $-1 < a \leqq 0$ and $\sigma_a(n) = \sum_{d|n} d^a$ be the sum of the a-th powers of positive divisors of the positive integer n, so that $\sigma_0(n) = d(n)$. Jutila [4] has investigated the exponential sum

$$D(X;\, h/k) = {\sum_{n \leq X}}' d(n) \cdot e(hn/k)$$

for large X, where h and k are co-prime integers with $1 \leqq k$, $e(t) = e^{2\pi it}$, and the symbol ${\sum_{n \leq X}}'$ denotes that if X is an integer, then the term corresponding to X is to be halved. Let $s = \sigma + it$ be a complex variable, and

$$E(s\,;\, h/k) = \sum_{n=1}^{\infty} d(n) \cdot e(hn/k) \cdot n^{-s}, \quad \mathrm{Re}(s) > 1.$$

The function $E(s\,;\, h/k)$ can be analytically continued to a meromorphic function in the whole complex plane (Estermann [2]). We put

$$(1.1) \quad \Delta(X;\, h/k) = D(X;\, h/k) - k^{-1} \cdot (\log X + 2\gamma - 1 - 2 \cdot \log k) \cdot X - E(0\,;\, h/k),$$

where γ is Euler's constant. Using the truncated Voronoi summation formula, Jutila [4] proved that if $1 \leqq k \leqq X$, and N is a positive integer such that $1 \leqq N \ll X$, then

$$(1.2) \quad \Delta(X;\, h/k) = (\pi\sqrt{2})^{-1} \cdot k^{\frac{1}{2}} \cdot X^{\frac{1}{4}} \cdot \sum_{n \leq N} d(n) \cdot e(-\bar{h}n/k) \cdot n^{-\frac{3}{4}} \cdot \cos\left(4\pi k^{-1} n^{\frac{1}{2}} X^{\frac{1}{2}} - \frac{1}{4}\pi\right) + O(k \cdot N^{-\frac{1}{2}} \cdot X^{\frac{1}{2}+\varepsilon}),$$

where the class

$$(1.3) \quad \bar{h} \pmod{k}$$

is defined by $h\bar{h} \equiv 1 \pmod{k}$.

The first purpose of this paper is to derive a formula of the Voronoi type (Theorem 1) for the exponential sum

$$(1.4) \quad D_a(X;\, h/k) = {\sum_{n \leq X}}' \sigma_a(n) \cdot e(hn/k).$$

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Theorem 2 gives an asymptotic formula for the mean square of the error term in the asymptotic formula for $D_a(X; h/k)$. Jutila's proof of the above formula (1.2) can be modified so as to give an analogous expression for $D_a(X; h/k)$, and such a modification is the basis of our proof.

We may appeal to an analogy between the function $E(s; h/k)$ and

$$(1.5) \qquad E_a(s; h/k) = \sum_{n=1}^{\infty} \sigma_a(n) \cdot e(hn/k) \cdot n^{-s}, \quad \mathrm{Re}(s) > 1.$$

The basic properties of the function $E_a(s; h/k)$ are given in Lemma 1. In what follows, ε is taken an arbitrarily small positive number, and not necessarily the same in each occurence. The constants implied by the symbols $\ll$ and $O(\ )$ always depend at most on ε and a. $\zeta(s)$ is the Riemann zeta-function.

To formulate the analogue of (1.1) for $D_a(X; h/k)$, we define that if $a = 0$, then

$$(1.6) \qquad \Delta_0(X; h/k) = \Delta(X; h/k),$$

and if $-1 < a < 0$, then

$$(1.7) \quad \Delta_a(X; h/k) = D_a(X; h/k) - k^{a-1} \cdot \zeta(1-a) \cdot X - k^{-1-a} \cdot (1+a)^{-1} \cdot \zeta(1+a) \cdot X^{1-a} - E_a(0; h/k),$$

which are the "right" analogy of (1.1) for $D_a(X; h/k)$. Then we have the following

Theorem 1. *For $-1 < a \leqq 0$, $k \leqq X$, and $1 \leqq N \ll X$ we have*

$$(1.8) \quad \Delta_a(X; h/k) = (\pi\sqrt{2})^{-1} \cdot k^{\frac{1}{2}} \cdot X^{\frac{1}{4}+\frac{1}{2}a} \cdot \sum_{n \leqq N} \sigma_a(n) \cdot e(-\bar{h}n/k) \cdot n^{-\frac{3}{4}-\frac{1}{2}a} \cdot \cos\left(4\pi k^{-1} n^{\frac{1}{2}} X^{\frac{1}{2}} - \frac{1}{4}\pi\right) + O(k \cdot N^{-\frac{1}{2}} \cdot X^{\frac{1}{2}+\varepsilon}).$$

Remark 1. As an immediate corollary of (1.8) with $N = k^{2/(3-2a)} \cdot X^{(1-2a)/(3-2a)}$ we have

$$(1.9) \qquad \Delta_a(X; h/k) \ll k^{(2-2a)/(3-2a)+\varepsilon} \cdot X^{1/(3-2a)+\varepsilon}$$

for $k \leqq X$, and $-1 < a \leqq 0$.

