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On SD graphs. I

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ON SD GRAPHS. I

To Hisao Tominaga on his 60th birthday

NOBORU ITO and MACHIO TADOKORO

1. Introduction. Let v, k and l be positive integers such that $v > 2k$ and $l(v-1) = k(k-1)$, and let $n = k-l$. Let V be the set of all $(-1, 1)$ vectors of size v. Let G be the graph whose vertex set is V such that two vertices α and β are adjacent in G if and only if the inner product of α and β equals $v-4n$. So α and β are adjacent in G if and only if α and β differ in exactly $2n$ coordinates. G will be called an SD graph.

The weight of a vertex α is the number of coordinates of α which are equal to -1 and it is denoted by $wt(\alpha)$. W_l denotes the set of all vertices of weight l. For two vertices α and β, $d(\alpha, \beta)$ denotes the distance in G between them. $D_l(\alpha)$ denotes the set of vertices β such that $d(\alpha, \beta) = l$. Let η denote the all one vector. Then obviously $D_1(\eta) = W_{2n}$.

The automorphism group $\mathfrak{G}$ of G contains the symmetric group $\mathfrak{S}$ on v coordinate positions of vectors and an elementary Abelian group $\mathfrak{E}$ of order 2^v consisting of sign changes of coordinates of vectors. $\mathfrak{E}$ acts regularly on V, and $\mathfrak{S}$ fixes η and acts transitively on $D_1(\eta)$. So G is a symmetric graph of valency $\binom{v}{2n}$.

Since $v > 2k$ we have $k > 2l$ and $n > l$. If we consider $v = 2n+l+(n^2-n)/l$ as a function of l for a fixed n, then v is steadily decreasing. Hence we have that $4n-1 \leq v \leq n^2+n+1$. We notice that two bounds correspond to the parameters of Hadamard designs and projective planes respectively. The case $v-4n < 1$ is studied to some extent in [1, 2]. So in the present paper it is assumed that $v > 4n$. Main purposes of the present paper are to determine diameters and automorphism groups of connected components of G.

Notation. Let $x_1, x_2, \ldots, x_m$ be vectors of size $l_1, l_2, \ldots, l_m$ respectively. Then $x = (x_1, x_2, \ldots, x_m)$ denotes a vector of size $l_1+l_2+\cdots+l_m$ such that the subvector of x consisting of the $(l_1+\cdots+l_{r-1}+1)$-st,...,$(l_1+\cdots+l_{r-1}+l_r)$-th coordinates of x equals x_r, $1 \leq r \leq m$. If $x_j = x_{j+1} = \ldots = x_{j+s}$, then the abbreviation $(x_j)_s$ will be used. Thus, for instance, $((1)_v) = \eta$.

2. Diameters of connected components of G. Let d be the diameter

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of the connected component of G containing η. Now $D_r(\eta)$ is invariant under $\mathfrak{S}$, $1 \leq r \leq d$, and $\mathfrak{S}$ is transitive on W_l, $0 \leq l \leq v$. So if $D_r(\eta) \cap W_l \neq \phi$, then W_l is contained in $D_r(\eta)$.

Lemma 1. $D_2(\eta) = \bigcup_{1 \leq i \leq 2n,\ i \neq n} W_{2i}$.

Proof. Let $\alpha_{2n} = ((-1)_{2n}, (1)_{v-2n})$. Then α_{2n} belongs to $D_1(\eta)$. Now let β be a vertex such that $d(\beta, \alpha_{2n}) = 1$ and that $wt(\beta) \neq 0, 2n$. In order to estimate $wt(\beta)$ we may assume that $\beta = ((-1)_e, (1)_f, (-1)_g, (1)_h)$, where $e+f = 2n$ and $g+h = v-2n$. Then we have that $f+g = 2n$ and $wt(\beta) = e+g$. So $e = g$ and $wt(\beta) = 4n-2f$, where $0 \leq f \leq 2n$ and $f \neq 0, n$.

