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Abstract

It is shown that any subset of a topological abelian monoid gives rise to a generalized homology theory that is closely related to the notion of labeled configuration space. Applications of the main theorem include generalizations of the classical Dold-Thom and the Barratt- Priddy-Quillen-Segal theorems.

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CONFIGURATION SPACES WITH PARTIALLY SUMMABLE LABELS AND HOMOLOGY THEORIES

Dedicated to the memory of Professor Katsuo Kawakubo

Kazuhisa SHIMAKAWA

Abstract. It is shown that any subset of a topological abelian monoid gives rise to a generalized homology theory that is closely related to the notion of labeled configuration space. Applications of the main theorem include generalizations of the classical Dold-Thom and the Barratt-Priddy-Quillen-Segal theorems.

Introduction

Let M be a topological abelian partial monoid, e.g. any subset of a topological abelian monoid. Following [2], we associate with any pointed space X the space $\widetilde{C}^M(X)$ of configurations of distinct points in X with labels in M. If M is taken to be the abelian monoid of non-negative integers $\mathbf{N}$ then we get a topological abelian monoid $\mathrm{SP}(X,*)$ generated by the points of X, with its basepoint $*$ identified with 0. On the other hand, if $M=\{1\}\subset\mathbf{N}$ then $\widetilde{C}^{\{1\}}(X)$ is the space of finite subsets of X, modulo the relation $*=$ null. Thus the notion of labeled configuration spaces generalizes both infinite symmetric products [1] and classical configuration spaces [7].

In general, $\widetilde{C}^M(X)$ is not a functor of X, nor does it preserve homotopy equivalences. Nevertheless we can show that if we replace $\widetilde{C}^M(X)$ by the limit of the inclusions $\widetilde{C}^M(\mathbf{R}^n\ltimes X)\subset\widetilde{C}^M(\mathbf{R}^{n+1}\ltimes X)$, where $\mathbf{R}^n\ltimes X=\mathbf{R}^n\times X/\mathbf{R}^n\times *$, then the correspondence

$$X\mapsto\widetilde{C}^M(\mathbf{R}^\infty\ltimes X)=\textstyle\bigcup_n\widetilde{C}^M(\mathbf{R}^n\ltimes X)$$

defines a generalized homology theory.

More precisely, we have

Main Theorem. *Given a topological abelian partial monoid M there exists a functor E^M, of the category of pointed spaces into itself, together with a natural transformation $E^M(X)\to\Omega E^M(\Sigma X)$ enjoying the following properties*:

(P1) $E^M(*)=*$.

(P2) *For any pointed space X, the map $E^M(X)\to\Omega E^M(\Sigma X)$ is a homotopy equivalence.*

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(P3) *For any pair of pointed spaces* (X, A)*, the sequence*

$$E^M(A) \to E^M(X) \to E^M(X \cup CA)$$

induced by the cofibration sequence $A \to X \to X \cup CA$ *is a homotopy fibration sequence.*

(P4) *If* X *is a pointed finite* CW*-complex then* $E^M(X)$ *has the weak homotopy type of a group completion of* $\widetilde{C}^M(\mathbf{R}^\infty \ltimes X)$*.*

Notice that $\widetilde{C}^M(\mathbf{R}^\infty \ltimes X)$ is an admissible H-space with respect to the multiplication induced by a linear isometry $\mathbf{R}^\infty \times \mathbf{R}^\infty \to \mathbf{R}^\infty$.

In short, the theorem says that there is a generalized homology theory $\tilde{h}_\bullet = \pi_\bullet E^M$ such that for a pointed finite CW-complex X,

$$\tilde{h}_\bullet(X) \cong \pi_\bullet \widetilde{C}^M(\mathbf{R}^\infty \ltimes X)$$

holds if $\widetilde{C}^M(\mathbf{R}^\infty \ltimes X)$ is grouplike. (This is always the case if X is connected).

Among the applications of the main theorem we have the following:

(1) Let M be an arbitrary topological abelian monoid. Then $\widetilde{C}^M(X)$ is a continuous functor of X and there is a sequence of natural isomorphisms of homology theories

$$\pi_\bullet E^M(X) \cong \pi_\bullet \Omega \widetilde{C}^M(\Sigma X) \cong \widetilde{H}_\bullet(X; \pi_\bullet G)$$

defined on the category of pointed CW-complexes; here G is the group completion of M and $\widetilde{H}_\bullet(X; \pi_\bullet G)$ is the ordinary homology of X with coefficients in the graded group $\pi_\bullet G$. Accordingly there holds a *generalized Dold-Thom isomorphism*

$$\pi_n \widetilde{C}^M(X) \cong \widetilde{H}_n(X; \pi_\bullet G) = \sum\nolimits_r \widetilde{H}_{n-r}(X; \pi_r G), \ n \geq 0,$$

if M is grouplike or if X has the homotopy type of a connected CW-complex.