Remark 2. In case $k = 1$, other formulas of the Voronoi type were studied first by Wigert [9], next by Cramér [1] and later by Oppenheim [5].

A plausible conjecture as to $\Delta_a(X; h/k)$ may be inferred from the following mean value theorem.

Theorem 2. *For* $1 \leqq k \leqq X$, *and* $-\frac{1}{2} < a \leqq 0$ *we have*

$$(1.10) \quad \int_1^X \left| \Delta_a(t; h/k) \right|^2 \cdot dt$$
$$= k \cdot [(6+4a)\pi^2]^{-1} \cdot \zeta(3/2-a) \cdot \zeta(3/2+a) \cdot \zeta^2(3/2) \cdot [\zeta(3)]^{-1} \cdot X^{\frac{3}{2}+a}$$
$$+ O(k^2 \cdot X^{1+\varepsilon}) + O(k^{\frac{3}{2}} \cdot X^{\frac{5}{4}+\frac{1}{2}a+\varepsilon}).$$

This argument is essentially similar to that of Cramér (see e.g. [3]).
Theorem 2 suggests the following

Conjecture. *For* $-\frac{1}{2} < a \leqq 0$, *and* $k^{2/(1+2a)} \leqq X$,

$$(1.11) \qquad \Delta_a(X; h/k) \ll k^{\frac{1}{2}} \cdot X^{\frac{1}{4}+\frac{1}{2}a+\varepsilon}.$$

2. Some lemmas.

Lemma 1. *The function* $E_a(s; h/k)$ *can be analytically continued to a meromorphic function, which is regular in the whole complex plane up to two simple poles at* $s = 1$, $1+a$ $(-1 < a < 0)$, *and is regular in the whole complex plane up to a double pole at* $s = 1$ $(a = 0)$. *The function* $E_a(s; h/k)$ *satisfies the functional equation*

$$(2.1) \quad E_a(s; h/k)$$
$$= \pi^{-1} \cdot [k/(2\pi)]^{1+a-2s} \cdot \Gamma(1-s) \cdot \Gamma(1+a-s) \cdot \Big[\cos\left(\frac{1}{2}\pi a\right)$$
$$\cdot E_a(1+a-s; \bar{h}/k) - \cos\left(\pi s - \frac{1}{2}\pi a\right) \cdot E_a(1+a-s; -\bar{h}/k) \Big],$$

and has for $-1 < a < 0$ *the Laurent expansions*

$$(2.2) \quad E_a(s; h/k) = \begin{cases} k^{-1+a} \cdot \zeta(1-a) \cdot (s-1)^{-1} + \cdots, & \text{at } s = 1, \\ k^{-1-a} \cdot \zeta(1+a) \cdot (s-a-1)^{-1} + \cdots, & \text{at } s = 1+a, \end{cases}$$

and has for $a = 0$ *the Laurent expansion*

$$(2.3) \quad E_a(s; h/k) = k^{-1} \cdot (s-1)^{-2} + k^{-1} \cdot (2\gamma - 2 \cdot \log k) \cdot (s-1)^{-1} + \cdots,$$
$$\text{at } s = 1.$$

Proof. We start from the following identity, valid for $\mathrm{Re}(s) > 1$:

$$(2.4) \qquad E_a(s; h/k) = \sum_{n=1}^{\infty} \sum_{t|n} t^a \cdot e(hn/k) \cdot n^{-s}$$
$$= \lim_{N\to\infty} \left(\sum_{t\le N} t^{a-s} \sum_{r\le N/t} e(hrt/k) \cdot r^{-s} \right)$$
$$= \sum_{1\le b\le k} \zeta(s-a;\, b,\, k) \cdot \zeta(s;\, e(hb/k)),$$

where

$$(2.5) \qquad \zeta(s;\, a,\, k) = \sum_{n\equiv a (\mathrm{mod}\ k)} n^{-s},$$

and

$$(2.6) \qquad \zeta(s;\, e(a/k)) = \sum_{n=1}^{\infty} e(an/k) \cdot n^{-s}.$$

The functional equations (2.5) and (2.6) have been investigated by Estermann [2]. We quote from [2] that