Lemma 2. *If $D_r(\eta) \neq \phi$, then $D_r(\eta) = \bigcup_{1 \leq i \leq 2n} W_{4(r-2)n+2i}$ for $r \geq 3$.*

Proof. First assume that $r = 3$. Let β be a vector of $D_3(\eta)$ and α a vertex of $D_2(\eta)$ such that $d(\beta, \alpha) = 1$. In order to estimate $wt(\beta)$ we may assume that $\beta = ((-1)_e, (1)_{v-e})$ and $\alpha = ((-1)_g, (1)_h, (-1)_i, (1)_j)$, where $e > 4n$, $g+h = e$, $i+j = v-e$ and $g+i = 2m$ with $1 \leq m \leq 2n$ and $m \neq n$. Since $h+i = 2n$, we have that $wt(\beta) = e = 2n-i+2m-i$. Let $s = m-n$. Then $wt(\beta) = 4n+2(s-i)$, where $i < s \leq n$. The case $r > 3$ is simpler.

From Lemmas 1 and 2 we have the following proposition.

Proposition 3. *G consists of two connected components E and O. E and O consist of all vectors of G of even and odd weights respectively. E and O are isomorphic. The diameter d of E satisfies the following inequalities:*

$$((v-2)/(4n))+2 \geq d \geq ((v-1)/(4n))+1.$$

Proof. Since $D_d(\eta) \neq \phi$, we have that $v \geq 4(d-2)n+2$. Since $D_{d+1}(\eta) = \phi$, we have that $v < 4(d-1)n+2$.

Corollary 4. $d = 2$ *if and only if* $v = 4n+1$ *(under the assumption that $v > 4n$).*

Corollary 5. *If $v = n^2+n+1$, then $d = (1/4)n+2$, $(1/4)(n+7)$, $(1/4)(n+6)$, or $(1/4)(n+5)$, according as $n \equiv 0, 1, 2$ or 3 (mod 4).*

Corollary 6. *The girth of G equals three.*

3. The automorphism group $\mathfrak{G}$ of G.

Lemma 7. *Let α and β be vertices of $D_2(\eta)$ such that $D_1(\alpha) \cap D_1(\eta) = D_1(\beta) \cap D_1(\eta)$. Then $\alpha = \beta$.*

Proof. By Lemma 1 and under the action of $\mathfrak{S}$ we may assume that $\alpha = ((-1)_{2i}, (1)_{v-2i})$ and $\beta = ((-1)_e, (1)_f, (-1)_g, (1)_h)$, where $1 \leq i \leq 2n$, $i \neq n$, $e+f = 2i$, $g+h = v-2i$, and $e+g = 2m$ with $1 \leq m \leq 2n$, $m \neq n$. First assume that $i < 2n$. Let $\gamma = ((-1)_i, (1)_i, (-1)_{2n-i}, (1)_{v-2n-i})$, $\gamma_1 = (1, (-1)_{i-1}, (1)_{i-1}, (-1)_{2n-i+1}, (1)_{v-2n-i})$ and $\gamma_2 = ((-1)_i, (1)_{i+1}, (-1)_{2n-i-1}, (1)_{v-2n-i-1}, -1)$. Then γ, γ_1 and γ_2 belong to $D_1(\eta) \cap D_1(\alpha)$. If $ef \neq 0$, then the inner products $(\beta, \gamma) \neq (\beta, \gamma_1)$. So either γ or γ_1 does not belong to $D_1(\beta)$. Thus we have that $ef = 0$. If $gh \neq 0$, then $(\beta, \gamma) \neq (\beta, \gamma_2)$. So either γ or γ_2 does not belong to $D_1(\beta)$. Thus we have that $gh = 0$. If $e = h = 0$, then $\beta = ((1)_{2i}, (-1)_{v-2i}) = -\alpha$. Since $v > 4n$, $(\gamma, \alpha) \neq (\gamma, \beta)$. So γ does not belong to $D_1(\beta)$. Thus we have that $f = 0$. If $h = 0$, then $\beta = -\eta$. Since $v > 4n$, this is a contradiction. So we get $g = 0$ and $\alpha = \beta$.

Now assume that $i = 2n$. We notice that γ_2 does not exist under this assumption. As above we get $ef = 0$. If $e = 0$, then $(\beta, \gamma) = h-g = v-4n-2g$. If β and γ are adjacent, then $(\beta, \gamma) = v-4n$. So we get $g = 0$ and $\beta = \eta$, which is absurd. So we have that $f = 0$. As above we get $g = 0$ and $\alpha = \beta$.