(2) Any pointed space M can be viewed as the subset of generators for the topological abelian monoid $\mathrm{SP}(M, *)$, and hence we can form $\widetilde{C}^M(X)$ and $E^M(X)$. There is a chain of natural weak equivalences

$$E^M(X) \to \Omega^\infty E^M(\Sigma^\infty X) \leftarrow \Omega^\infty \Sigma^\infty (X \wedge M)$$

inducing a natural isomorphism of homology theories

$$\pi_\bullet E^M(X) \cong \pi_\bullet^S(X \wedge M).$$

This can be regarded as a generalization of the Barratt-Priddy-Quillen theorem (the case $M = X = S^0$). It follows by (P4) that $\widetilde{C}^M(\mathbf{R}^n \ltimes X)$ approximates the homotopy type of $\Omega^\infty \Sigma^\infty (X \wedge M)$ as $n \to \infty$ if X is connected.

The paper is organized as follows. Section 1 introduces the notion of labeled configuration spaces. Section 2 provides a construction of $E^M(X)$ and proves the first three properties in the statement of the main theorem. Section 3 provides several applications of our theory, including certain generalizations of the classical Dold-Thom and the Barratt-Priddy-Quillen theorems. Section 4 gives the main part of the proof of the last property (P4). Finally, Section 5 completes the proof of (P4) by proving the key proposition, Proposition 4.7.

Throughout the paper we shall work in the category of compactly generated spaces. Basepoints are generically denoted by $*$ and are assumed to be non-degenerate.

1. Labeled configuration spaces

By a *topological abelian partial monoid* we shall mean a topological space M equipped with subsets $D_n \subset M^n$, $n \geq 2$, and operations $D_n \to M$, $(a_1, \ldots, a_n) \mapsto a_1 + \cdots + a_n$, enjoying the following property:

If $a_1, \ldots, a_n \in M$ and $I_1, \ldots, I_r$, $J_1, \ldots, J_s$ are subsets of $\{1, 2, \ldots, n\}$ such that $I_1 \coprod \cdots \coprod I_r = J_1 \coprod \cdots \coprod J_s = \{1, 2, \ldots, n\}$ holds then we have

$$\textstyle\sum_{i \in I_1} a_i + \cdots + \sum_{i \in I_r} a_i = \sum_{j \in J_1} a_j + \cdots + \sum_{j \in J_s} a_j$$

whenever both sides of the equation are defined.

We always assume that M is embeddable in a topological abelian partial monoid with unit element 0 and we write $\overline{M} = M \cup \{0\}$. Observe that any subset of a topological abelian monoid (with unit) is a abelian partial monoid. In particular, any topological space Y can be viewed as an abelian partial monoid with $D_n = \emptyset$, for Y is the set of generators for the topological abelian monoid $\mathrm{SP}(Y)$.

To each space (without basepoint) X we assign a topological category $\mathcal{Q}^M(X)$ defined as follows:

(1) Objects of $\mathcal{Q}^M(X)$ are the points of $\coprod_{p \geq 0} X^p \times \overline{M}^p$.

(2) A morphism from $(x_1, \ldots, x_p, a_1, \ldots, a_p)$ to $(y_1, \ldots, y_q, b_1, \ldots, b_q)$ is a map of finite sets $\theta\colon \{1, \ldots, p\} \to \{1, \ldots, q\}$ such that

(a) $x_i = y_{\theta(i)}$ for $1 \leq i \leq p$, and
(b) $b_j = \sum_{i \in \theta^{-1}(j)} a_i$ for $1 \leq j \leq q$ (meaning that $b_j = 0$ if $\theta^{-1}(j) = \emptyset$).