$$(2.7) \qquad \zeta(s;\, a,\, k) = G(s) \cdot k^{-s} \cdot \left[e\left(\frac{1}{4}s\right) \cdot \zeta(1-s;\, e(a/k)) - e\left(-\frac{1}{4}s\right) \cdot \zeta(1-s;\, e(-a/k)) \right],$$

and

$$(2.8) \qquad \zeta(s;\, e(a/k)) = G(s) \cdot k^{1-s} \cdot \left[e\left(\frac{1}{4}s\right) \cdot \zeta(1-s;\, -a,\, k) - e\left(-\frac{1}{4}s\right) \cdot \zeta(1-s;\, a,\, k) \right],$$

where

$$G(s) = -i(2\pi)^{s-1} \cdot \Gamma(1-s).$$

Applying (2.7) and (2.8) to (2.4), we obtain

$$(2.9) \qquad E_a(s; h/k)$$
$$= 2G(s) \cdot G(s-a) \cdot k^{1+a-2s} \cdot \left[\cos\left(\pi s - \frac{1}{2}\pi a\right) \sum_{1\le b\le k} \zeta(1+a-s;\, e(b/k)) \cdot \zeta(1-s;\, -hb,\, k) - \cos\left(\frac{1}{2}\pi a\right) \sum_{1\le b\le k} \zeta(1+a-s;\, e(b/k)) \cdot \zeta(1-s;\, hb,\, k) \right].$$

Hence, by (1.3), we have proved the functional equation $E_a(s; h/k)$.

It easily follows from (2.4) that $E_a(s; h/k)$ is regular everywhere except at $s=1, 1+a\ (-1<a<0)$ or $s=1\ (a=0)$. We shall investigate the Laurent expansion in the neighbourhood of these points.

In case of the simple pole $s=1$, by the property of (2.6) and $(h, k)=1$, we have

$$\begin{aligned} &E_a(s; h/k)-\zeta(s)\cdot\zeta(s-a; k, k)\\ &=\sum_{1\leq b\leq k-1}\zeta(s; e(hb/k))\cdot\zeta(s-a; b, k). \end{aligned}$$

Since $\zeta(s; e(hb/k))$ is regular at $s=1$ for $1\leqq b\leqq k-1$, the above identity means that $E_a(s; h/k)$ has the same meromorphic part as $\zeta(s)\cdot\zeta(s-a; k, k)$ in a neighbourhood of $s=1$. And in case of the simple pole $s=1+a$, by (2.5), we have

$$\begin{aligned} &E_a(s; h/k)-\zeta(s-a; k, k)\sum_{1\leq b\leq k}\zeta(s; e(hb/k))\\ &=\sum_{1\leq b\leq k}[\zeta(s-a; b, k)-\zeta(s-a; k, k)]\cdot\zeta(s; e(hb/k)). \end{aligned}$$

The above identity means that $E_a(s; h/k)$ has the same meromorphic part as

$$k^{1+a-2s}\cdot\zeta(s)\cdot\zeta(s-a)$$

in a neighbourhood of $s=1+a$. Thus we have proved the Laurent expansions (2.2). Lastly in case of the double pole $s=1$, the Laurent expansion (2.3) has been investigated in detail by Estermann [1]. Therefore we have proved Lemma 1.

Lemma 2. *Let $F(t)$ and $G(t)$ be real functions, $G(t)/F'(t)$ monotonic, $|F'(t)/G(t)|\geqq m>0$, throughout the interval $[a, b]$. Then*

$$\left|\int_a^b G(t)\cdot\exp(iF(t))\cdot dt\right|<4/m.$$

The proof of this lemma depends on Titchmarsh [7].

3. Proof of Theorem 1. Let T be a parameter given by

$$k^2T^2\cdot(4\pi^2X)^{-1}=N+\frac{1}{2}. \tag{3.1}$$

By Perron's formula (see e.g. [7]), and (3.1), we have

$$\sum_{n\leq X}{}' \sigma_a(n)\cdot e(hn/k)$$

$$=(2\pi i)^{-1}\cdot\int_{1+\varepsilon-iT}^{1+\varepsilon+iT} E_a(s;\ h/k)\cdot X^s\cdot s^{-1}\cdot ds+O(k\cdot N^{-\frac{1}{2}}\cdot X^{\frac{1}{2}+\varepsilon}).$$

The above integral is evaluated by the theorem of residues using the rectangular contour with vertices at $1+\varepsilon\pm iT$, $a-\varepsilon\pm iT$. By the equation (2.1), and the Phragmén-Lindelöf principle, we have

$$(3.2)\qquad E_a(s;\ h/k)\ll(k|t|)^{1+\varepsilon-\sigma},\ \text{for } a-\varepsilon\leqq\sigma\leqq 1+\varepsilon,\ |t|\geqq 1.$$