Lemma 8. *Let σ be an automorphism of G such that σ fixes $D_1(\eta)$ and that the restriction of σ to $D_1(\eta)$ is trivial. Then σ is the identity automorphism.*

Proof. Let α be a vertex of G such that $D_1(\alpha) = D_1(\eta)$. Then under the action of $\mathfrak{S}$ we may assume that $\alpha = ((-1)_i, (1)_{v-i})$. If $2n \geq i$, then let $\alpha_1 = ((-1)_{2n}, (1)_{v-2i})$. α_1 belongs to $D_1(\eta)$ and $(\alpha, \alpha_1) = i-(2n-i)+v-2n = v-4n+i$. Thus $i = 0$ and $\alpha = \eta$. If $2n < i$, then $(\alpha, \alpha_1) = 2n-(i-2n)+v-i = v+4n-2i$. Thus $i = 4n$. Let $\alpha_2 = ((1)_{v-2n}, (-1)_{2n})$. Then α_2 belongs to $D_1(\eta)$. If $v-2n \geq 4n$, then $(\alpha, \alpha_2) \leq v-8n$ and α_2 does not belong to $D_1(\alpha)$. If $v-2n < 4n$, then $(\alpha, \alpha_2) = -(v-2n)+(4n-v+2n)-(v-4n) = -3v+12n$. Thus if α_2 belongs to $D_1(\alpha)$, then $v = 4n$ which is against the assumption. So σ fixes η.

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If α and β are two distinct vertices of $D_2(\eta)$, then by Lemma 7 there exists a vertex γ of $D_1(\eta)$ such that γ is adjacent with exactly one of α and β. If $\beta=\alpha\sigma$, then σ destroys the adjacency. Thus σ restricted to $D_2(\eta)$ is trivial. Now since G is vertex-transitive, we may apply an induction argument to complete the proof.

Let α_{2i} be a vertex of $D_2(\eta)$ of weight $2i$, $1\le i\le 2n$, $i\neq n$. Then it is easy to see that $D_1(\alpha_{2i})\cap D_1(\eta)$ consists of $\binom{2i}{i}\binom{v-2i}{2n-i}$ vertices. Put $A(i)=\binom{2i}{i}\binom{v-2i}{2n-i}$, $1\le i\le 2n$.

Lemma 9. *If $v=4n+1$, then $A(i)=A(2n-i+1)$ for $1\le i\le n$ and $A(1)>A(i)$ for $2\le i\le 2n-1$. If $v=4n+2$, then $A(1)>A(i)$ for $2\le i\le 2n$.*

Proof. Let $v=4n+1$. Then $A(i)-A(2n-i+1)=\binom{2i}{i}\binom{4n-2i+1}{2n-i}-\binom{4n-2i+2}{2n-i+1}\binom{2i-1}{i-1}=\binom{2i-1}{i}\binom{4n-2i+1}{2n-i}-\binom{4n-2i+1}{2n-i}\binom{2i-1}{i-1}=0$. We have that $A(i+1)/A(i)=(2i+1)(2n-i+1)/(i+1)(4n-2i+1)$. Let $B(i)=(i+1)(4n-2i+1)-(2i+1)(2n-i+1)$. Then $B(i)=2n-2i$. So $A(1)>A(i)$ for $2\le i\le n$.

Now assume that $v=4n+2$. Then $A(i+1)/A(i)=(2i+1)(2n-1)\cdot(2n-i+2)/(i+1)(2n-i+1)(4n-2i+1)$. Let $B(i)=(i+1)(2n-i+1)\cdot(4n-2i+1)-(2i+1)(2n-i)(2n-i+2)=2i^2-6ni+4n^2+2n+1$. We have that $A(i)>A(i+1)$ if and only if $B(i)>0$. $B(i)$ is quadratic with respect to i and takes the minimum at $i=3n/2$. Since $B(2n-2)<0$ and $B(2n-1)>0$, we have only to compare $A(1)$ with $A(2n-1)$. Now $A(1)/A(2n-1)=(4n-1)/(3n)>1$. This completes the proof.

Lemma 10. *If $v\ge 4n+2$, then $A(1)>A(i)$ for $2\le i\le 2n$.*

Proof. Let $C(v)=A(1)-A(i)=2\binom{v-2}{2n-1}-\binom{2i}{i}\binom{v-2i}{2n-i}$. By Lemma 9 $C(4n+2)>0$. So we use an induction argument on v. Assume that $C(v)>0$. Then we have that $C(v+1)=2\binom{v-1}{2n-1}-\binom{2i}{i}\binom{v-2i+1}{2n-i}$

$$=\frac{v-1}{v-2n}\cdot 2\binom{v-2}{2n-1}-\binom{2i}{i}\binom{v-2i+1}{2n-i}$$

$$> \frac{v-1}{v-2n}\binom{2i}{i}\binom{v-2i}{2n-i}-\binom{2i}{i}\binom{v-2i+1}{2n-i}$$
$$=\frac{(v-1)(v-2n-i+1)}{(v-2n)(v-2i+1)}\binom{2i}{i}\binom{v-2i+1}{2n-i}-\binom{2i}{i}\binom{v-2i+1}{2n-i}.$$

Since $(v-1)(v-2n-i+1)-(v-2n)(v-2i+1)=(v-4n+1)(i-1)>0$, we have the assertion.