Let us denote this morphism by $(\theta^* y, a) \xrightarrow{\theta} (y, \theta_* a)$, where $\theta^*\colon X^q \to X^p$ and $\theta_*\colon A^p \to A^q$ are given by the formulas

$$\theta^* y = \left(y_{\theta(1)}, \ldots, y_{\theta(p)}\right), \quad \theta_* a = \left(\textstyle\sum_{i \in \theta^{-1}(1)} a_i, \ldots, \sum_{i \in \theta^{-1}(q)} a_i\right).$$

Then $\operatorname{Mor}\mathcal{Q}^M(X)$ can be identified with the subset of

$$\coprod_{p,q\geq 0} X^q \times \operatorname{Map}(\{1,\dots,p\},\{1,\dots,q\}) \times \overline{M}^p$$

consisting of those triples (y,θ,a) such that every component of $\theta_* a$ is in $\overline{M}$.

(3) The composition of two morphisms $(\theta^* y, a) \xrightarrow{\theta} (y, \theta_* a)$ and $(\psi^* z, b) \xrightarrow{\psi} (z, \psi_* b)$, where $y = \psi^* z$ and $b = \theta_* a$, is defined to be the morphism

$$(\theta^* y, a) = ((\psi\theta)^* z, a) \xrightarrow{\psi\theta} (z, (\psi\theta)_* a) = (z, \psi_* b).$$

In other words, we have $(z,\psi,b)(y,\theta,a) = (z,\psi\theta,a)$ whenever $y = \psi^* z$ and $b = \theta_* a$. Note that $(\psi\theta)_* = \psi_*\theta_*$ holds, for A is abelian.

Definition 1.1. Given a space X, let $B^M(X)$ denote the identification space of $\operatorname{Ob}\mathcal{Q}^M(X)$ with respect to the least equivalence relation such that two objects are equivalent if there is a morphism between them. We denote by $C^M(X)$ the subspace of $B^M(X)$ consisting of those classes $[x,a]$ such that the components of x, say $x_1,\dots,x_p$, are mutually distinct in X.

Each point of $C^M(X)$ has a representative of the form $(x_1,\dots,x_p, a_1,\dots,a_p)$ such that $a_i \neq 0$ and $x_i \neq x_j$ if $i \neq j$. Such a representative will be called *reduced*. Clearly reduced representatives are isomorphic with each other through permutations of indices, and hence determine a unique finite subset $\{(x_1,a_1),\dots,(x_p,a_p)\}$ of $X \times M$ such that $a_i \neq 0$ and $x_i \neq x_j$ if $i \neq j$. Thus we may regard $C^M(X)$ as the configuration space of finite distinct points in X with labels in M, modulo the relation $(x,0) = \text{null}$. It enjoys the property that $\{(x_1,a_1),\dots,(x_p,a_p)\}$ converges to $\{(x,a)\}$ as $x_i \to x$ if, and only if, $a_1 + \cdots + a_p$ converges to $a \in \overline{M}$. (Compare [2, §3] as well as [9]).

Note that we have $C^M(X) = C^{\overline{M}}(X)$. Note also that $C^M(X)$ is not necessarily a functor of X.

Example 1.2. (1) Let M be the subset $\{1\}$ of the abelian monoid $\mathbf{N}$. Then

$$C^{\{1\}}(X) = \coprod_{n\geq 0} C_n(X),$$

where $C_n(X)$ is the configuration spaces of n distinct points in X.

(2) For $M = \mathbf{N}$, we have

$$C^{\mathbf{N}}(X) = \coprod_{n\geq 0} \operatorname{SP}^n(X)$$

where $\operatorname{SP}^n(X)$ is the n-fold symmetric products of X. In other words, $C^{\mathbf{N}}(X)$ is the topological free abelian monoid generated by the points of X. If X is the complex line $\mathbf{C}$ then under the correspondence

$$\{(z_1,n_1),\dots,(z_p,n_p)\} \mapsto (z-z_i)^{n_i}\cdots(z-z_p)^{n_p},$$

$C^{\mathbf{N}}(\mathbf{C})$ can be identified with the space of monic polynomials over $\mathbf{C}$.

(3) Let $M = \mathbf{N} \times 0 \cup 0 \times \mathbf{N} \subset \mathbf{N}^2$. Then $C^M(\mathbf{C})$ is identified with the space of pairs (p, q) of coprime monic polynomials. Clearly we have $C^M(\mathbf{C}) = \coprod_{m,n\geq 0} C^M_{m,n}(\mathbf{C})$, where $C^M_{m,n}(\mathbf{C})$ is the component consisting of those (p, q) such that $\deg p = m$ and $\deg q = n$. In particular, $C^M_{n,n}(\mathbf{C})$ is identified with the space of rational algebraic maps of degree n of the Riemann sphere $S^2 = \mathbf{C} \cup \infty$ into itself. (See [10]).