By (3.1) and (3.2), the integrals over the horizontal sides are

$$\ll k\cdot N^{-\frac{1}{2}}\cdot X^{\frac{1}{2}+\varepsilon}.$$

If $-1<a<0$, by (2.2), the residues of the integrand at $s=1,\ 1+a$ yield the second and third terms on the right-hand side of (1.7), and the residue at $s=0$ gives $E_a(0;\ h/k)$. If $a=0$, by (2.3), the residue of the integrand at $s=1$ yields the second term on the right-hand side of (1.6), the residue at $s=0$ gives $E_0(0;\ h/k)=E(0;\ h/k)$. Hence we have, for $-1<a\leqq 0$,

$$\Delta_a(X;\ h/k)$$

$$=(2\pi i)^{-1}\cdot\int_{a-\varepsilon-iT}^{a-\varepsilon+iT} E_a(s;\ h/k)\cdot X^s\cdot s^{-1}\cdot ds+O(k\cdot N^{-\frac{1}{2}}\cdot X^{\frac{1}{2}+\varepsilon}).$$

After substituting the expression (2.1) in the right-hand side, this becomes

$$(3.3)\qquad \Delta_a(X;\ h/k)$$

$$=(\pi i)^{-1}\cdot(2\pi)^{-a-2}\cdot k^{a+1}\cdot\cos\left(\frac{1}{2}\pi a\right)\cdot\sum_{n=1}^{\infty}\sigma_a(n)\cdot e(-\bar{h}n/k)\cdot n^{-a-1}\cdot \mathrm{I}_1$$

$$-(\pi i)^{-1}\cdot(2\pi)^{-a-2}\cdot k^{a+1}\cdot\sin\left(\frac{1}{2}\pi a\right)\cdot\sum_{n=1}^{\infty}\sigma_a(n)\cdot e(-\bar{h}n/k)\cdot n^{-a-1}\cdot \mathrm{I}_2$$

$$+\pi^{-2}\cdot(2\pi)^{-a-1}\cdot k^{a+1}\cdot\cos\left(\frac{1}{2}\pi a\right)\cdot\sum_{n=1}^{\infty}\sigma_a(n)\cdot\sin(2\pi\bar{h}n/k)\cdot n^{-a-1}\mathrm{I}_3$$

$$+O(k\cdot N^{-\frac{1}{2}}\cdot X^{\frac{1}{2}+\varepsilon}),$$

where

$$I_1 = \int_{a-\varepsilon-iT}^{a-\varepsilon+iT} \Gamma(1-s)\cdot\Gamma(1+a-s)\cdot[1-\cos(\pi s)]\cdot(4\pi^2k^{-2}nX)^s\cdot s^{-1}\cdot ds,$$

$$I_2 = \int_{a-\varepsilon-iT}^{a-\varepsilon+iT} \Gamma(1-s)\cdot\Gamma(1+a-s)\cdot\sin(\pi s)\cdot(4\pi^2k^{-2}nX)^s\cdot s^{-1}\cdot ds,$$

$$I_3 = \int_{a-\varepsilon-iT}^{a-\varepsilon+iT} \Gamma(1-s)\cdot\Gamma(1+a-s)\cdot(4\pi^2k^{-2}nX)^s\cdot s^{-1}\cdot ds,$$

The integral I_3 is estimated by the Stirling formula as

$$\ll k^{-2a}\cdot N^{\varepsilon}\cdot X^{a+\varepsilon}\cdot n^{a-\varepsilon}. \tag{3.4}$$

And, it is easily shown that the contribution of (3.4) to (3.3) is

$$\ll k^{-a+1}\cdot N^{\varepsilon}\cdot X^{a+\varepsilon}.$$

Hence we have

$$\begin{aligned}(3.5)\quad & \Delta_a(X;\, h/k) \\ &= (\pi i)^{-1}\cdot(2\pi)^{-a-2}\cdot k^{a+1}\cdot\cos\left(\frac{1}{2}\pi a\right)\cdot\sum_{n=1}^{\infty}\sigma_a(n)\cdot e(-\bar{h}n/k)\cdot n^{-a-1}\cdot I_1 \\ &\quad -(\pi i)^{-1}\cdot(2\pi)^{-a-2}\cdot k^{a+1}\cdot\sin\left(\frac{1}{2}\pi a\right)\cdot\sum_{n=1}^{\infty}\sigma_a(n)\cdot e(-\bar{h}n/k)\cdot n^{-a-1}\cdot I_2 \\ &\quad + O(k\cdot N^{-\frac{1}{2}}\cdot X^{\frac{1}{2}+\varepsilon}).\end{aligned}$$