Remark. By Lemma 10, we see that if an automorhpism σ of $\mathfrak{G}$ leaves η invariant, then σ leaves W_2 invariant.

Lemma 11. *Let α and β be two distinct vertices of $D_1(\eta)$. Then we have that $D_1(\alpha)\cap W_2 \neq D_1(\beta)\cap W_2$.*

Proof. Under the action of $\mathfrak{S}$ we may assume that $\alpha=((-1)_{2n}, (1)_{v-2n})$ and $\beta=((-1)_e, (1)_{2n-e}, (-1)_{2n-e}, (1)_{v-4n+e})$, where $2n>e$. Let $\gamma=((1)_{2n-1}, -1, (1)_{v-2n-1}, -1)$. We have that $(\alpha, \gamma)=v-4n$ and hence γ belongs to $D_1(\alpha)$. We have that $(\beta, \gamma)=v-4n+4$ and hence γ does not belong to $D_1(\beta)$.

Assume that $v\geq 4n+2$.

Lemma 12. *Let σ be an automorphism of G such that $\eta\sigma=\eta$. If σ restricted to W_2 is trivial, then σ is trivial.*

Proof. Now deny the assertion. Then by Lemma 8 there exist two distinct vertices α and β of $D_1(\eta)$ such that $\beta=\alpha\sigma$. By Lemma 11 σ destroys the adjacency.

Lemma 13. *Let $\omega(i, j)$ be a vertex in W_2 such that the i-th and j-th coordinates equal -1, where $1\leq i, j\leq v$ and $i\neq j$. Then*

(i) *$D_1(\omega(i, j_1))\cap D_1(\omega(i, j_2))\cap D_1(\eta)$ consists of $\binom{v-2}{2n-1}$ vertices, where $j_1\neq j_2$, and $D_1(\omega(i_1, j_1))\cap D_1(\omega(i_2, j_2))\cap D_1(\eta)$ consists of $4\binom{v-4}{2n-2}$ vertices, where i_1, j_1, i_2 and j_2 are distinct.*

(ii) *$D_1(\omega(i, j_1))\cap D_1(\omega(i, j_2))\cap D_2(\eta)$ consists of $\binom{v-2}{2n-1}$ vertices, where $j_1\neq j_2$, and $D_1(\omega(i_1, j_1))\cap D_1(\omega(i_2, j_2))\cap D_2(\eta)$ consists of*

$2\binom{v-4}{2n-2}$ *vertices, where* i_1, j_1, i_2 *and* j_2 *are distinct.*

Proof. (i) Let α be a vertex of $D_1(\omega(i, j_1)) \cap D_1(\omega(i, j_2)) \cap D_1(\eta)$. Then the i-th, j_1-th and j_2-th coordinates of α equal either -1, 1 and 1, or 1, -1 and -1 respectively. Let β be a vertex of $D_1(\omega(i_1, j_1)) \cap D_1(\omega(i_2, j_2)) \cap D_1(\eta)$. Then the i_1-th, j_1-th, i_2-th and j_2-th coordinates of β equal either $-1, 1, -1$ and 1, or -1, 1, 1 and -1, or 1, -1, -1 and 1, or 1, $-1, 1$ and -1 respectively. Since $wt(\alpha) = wt(\beta) = 2n$, we obtain the assertion.

(ii) Let α be a vertex of $D_1(\omega(i, j_1)) \cap D_1(\omega(i, j_2)) \cap D_2(\eta)$. Then, since $wt(\alpha) \neq 2n$, the i-th, j_1-th and j_2-th coordinates of α should be equal. Let β be a vertex of $D_1(\omega(i_1, j_1)) \cap D_1(\omega(i_2, j_2)) \cap D_2(\eta)$. Then, since $wt(\beta) \neq 2n$, the i_1-th, i_2-th, j_1-th and j_2-th coordinates of β should be equal. So we obtain the assertion.