(4) The example above can be generalized in several ways. If we take $M = \{(n_1, \ldots, n_m) \mid n_1 \cdots n_m = 0\}$ then $C^M(\mathbf{C})$ is the space of m-tuples of polynomials with no common root; its component consisting of those $(p_1, \ldots, p_m)$ with $\deg p_1 = \cdots = \deg p_m = n$ is identified with the space of rational algebraic maps of degree n of S^2 into the m-dimensional complex projective space $\mathbf{P}^m$. (See [10]). On the other hand, if $M = \bigcup_{k=1}^m \{0\}^{k-1} \times \mathbf{N} \times \{0\}^{m-k}$, then $C^M(\mathbf{C})$ is the space of m-tuples $(p_1, \ldots, p_m)$ of mutually coprime monic polynomials. (See [3]).

(5) Let $M = \mathbf{Z}$. Then $C^{\mathbf{Z}}(X) = \mathrm{AG}(X)$ is the topological free abelian group generated by the points of X. More generally, if R is a topological ring then $C^R(X)$ is the topological free R-module generated by the points of X. $\square$

We next introduce a pointed version of the notion above. For $p \geq 0$, let $\mathbf{p}$ denote the pointed finite set $\{0, 1, \ldots, p\}$ based at 0. If X is a pointed space, we have a topological category $\widetilde{\mathcal{Q}}^M(X)$ defined as follows:

(1) Objects of $\widetilde{\mathcal{Q}}^M(X)$ are the points of $\coprod_{p\geq 0} X^p \times \overline{M}^p$.

(2) A morphism from $(x_1, \ldots, x_p, a_1, \ldots, a_p)$ to $(y_1, \ldots, y_q, b_1, \ldots, b_q)$ is a basepoint preserving map $\theta\colon \mathbf{p} \to \mathbf{q}$ such that

(a) $x_i = y_{\theta(i)}$ for $1 \leq i \leq p$ (meaning that $x_i = *$ if $\theta(i) = 0$), and
(b) $b_j = \sum_{i\in\theta^{-1}(j)} a_i$ for $1 \leq j \leq q$.

Again, we denote this morphism by $(\theta^* y, a) \xrightarrow{\theta} (y, \theta_* a)$, where $\theta^*\colon X^q \to X^p$ and $\theta_*\colon A^p \to A^q$ are given by the formula (a) and (b) above. Then $\mathrm{Mor}\, \widetilde{\mathcal{Q}}^M(X)$ can be identified with the subset of

$$\coprod_{p,q\geq 0} X^q \times \mathrm{Map}_0(\mathbf{p}, \mathbf{q}) \times \overline{M}^p$$

consisting of those triples (y, θ, a) such that every component of $\theta_* a$ is in $\overline{M}$.

(3) The composition of morphisms is given by $(z, \psi, b)(y, \theta, a) = (z, \psi\theta, a)$, where $y = \psi^* z$ and $b = \theta_* a$.

One easily verifies that if $X_+ = X \coprod *$ is a space with disjoint basepoint $*$ then the inclusion $\mathcal{Q}^M(X) \xrightarrow{I} \widetilde{\mathcal{Q}}^M(X_+)$ has a natural left adjoint

$\widetilde{\mathcal{Q}}^M(X_+) \xrightarrow{R} \mathcal{Q}^M(X)$ such that $RI = \mathrm{Id}$. Thus the functor $\widetilde{\mathcal{Q}}^M$ can be regarded as a natural extension of $\mathcal{Q}^M$.

Definition 1.3. Given a pointed space X, let $\widetilde{B}^M(X)$ denote the identification space of $\mathrm{Ob}\,\widetilde{\mathcal{Q}}^M(X)$ with respect to the least equivalence relation such that two objects are equivalent if there is a morphism between them. We define $\widetilde{C}^M(X)$ to be the subspace of $\widetilde{B}^M(X)$ consisting of those classes $[x, a]$ such that the components $x_1, \ldots, x_p$ of x are mutually distinct in X. $\widetilde{C}^M(X)$ has the basepoint $\emptyset$ represented by $X^0 \times \overline{M}^0$.