Now, we consider the contribution of the terms with $n > N$ in the first two terms of the right-hand side of (3.5). Each of the integrals I_1 and I_2 can be divided into the three parts ;

$$\begin{aligned}(3.6)\quad & \left[\int_{a-\varepsilon+i}^{a-\varepsilon+iT} + \int_{a-\varepsilon-i}^{a-\varepsilon+i} + \int_{a-\varepsilon-iT}^{a-\varepsilon-i}\right]\cdot\Gamma(1-s)\cdot\Gamma(1+a-s)\cdot[1-\cos(\pi s)] \\ & \cdot(4\pi^2k^{-2}nX)^s\cdot s^{-1}\cdot ds = I_{1,1}+I_{1,2}+I_{1,3},\end{aligned}$$

and

$$\begin{aligned}(3.7)\quad & \left[\int_{a-\varepsilon+i}^{a-\varepsilon+iT} + \int_{a-\varepsilon-i}^{a-\varepsilon+i} + \int_{a-\varepsilon-iT}^{a-\varepsilon-i}\right]\cdot\Gamma(1-s)\cdot\Gamma(1+a-s)\cdot\sin(\pi s) \\ & \cdot(4\pi^2k^{-2}nX)^s\cdot s^{-1}\cdot ds = I_{2,1}+I_{2,2}+I_{2,3}, \text{ say.}\end{aligned}$$

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The integral $I_{1,1}$ is estimated by Lemma 2 and the Stirling formula as

$$\ll k^{-a}\cdot N^{-\frac{1}{2}a+\varepsilon}\cdot X^{\frac{1}{2}a}\cdot n^{a-\varepsilon}.$$

By similar estimations, The integrals $I_{1,3}$, $I_{2,1}$, and $I_{2,3}$ are

$$\ll k^{-a}\cdot N^{-\frac{1}{2}a+\varepsilon}\cdot X^{\frac{1}{2}a}\cdot n^{a-\varepsilon}.$$

And we easily estimate that the integrals $I_{1,2}$ and $I_{2,2}$ are

$$\ll k^{-2a}\cdot N^{\varepsilon}\cdot X^{a+\varepsilon}\cdot n^{a-\varepsilon}.$$

Hence, it is easily shown that the contribution of (3.6) and (3.7) to (3.5) is

$$\ll k\cdot N^{-\frac{1}{2}a+\varepsilon}\cdot X^{\frac{1}{2}a}.$$

Hence we have

$$\begin{aligned}(3.8)\quad & \Delta_a(X;\, h/k) \\ &= (\pi i)^{-1}\cdot(2\pi)^{-a-2}\cdot k^{a+1}\cdot\cos\left(\frac{1}{2}\pi a\right)\cdot\sum_{n\leq N}\sigma_a(n)\cdot e(-\bar{h}n/k)\cdot n^{-a-1}\cdot I_1 \\ &\quad -(\pi i)^{-1}\cdot(2\pi)^{-a-2}\cdot k^{a+1}\cdot\sin\left(\frac{1}{2}\pi a\right)\cdot\sum_{n\leq N}\sigma_a(n)\cdot e(-\bar{h}n/k)\cdot n^{-a-1}\cdot I_2 \\ &\quad +O(k\cdot N^{-\frac{1}{2}}\cdot X^{\frac{1}{2}+\varepsilon}).\end{aligned}$$

Next, each of the integrals I_1 and I_2 can be divided into the five parts;

$$\begin{aligned}(3.9)\quad & \left[\int_{-i\infty}^{i\infty}-\left(\int_{iT}^{i\infty}+\int_{-i\infty}^{-iT}+\int_{-iT}^{a-\varepsilon-iT}+\int_{a-\varepsilon+iT}^{iT}\right)\right]\Gamma(1-s)\Gamma(1+a-s) \\ &\cdot[1-\cos(\pi s)](4\pi^2k^{-2}nX)^s s^{-1}ds \\ &= I_{1,4}-(I_{1,5}+I_{1,6}+I_{1,7}+I_{1,8}),\end{aligned}$$

and

$$\begin{aligned}(3.10)\quad & \left[\int_{-i\infty}^{i\infty}-\left(\int_{iT}^{i\infty}+\int_{-i\infty}^{-iT}+\int_{-iT}^{a-\varepsilon-iT}+\int_{a-\varepsilon+iT}^{iT}\right)\right]\cdot\Gamma(1-s)\cdot\Gamma(1+a-s) \\ &\cdot\sin(\pi s)\cdot(4\pi^2k^{-2}nX)^s s^{-1}ds \\ &= I_{2,4}-(I_{2,5}+I_{2,6}+I_{2,7}+I_{2,8}),\quad \text{say.}\end{aligned}$$

Firstly, we calculate the integrals $I_{1,4}$ and $I_{2,4}$.