Lemma 14. *Let* σ *be an automorphism of* G *such that* $\eta\sigma = \eta$ *and* $\omega(1, 2)\sigma = \omega(1, 2)$. *Then* $\{\omega(1, 3), \ldots, \omega(1, v), \omega(2, 3), \ldots, \omega(2, v)\}$ *is invariant under* σ.

Proof. This follows from Lemma 13.

Lemma 15. *Let* σ *be an automorphism of* G *such that* $\eta\sigma = \eta$ *and* $\omega(1, 2)\sigma = \omega(1, 2)$. *If* σ *restricted to* $\{\omega(1, 3), \ldots, \omega(1, v), \omega(2, 3), \ldots, \omega(2, v)\}$ *is trivial, then* σ *is trivial.*

Proof. Deny. Then by Lemma 12 we may assume that $\omega(3, 4)\sigma = \omega(3, 5)$ or $\omega(5, 6)$. So $D_1(\omega(3, 4)) \cap D_1(\omega(1, 4)) \cap D_l(\eta)$ moves to $D_1(\omega(3, 5)) \cap D_1(\omega(1, 4)) \cap D_l(\eta)$ or $D_1(\omega(5, 6)) \cap D_1(\omega(1, 4)) \cap D_l(\eta)$, where $l = 1, 2$. By Lemma 13 this is a contradiction.

Lemma 16. *Let* σ *be an automorphism of* G *such that* $\eta\sigma = \eta$, $\omega(1, 2)\sigma = \omega(1, 2)$ *and* $\omega(1, 3)\sigma = \omega(1, 3)$. *Then* $\{\omega(1, 4), \ldots, \omega(1, v)\}$ *is invariant under* σ.

Proof. Otherwise, by Lemma 14 we may assume that $\omega(1, 4)\sigma = \omega(2, 4)$ or $\omega(2, 5)$. So we may follow the proof of Lemma 15.

Lemma 17. *Let* σ *be an automorphism of* G *such that* $\eta\sigma = \eta$, $\omega(1, 2)\sigma = \omega(1, 2)$ *and* $\omega(1, 3)\sigma = \omega(1, 3)$. *If* σ *rectriced to* $\{\omega(1, 4), \ldots, \omega(1, v)\}$ *is trivial, then* σ *is trivial.*

Proof. Deny. Then by Lemma 15 we may assume that $\omega(2, i)\sigma =$

$\omega(2, j)$, where $i \neq j$. So we may follow the proof of Lemma 15 to get a contradiction.

Proposition 18. $\mathfrak{G} = \mathfrak{C}\mathfrak{S}$.

Proof. First we notice that the normalizer of $\mathfrak{C}$ contains $\mathfrak{S}$. So the product $\mathfrak{C}\mathfrak{S}$ is a subgroup of $\mathfrak{G}$.

Now let σ be an automorphism of G outside $\mathfrak{C}\mathfrak{S}$. Since $\mathfrak{C}$ is transitive on V, we may assume that $\eta\sigma = \eta$. Since $\mathfrak{S}$ is transitive on W_2, by Lemma 10 we may assume that $\omega(1, 2)\sigma = \omega(1, 2)$. Let $\mathfrak{S}_{\{1,2\}}$ denote the stabilizer of $\{1, 2\}$ in $\mathfrak{S}$. Then $\mathfrak{S}_{\{1,2\}}$ is transitive on $\{\omega(1, 3), \ldots, \omega(1, v), \omega(2, 3), \ldots, \omega(2, v)\}$. So by Lemma 14 we may assume that $\omega(1, 3)\sigma = \omega(1, 3)$. By Lemma 16 σ leaves $\{\omega(1, 4), \ldots, \omega(1, v)\}$ invariant. The stabilizer $\mathfrak{S}_{1,2,3}$ of 1, 2 and 3 in $\mathfrak{S}$ acts as the symmetric group on $\{\omega(1, 4), \ldots, \omega(1, v)\}$. So we may assume that σ is trivial on $\{\omega(1, 4), \ldots, \omega(1, v)\}$. By Lemma 17 σ is trivial, which is a contradiction.

Now let $\mathfrak{D}$ be the subgroup of $\mathfrak{C}$ of order 2^{v-1} consisting of sign changes of even number of coordinates of vectors. Then the automorphism group $\mathfrak{E}$ of E obviously equals the product $\mathfrak{D}\mathfrak{S}$; $\mathfrak{E} = \mathfrak{D}\mathfrak{S}$.