Each point of $\widetilde{C}^M(X)$ can be identified with a finite subset $\{(x_1, a_1), \ldots, (x_p, a_p)\}$ of $X \times M$ such that $a_i \neq 0$, $x_i \neq *$, and $x_i \neq x_j$ if $i \neq j$. In addition to the property similar to that of $C^M(X)$, $\widetilde{C}^M(X)$ enjoys the property that $\{(x_i, a_i)\}$ converges to $\emptyset$ as $x_i \to *$. (That is to say, the basepoint $*$ is a 'vanishing point' of $\widetilde{C}^M(X)$).

The following is immediate from the definitions.

Proposition 1.4. *For any space (without basepoint) X, the natural map $C^M(X) \to \widetilde{C}^M(X_+)$ induced by the inclusion $X \subset X \coprod * = X_+$ is a homeomorphism.*

Example 1.5. (1) Any pointed space M can be viewed as the set of generators for the topological abelian monoid $\mathrm{SP}(M, *)$ generated by the points of M, where $*$ is identified with 0. The corresponding $\widetilde{C}^M(X)$ is the space of finite subsets of X "labeled by M" in the sense of [7].

(2) For any pointed space X, $\widetilde{C}^{\mathbf{N}}(X)$ is identified with $\mathrm{SP}(X, *)$. In particular, if X is the Riemann sphere $S^2 = \mathbf{C} \cup \infty$ then $\mathrm{SP}(S^2, \infty)$ can be identified with the infinite dimensional complex projective space $\mathbf{P}^\infty$ formed from the vector space $\mathbf{C}[z]$ of polynomials by mapping $\{(z_i, n_i)\}$, where $z_i = [u_i, v_i] \in \mathbf{P}^1$, to the class of the polynomial $\prod(u_i - v_i z)$.

(3) Let $M = \mathbf{N} \times 0 \cup 0 \times \mathbf{N} \subset \mathbf{N}^2$. Then there is a homotopy equivalence

$$\mathbf{P}^\infty \vee \mathbf{P}^\infty = \widetilde{C}^{\mathbf{N}}(S^2) \vee \widetilde{C}^{\mathbf{N}}(S^2) \xrightarrow{\simeq} \widetilde{C}^M(S^2)$$

induced by the inclusions $\mathbf{N} \to \mathbf{N} \times 0$ and $\mathbf{N} \to 0 \times \mathbf{N}$. Proposition (1.5) of [10] says that there is a homotopy equivalence $\widehat{Q} \xrightarrow{\simeq} \Omega^2 \widetilde{C}^M(S^2)$, where $\widehat{Q}$ is a 'group completion' of the H-space $C^M(\mathbf{C}) \cong \widetilde{C}^M(\mathbf{R}^2 \ltimes S^0)$. This equivalence is a prototype of our main theorem. $\square$

Remark 1.6. To any subset M of a (not necessarily abelian) topological monoid A we can assign a variant of $\widetilde{C}^M(X)$ constructed from the subcategory of $\widetilde{\mathcal{Q}}^M(X)$ whose morphisms are those $(\theta^* y, a) \xrightarrow{\theta} (y, \theta_* a)$ such that θ is order-preserving. Here *order-preserving* means that $0 < \theta(i) < \theta(j)$ implies

$i < j$. For example, if $M = \mathbf{N}$ then we get the *reduced product space* X_∞ in the sense of James [4].

2. Homology theories associated with labeled configuration spaces

In this section we construct a generalized homology theory starting from the classifying space of the category $\widetilde{\mathcal{Q}}^M(X)$. Recall that if $\mathcal{X}$ is a topological category, its classifying space $|\mathcal{X}|$ is defined to be the realization of the simplicial space $N\mathcal{X} = \{N_n\mathcal{X}\}$ whose n-simplexes are chains of arrows $x_0 \to x_1 \to \cdots \to x_n$.

Given a pointed space X, let $\widetilde{Q}^M(X) = |\widetilde{\mathcal{Q}}^M(X)|$ with the basepoint $\emptyset = X^0 \times \overline{M}^0$, and put

$$F^M(X) = \widetilde{Q}^M(X)/\widetilde{Q}^M(*), \quad E^M(X) = \Omega F^M(\Sigma X).$$

Here we write $\Sigma Z = S^1 \wedge Z$ and $\Omega Z = \mathrm{Map}_0(S^1, Z)$ for a pointed space Z.