Applying Mellin's inversion formula (see e.g. [6]), we obtain that if $-1 < a < \frac{1}{2}$, then

$$\begin{aligned} I_{1,4} &= -2^{a+2}\cdot(4\pi^2k^{-2}nX)^{1+a}\cdot\int_{2+2a-i\infty}^{2+2a+i\infty} 2^{s-a-3}\cdot\Gamma\left(\frac{1}{2}s\right)\cdot\Gamma\left(\frac{1}{2}s-a-1\right) \\ &\quad \cdot(4\pi k^{-1}n^{\frac{1}{2}}X^{\frac{1}{2}})^{-s}\cdot ds + 2^{a+2}\pi(4\pi^2k^{-2}nX)^{1+a}\cdot\int_{2+2a-i\infty}^{2+2a+i\infty} 2^{s-a-2} \\ &\quad \cdot\Gamma\left(\frac{1}{2}s\right)\cdot\Gamma\left(\frac{1}{2}s-a-1\right)\cdot\pi^{-1}\cdot\cos\left(\frac{1}{2}\pi s-\pi a-\pi\right) \\ &\quad \cdot(4\pi k^{-1}n^{\frac{1}{2}}X^{\frac{1}{2}})^{-s}\cdot ds \\ &= -2i(2\pi)^{a+2}\cdot k^{-a-1}\cdot(nX)^{\frac{1}{2}a+\frac{1}{2}} \\ &\quad \cdot\left[K_{a+1}(4\pi k^{-1}n^{\frac{1}{2}}X^{\frac{1}{2}})+\frac{1}{2}\pi Y_{a+1}(4\pi k^{-1}n^{\frac{1}{2}}X^{\frac{1}{2}})\right], \end{aligned}$$

and

$$\begin{aligned} I_{2,4} &= 2^{a+1}\pi(4\pi^2k^{-2}nX)^{1+a}\int_{2+2a-i\infty}^{2+2a+i\infty} 2^{s-a-2}\Gamma\left(\frac{1}{2}s\right)\left[\Gamma\left(2+a-\frac{1}{2}s\right)\right]^{-1} \\ &\quad \cdot(4\pi k^{-1}n^{\frac{1}{2}}X^{\frac{1}{2}})^{-s}ds \\ &= \pi i(2\pi)^{2+a}\cdot k^{-a-1}\cdot(nX)^{\frac{1}{2}+\frac{1}{2}a}\cdot J_{a+1}(4\pi k^{-1}n^{\frac{1}{2}}X^{\frac{1}{2}}), \end{aligned}$$

where K_{a+1}, Y_{a+1}, and J_{a+1} are Bessel functions (see e.g. [8]). Next, we estimate $I_{1,5}$, $I_{1,6}$, $I_{2,5}$, and $I_{2,6}$. We must be divide into the following two cases with a view to satisfying the monotone condition in Lemma 2. If $n \leqq \left(N+\frac{1}{2}\right)\cdot\exp(-2/a)$, we have, by Lemma 2 and the Stirling formula,

$$I_{1,5} \ll T^a\cdot\left[\log\left(n\cdot\left(N+\frac{1}{2}\right)^{-1}\right)\right]^{-1}.$$

By similar estimations, The integrals $I_{1,6}$, $I_{2,5}$, and $I_{2,6}$ are

$$\ll T^{a}\cdot\left[\log\left(n\cdot\left(N+\frac{1}{2}\right)^{-1}\right)\right]^{-1}.$$

If $n>\left[N+\frac{1}{2}\right]\cdot\exp(-2/a)$, we have, by Lemma 2 and the Stirling formula,

$$\begin{aligned} \mathrm{I}_{1,5} &= c\left[\int_{T}^{T_0}+\int_{T_0}^{\infty}\right]\cdot G(t)\cdot\exp(iF(t))\cdot[1+O(t^{-1})]\cdot dt \\ &\ll T_0^{a}\cdot[\log(4\pi^2k^{-2}T_0^{-2}nX)]^{-1} \\ &\ll T^{a}\cdot\left[\log\left(n\cdot\left(N+\frac{1}{2}\right)^{-1}\right)\right]^{-1}, \end{aligned}$$

where c is a constant,

$$\begin{aligned} F(t) &= 2t(-\log kt+\log 2\pi+1)+t\cdot\log nX, \\ G(t) &= t^{a}, \end{aligned}$$

and

$$T_0 = 2\pi k^{-1}n^{\frac{1}{2}}X^{\frac{1}{2}}\cdot\exp(1/a).$$

By similar estimations, the integrals $\mathrm{I}_{1,6}$, $\mathrm{I}_{2,5}$, and $\mathrm{I}_{2,6}$ are

$$\ll T^{a}\cdot\left[\log\left(n\cdot\left(N+\frac{1}{2}\right)^{-1}\right)\right]^{-1}.$$