4. The case v = 4n+1. In this section we assume that $v = 4n+1$. Let σ be an automorphism of G outside $\mathfrak{C}\mathfrak{S}$. Since $\mathfrak{S}$ is transitive on V, we may assume that $\eta\sigma = \eta$. By Lemma 9 $W_2 \cup W_{4n}$ is invariant under σ. W_2 and W_{4n} contains $\binom{v}{2}$ and v vertices respectively. So under the action of $\mathfrak{S}$ we may assume that $\omega(1, 2)\sigma$ belongs to W_2. Since $\mathfrak{S}$ is transitive on W_2, we may assume that $\omega(1, 2)\sigma = \omega(1, 2)$.

Lemma 19. *Let $\omega(i)$ be a vertex of W_{4n} such that the i-th coordinate equals* 1, $1 \leq i \leq 4n+1$. *Then*

(i) $D_1(\omega(i, j)) \cap D_1(\omega(i)) \cap D_1(\eta)$ *consists of* $\binom{4n-1}{2n}$ *vertices.*

(ii) $D_1(\omega(i, j)) \cap D_1(\omega(l)) \cap D_1(\eta)$ *consists of* $2\binom{4n-2}{2n-1}$ *vertices, where* $l \neq i, j$.

Proof. We may assume that $i = 1$, $j = 2$ and $l = 3$. Let α be a vertex of $D_1(\eta)$ adjacent with $\omega(1, 2)$. Then we see that the first two coordinates of α are distinct. If α is adjacent with $\omega(1)$, too, then the first coordinate of α must be equal to 1. So we get (i). If α is adjacent with $\omega(3)$, then the third coordinate of α must be equal to 1. So we get (ii).

Here we notice that Lemma 13 holds for the case $v = 4n+1$, and that $\binom{4n-1}{2n-1} = \binom{4n-1}{2n}$ and $2\binom{4n-2}{2n-1} = 4\binom{4n-3}{2n-2}$.

Lemma 20. *Let τ be an automorphism of G such that $\eta\tau = \eta$ and $\omega(1, 2)\tau = \omega(1, 2)$. Then $\{\omega(1), \omega(2), \omega(1, 3), \dots, \omega(1, v), \omega(2, 3), \dots, \omega(2, v)\}$ is invariant under τ.*

Proof. This is immediate from Lemmas 13 and 19.

Now we go back to our σ. Since $\mathfrak{S}_{\{1,2\}}$ is transitive on $\{\omega(1, 3), \dots, \omega(1, v), \omega(2, 3), \dots, \omega(2, v)\}$, under the action of $\mathfrak{S}$ we may assume that $\omega(1, 3)\sigma = \omega(1, 3)$. So by Lemma 20 $\omega(2, 3)\sigma = \omega(2, 3)$ and $\{\omega(1), \omega(1, 4), \dots, \omega(1, v)\}$ is invariant under σ. Since $\mathfrak{S}_{1,2,3}$ acts as the symmetric group on $\{\omega(1, 4), \dots, \omega(1, v)\}$, under the action of $\mathfrak{S}$ we may assume that $\omega(1, i)\sigma = \omega(1, i)$, $4 \le i \le v-1$ and $\{\omega(1), \omega(1, v)\}$ is invariant under σ. Then by Lemma 20 we have that $\omega(i, j)\sigma = \omega(i, j)$, $4 \le i, j \le v-1$ and $\{\omega(i), \omega(i, v)\}$, $1 \le i \le v-1$, invariant under σ. Hence we also have that $\omega(v)\sigma = \omega(v)$.

Finally, let $\alpha = ((1)_{2n-1}, (-1)_{2n+2})$. Then since α and $\omega(2n, i)$, $2n+1 \le i \le 4n$, are adjacent, the $2n$-th, ..., the $4n$-th coordinates of $\alpha\sigma$ must be equal to -1. Further since α and $\omega(v)$ are not adjacent, the $(4n+1)$-st coordinate of $\alpha\sigma$ must be equal to -1. Since by Lemma 9 W_{2n+2} is invariant under σ, we have that $\alpha\sigma = \alpha$. Now $\omega(2n)$ is not adjacent with α. Therefore $\omega(2n, 4n+1)$ and $\omega(2n)$ are fixed by σ. This implies that σ is trivial. This is a contradiction. So Proposition 18 holds for the case $v = 4n+1$.

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