As $F^M(*) = *$, the continuous map

$$\mathrm{Map}_0(X, Y) \to \mathrm{Map}_0(F^M(X), F^M(Y)), \quad f \mapsto F^M(f)$$

preserves basepoints. Hence there are natural maps

$$\mu_{Y,X} \colon Y \wedge F^M(X) \to F^M(Y \wedge X), \quad \mu_{Y,X} \colon Y \wedge E^M(X) \to E^M(Y \wedge X)$$

induced, by adjunction, from the composite maps

$$Y \xrightarrow{l} \mathrm{Map}_0(X, Y \wedge X) \xrightarrow{T} \mathrm{Map}_0(T(X), T(Y \wedge X)), \quad T = F^M, E^M,$$

where l is given by $l(y)(x) = y \wedge x$. It follows that

Proposition 2.1. *Both F^M and E^M preserve homotopies. In particular, if $X \simeq Y$ then we have $F^M(X) \simeq F^M(Y)$ and $E^M(X) \simeq E^M(Y)$.*

We now state the main result of this section. Recall from [5] that a (homology) group completion $g \colon Y \to Z$ is an H-map between admissible (e.g. homotopy associative and homotopy commutative) H-spaces such that $\pi_0 Z$ is the universal group associated with $\pi_0 Y$ and $H_\bullet(Z; k)$ is the localization of the Pontrjagin ring $H_\bullet(Y; k)$ at its submonoid $\pi_0 Y$ for every ring of coefficients k. For any X, $\widetilde{Q}^M(X)$ is an admissible H-space since $\widetilde{\mathcal{Q}}^M(X)$ is a permutative category with respect to the operation

$$\widetilde{\mathcal{Q}}^M(X) \times \widetilde{\mathcal{Q}}^M(X) \xrightarrow{\oplus} \widetilde{\mathcal{Q}}^M(X), \quad (x, a) \oplus (y, b) = (x, y, a, b).$$

Theorem 2.2. *Let X be a pointed space. Then*

(1) *the composite $\nu \colon \widetilde{Q}^M(X) \to E^M(X)$ of the natural map $\widetilde{Q}^M(X) \to F^M(X)$ with the map $F^M(X) \to \Omega F^M(\Sigma X) = E^M(X)$ adjoint to $\mu_{S^1,X}$ is a group completion map,*

(2) *the map* $E^M(X) \to \Omega E^M(\Sigma X)$ *adjoint to* $\mu_{S^1,X}$ *is a homotopy equivalence, and*
(3) *for every pointed subspace* A *of* X, $E^M(A) \to E^M(X) \to E^M(X \cup CA)$ *is a homotopy fibration sequence, that is, the natural map from* $E^M(A)$ *to the homotopy fiber of* $E^M(X) \to E^M(X \cup CA)$ *is a homotopy equivalence.*

This implies

Corollary 2.3. *For every pointed space* X, $E^M(X)$ *has a structure of an infinite loop space.*

and also,

Corollary 2.4. *The correspondence* $X \mapsto \pi_\bullet E^M(X)$, *together with the natural equivalence* $\pi_\bullet E^M(X) \cong \pi_{\bullet+1} E^M(\Sigma X)$ *induced by* $E^M(X) \xrightarrow{\sim} \Omega E^M(\Sigma X)$, *defines a generalized homology theory on the category of pointed spaces.*

Observe that, under the natural transformation

$$\lim_k \pi_{n+k}(X \wedge E^M(S^k)) \to \lim_k \pi_{n+k} E^M(X \wedge S^k) \cong \pi_n E^M(X)$$

induced by the maps $X \wedge E^M(S^k) \to E^M(X \wedge S^k)$, $\pi_\bullet E^M$ is equivalent to the generalized homology theory associated with the Ω-spectrum $\{E^M(S^n)\}$. Clearly $\pi_\bullet E^M$ satisfies the wedge axiom.

To prove Theorem 2.2 we need several lemmas similar to those used in the proof of Theorem 1.7 of [12].

Lemma 2.5. *If* (X, A) *is an* NDR *pair then so is* $(\widetilde{Q}^M(X), \widetilde{Q}^M(A))$.

Proof. By definition, (X, A) is an NDR pair if there are a map $u\colon X \to I$ such that $A = u^{-1}(0)$ and a homotopy $h\colon I \times X \to X$ such that $h(0, x) = x$ for $x \in X$, $h(t, a) = a$ for $(t, a) \in I \times A$, and $h(1, x) \in A$ for $x \in u^{-1}[0, 1)$.