Lastly, by easy estimations, we estimate that the integrals $\mathrm{I}_{1,7}$, $\mathrm{I}_{1,8}$, $\mathrm{I}_{2,7}$, and $\mathrm{I}_{2,8}$ are

$$\ll k^{-a}\cdot N^{-\frac{1}{2}a+\varepsilon}\cdot X^{\frac{1}{2}a}\cdot n^{a-\varepsilon}.$$

Hence, by the results (3.9) and (3.10), we obtain

$$\begin{aligned} &\Delta_a(X;\,h/k) \\ &= -2\pi^{-1}\cos\left(\frac{1}{2}\pi a\right)\cdot X^{\frac{1}{4}+\frac{1}{2}a}\cdot\sum_{n\leq N}\sigma_a(n)\cdot e(-\bar{h}n/k)\cdot n^{-\frac{1}{4}-\frac{1}{2}a} \\ &\quad\cdot\left[K_{a+1}(4\pi k^{-1}n^{\frac{1}{2}}X^{\frac{1}{2}})+\frac{1}{2}\pi Y_{a+1}(4\pi k^{-1}n^{\frac{1}{2}}X^{\frac{1}{2}})\right]-\sin\left(\frac{1}{2}\pi a\right) \\ &\quad\cdot X^{\frac{1}{4}+\frac{1}{2}a}\cdot\sum_{n\leq N}\sigma_a(n)\cdot e(-\bar{h}n/k)\cdot n^{-\frac{1}{2}-\frac{1}{2}a}\cdot J_{a+1}(4\pi k^{-1}n^{\frac{1}{2}}X^{\frac{1}{2}}) \\ &\quad+O(k\cdot N^{-\frac{1}{2}}\cdot X^{\frac{1}{2}+\varepsilon}). \end{aligned}$$

By the usual asymptotic formulas for Bessel functions [8], for $1\leqq N\ll X$, we have

$$\begin{aligned}&\Delta_a(X;h/k)\\&=(\pi\sqrt{2})^{-1}\cdot k^{\frac{1}{2}}\cdot X^{\frac{1}{4}+\frac{1}{2}a}\cdot\sum_{n\leq N}\sigma_a(n)\cdot e(-\bar{h}n/k)\cdot n^{-\frac{3}{4}-\frac{1}{2}a}\cdot\\&\quad\cdot\cos\left(4\pi k^{-1}n^{\frac{1}{2}}X^{\frac{1}{2}}-\frac{1}{4}\pi\right)+O(k\cdot N^{-\frac{1}{2}}\cdot X^{\frac{1}{2}+\varepsilon}).\end{aligned}$$

We have proved Theorem 1.

4. Proof of Theorem 2. It will be sufficient to prove the corresponding formula for the integral over $\left[\frac{1}{2}X,\ X\right]$ and then to replace $\frac{1}{2}X$ by $\frac{1}{4}X$, $X/8$, and so on, and to add up all the results. We start from the result of Theorem 1 with $N=X$. By integrating term by term and using the first mean-value theorem for integrals, we obtain

$$\begin{aligned}(3.1)\quad&\int_{X/2}^{X}\left|\Delta_a(t;\,h/k)\right|^2\cdot dt\\&=(2\pi^2)^{-1}\cdot k\cdot\sum_{m,n\leq X}\sigma_a(m)\cdot\sigma_a(n)\cdot(mn)^{-\frac{3}{4}-\frac{1}{2}a}\cdot\int_{X/2}^{X}t^{\frac{1}{2}+a}\cdot e(-\bar{h}m/k)\\&\quad\cdot e(\bar{h}n/k)\cdot\cos\left(4\pi k^{-1}m^{\frac{1}{2}}t^{\frac{1}{2}}-\frac{1}{4}\pi\right)\cdot\cos\left(4\pi k^{-1}n^{\frac{1}{2}}t^{\frac{1}{2}}-\frac{1}{4}\pi\right)\cdot dt\\&\quad+O\left(k^{\frac{3}{2}}\cdot X^{\frac{1}{4}+\frac{1}{2}a+\varepsilon}\cdot\int_{X/2}^{X}\left|\sum_{n\leq X}\sigma_a(n)\cdot e(-\bar{h}n/k)\cdot n^{-\frac{3}{4}-\frac{1}{2}a}\right.\right.\\&\quad\left.\left.\cdot\cos\left(4\pi k^{-1}n^{\frac{1}{2}}t^{\frac{1}{2}}-\frac{1}{4}\pi\right)\right|dt\right)+O(k^2\cdot X^{1+\varepsilon}).\end{aligned}$$