Let Δ^n be the standard n-simplex with vertices $e_0, e_1, \ldots, e_n$. Given a sequence of real numbers $\nu = (\nu_0, \ldots, \nu_n)$ such that $0 \le \nu_i \le 1$, let w_ν be the map $\Delta^n \to I$ which sends the barycenter b_σ of a face $\sigma = |e_{i_0} \cdots e_{i_k}|$ to $\max\{\nu_{i_0}, \ldots, \nu_{i_k}\}$ and is linear on each face $|b_{\sigma_0} \cdots b_{\sigma_k}|$ of the barycentric subdivision $\operatorname{Sd} \Delta^n$ of Δ^n; that is,

$$w_\nu(s_0 b_{\sigma_0} + \cdots + s_k b_{\sigma_k}) = s_0 w_\nu(b_{\sigma_0}) + \cdots + s_k w_\nu(b_{\sigma_k}) \text{ if } s_0 + \cdots + s_k = 1.$$

Let δ_i denote the inclusion of Δ^{n-1} as the i-th face of Δ^n and σ_i the i-th projection $\Delta^{n+1} \to \Delta^n$. Then we have

$$w_\nu \delta_i = w_{d_i \nu}, \quad w_\nu \sigma_i = w_{s_i \nu},$$

where

$$d_i\nu = (\nu_0, \dots, \nu_{i-1}, \nu_{i+1}, \dots, \nu_n),$$
$$s_i\nu = (\nu_0, \dots, \nu_i, \nu_i, \nu_{i+1}, \dots, \nu_n).$$

Now we can define $\tilde{u}\colon \widetilde{Q}^M(X) \to I$ by the formula

$$\tilde{u}[(x^0, a^0) \xrightarrow{\theta_1} \cdots \xrightarrow{\theta_n} (x^n, a^n), s] = w_\nu(s),$$
$$\nu = \left(\max_{1\le j\le p_0} u(x_j^0), \dots, \max_{1\le j\le p_n} u(x_j^n) \right),$$

where $x^i = (x_j^i) \in X^{p_i}$ for $1 \le i \le n$. Clearly we have $\tilde{u}^{-1}(0) = \widetilde{Q}^M(A)$.

We also have a homotopy $\tilde{h}\colon I \times \widetilde{Q}^M(X) \to \widetilde{Q}^M(X)$ defined by the formula

$$\tilde{h}(t, [(x^0, a^0) \xrightarrow{\theta_1} \cdots \xrightarrow{\theta_n} (x^n, a^n), s])$$
$$= [(h_t(x^0), a^0) \xrightarrow{\theta_1} \cdots \xrightarrow{\theta_n} (h_t(x^n), a^n), (k_\nu)_t(s)].$$

Here $h_t(x^i) = (h(t, x_j^i))$ and k_ν is the homotopy $I \times \Delta^n \to \Delta^n$ characterized by the following properties:

(1) if $w_\nu(b_\sigma) < 1$ or if $w_\nu(b_\tau) = 1$ holds for every subface τ of σ, then $k_\nu(t, b_\sigma) = b_\sigma$ for all $t \in I$,
(2) if $w_\nu(b_\sigma) = 1$ and if there is a subface τ of σ such that $w_\nu(b_\tau) < 1$, then $k(t, b_\sigma) = (1-t)b_\sigma + tb_{\tau_0}$, where τ_0 is the largest subface such that $w_\nu(b_{\tau_0}) < 1$,
(3) for each $t \in I$, $(k_\nu)_t\colon \Delta^n \to \Delta^n$ is linear on each face of $\operatorname{Sd}\Delta^n$.

One easily observes that $(\tilde{h}, \tilde{u})$ represents $(\widetilde{Q}^M(X), \widetilde{Q}^M(A))$ as an NDR pair. □

Lemma 2.6. *$\widetilde{Q}^M(*)$ is contractible.*

Proof. Let I be the inclusion of the basepoint $\emptyset$ into $\widetilde{\mathcal{Q}}^M(*)$ and R the unique functor $\widetilde{\mathcal{Q}}^M(*) \to \{\emptyset\}$. Obviously we have $RI = \mathrm{Id}$. On the other hand, there is a natural transformation $IR \to \mathrm{Id}$ given by the evident morphisms $(*, \dots, *, a_1, \dots, a_p) \to \emptyset$. This implies that $|\widetilde{\mathcal{Q}}^M(*)|$ is contractible to $\emptyset$. □

Consequently we have

Corollary 2.7. *If X has a non-degenerate basepoint $*$ then the natural map*

$$\widetilde{Q}^M(X) \to \widetilde{Q}^M(X)/\widetilde{Q}^M(*) = F^M(X)$$

is a homotopy equivalence.