In the first sum in the right-hand side of (3.1) we distinguish the cases $m=n$ and $m\neq n$ contribute

$$\begin{aligned}&(2\pi^2)^{-1}\cdot k\cdot\sum_{n\leq X}\sigma_a^2(n)\cdot n^{-\frac{3}{2}-a}\cdot\int_{X/2}^{X}t^{\frac{1}{2}+a}\cdot\cos^2\left(4\pi k^{-1}n^{\frac{1}{2}}t^{\frac{1}{2}}-\frac{1}{4}\pi\right)\cdot dt\\&=[(6+4a)\pi^2]^{-1}k\cdot\left[X^{\frac{3}{2}+a}-\left(\frac{1}{2}X\right)^{\frac{3}{2}+a}\right]\cdot\sum_{n=1}^{\infty}\sigma_a^2(n)\cdot n^{-\frac{3}{2}-a}\\&\quad+O(k\cdot X^{1+\varepsilon})+O(k^2\cdot X^{1+a+\varepsilon}).\end{aligned}$$

It is seen that the terms in (3.1) for which $m \neq n$ are a multiple of

$$k \sum_{m \neq n \leq X} \sigma_a(m)\sigma_a(n)e(-\bar{h}m/k)e(\bar{h}n/k)(mn)^{-\frac{3}{4}-\frac{1}{2}a}$$

$$\int_{X/2}^{X} t^{\frac{1}{2}+a}\cos(4\pi k^{-1}m^{\frac{1}{2}}t^{\frac{1}{2}}-4\pi k^{-1}n^{\frac{1}{2}}t^{\frac{1}{2}})dt$$

$$+k \sum_{m \neq n \leq X} \sigma_a(m)\sigma_a(n)e(-\bar{h}m/k)e(\bar{h}n/k)(mn)^{-\frac{3}{4}-\frac{1}{2}a}$$

$$\int_{X/2}^{X} t^{\frac{1}{2}+a}\sin(4\pi k^{-1}m^{\frac{1}{2}}t^{\frac{1}{2}}+4\pi k^{-1}n^{\frac{1}{2}}t^{\frac{1}{2}})dt$$

$$= S_1+S_2, \text{ say.}$$

Estimating the integral in S_2 by Lemma 2 we have

$$S_2 \ll k^2\cdot X^{1+a}\cdot \sum_{m<n\leq X} \sigma_a(m)\cdot\sigma_a(n)\cdot(mn)^{-\frac{3}{4}-\frac{1}{2}a}\cdot(m^{\frac{1}{2}}+n^{\frac{1}{2}})^{-1}$$
$$\ll k^2\cdot X^{1+a}.$$

Analogously, we obtain

$$S_1 \ll k^2\cdot X^{1+a}\cdot\left(\sum_{\substack{n\leq m/2\\ m\leq X}}+\sum_{\substack{m/2<n\\ m\leq X}}\right)\cdot\sigma_a(m)\cdot\sigma_a(n)\cdot(mn)^{-\frac{3}{4}-\frac{1}{2}a}\cdot(m^{\frac{1}{2}}-n^{\frac{1}{2}})^{-1}$$
$$= k^2\cdot X^{1+a}\cdot[S_{1,1}+S_{1,2}], \text{ say.}$$

By partial summation formula, we have

$$S_{1,1} \ll \sum_{m\leq X}\sigma_a(m)\cdot m^{-\frac{5}{4}-\frac{1}{2}a}\cdot\sum_{n\leq m/2}\sigma_a(n)\cdot n^{-\frac{3}{4}-\frac{1}{2}a} \ll X^{-a+\varepsilon},$$
$$S_{1,2} \ll \sum_{m\leq X}\sigma_a(m)\cdot m^{-1-a}\cdot\sum_{m/2<n<m}\sigma_a(n)\cdot(m-n)^{-1} \ll X^{-a+\varepsilon}.$$

Therefore the first sum in (3.1) is equal to

$$k\cdot[(6+4a)\pi^2]^{-1}\cdot\left[X^{\frac{3}{2}+a}-\left(\frac{1}{2}X\right)^{\frac{3}{2}+a}\right]\cdot\sum_{n=1}^{\infty}\sigma_a^2(n)\cdot n^{-\frac{3}{2}-a}$$
$$+O(k^2\cdot X^{1+a+\varepsilon})+O(k\cdot X^{1+\varepsilon}).$$

The first O-term in (3.1) is estimated by the Cauchy-Schwarz inequality as

$$\ll k^{\frac{3}{2}}X^{\frac{5}{4}+\frac{1}{2}a+\varepsilon}$$

when we square out the modulus under the integral sign and treat the terms $m=n$ and $m\neq n$ similarly as before. We have proved Theorem 2.

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