Lemma 2.8. *For any X, Y the natural map $\widetilde{Q}^M(X \vee Y) \to \widetilde{Q}^M(X) \times \widetilde{Q}^M(Y)$ induced by the evident projections is a homotopy equivalence.*

Proof. Let π denote the composite

$$\widetilde{\mathcal{Q}}^M(X) \times \widetilde{\mathcal{Q}}^M(Y) \xrightarrow{I_X \times I_Y} \widetilde{\mathcal{Q}}^M(X \vee Y)^2 \xrightarrow{\oplus} \widetilde{\mathcal{Q}}^M(X \vee Y),$$

where I_X and I_Y are induced by the inclusions $X \to X \vee Y$ and $Y \to X \vee Y$ respectively. Then π is an adjoint inverse to $\widetilde{\mathcal{Q}}^M(X \vee Y) \to \widetilde{\mathcal{Q}}^M(X) \times \widetilde{\mathcal{Q}}^M(Y)$; hence its realization gives a homotopy inverse to $\widetilde{Q}^M(X \vee Y) \to \widetilde{Q}^M(X) \times \widetilde{Q}^M(Y)$. □

Now suppose $X_\bullet$ is a pointed simplicial space. Then the maps $X_n \times \Delta^n \to |X_\bullet|$, $(x, s) \mapsto [x, s]$, factor through $X_n \wedge \Delta^n_+ = X_n \times \Delta^n / * \times \Delta^n$ and we obtain

$$\widetilde{Q}^M(X_n) \times \Delta^n \to \widetilde{Q}^M(X_n) \wedge \Delta^n_+ \to \widetilde{Q}^M(X_n \wedge \Delta^n_+) \to \widetilde{Q}^M(|X_\bullet|)$$

which, in turn, induce a pointed map $|\widetilde{Q}^M(X_\bullet)| \to \widetilde{Q}^M(|X_\bullet|)$.

Lemma 2.9. *Given a pointed simplicial space $X_\bullet$, the map $|\widetilde{Q}^M(X_\bullet)| \to \widetilde{Q}^M(|X_\bullet|)$ is a homeomorphism.*

Proof. As is well known, $|\widetilde{Q}^M(X_\bullet)| = \Big|[m] \mapsto |\widetilde{\mathcal{Q}}^M(X_m)|\Big|$ is homeomorphic to the successive realization

$$\Big|[n] \mapsto |N_n \widetilde{\mathcal{Q}}^M(X_\bullet)|\Big| = \Big|[n] \mapsto |[m] \mapsto N_n \widetilde{\mathcal{Q}}^M(X_m)|\Big|$$

of the bisimplicial space $([m], [n]) \mapsto N_n \widetilde{\mathcal{Q}}^M(X_m)$. To prove the lemma, it suffices to show that the natural map $|N_n \widetilde{\mathcal{Q}}^M(X_\bullet)| \to N_n \widetilde{\mathcal{Q}}^M(|X_\bullet|)$ is a homeomorphism for $n \geq 0$. But this follows from the fact that $|X^p_\bullet| \to |X_\bullet|^p$ is a homeomorphism for $p \geq 0$. □

We are ready to prove Theorem 2.2. In fact, we can prove a more general result. A functor T of the category of pointed spaces into itself is called *continuous* if the function

$$\mathrm{Map}_0(X, Y) \to \mathrm{Map}_0(T(X), T(Y)), \quad f \mapsto T(f)$$

is continuous for any X and Y. If T is continuous and $T(*) = *$ then there are pairings

$$\mu_{Y,X} \colon Y \wedge T(X) \to T(Y \wedge X), \quad \mu'_{Y,X} \colon T(Y) \wedge X \to T(Y \wedge X)$$

obtained, by adjunction, from the composite

$$Y \xrightarrow{l} \mathrm{Map}_0(X, Y \wedge X) \xrightarrow{T} \mathrm{Map}_0(T(X), T(Y \wedge X)),$$

where l is given by $l(y)(x) = y \wedge x$, and the similar one

$$X \to \mathrm{Map}_0(T(Y), T(Y \wedge X)).$$

It follows, in particular, that any homotopy $h\colon I_+ \wedge X \to Y$ induces one $I_+ \wedge T(X) \to T(Y)$ between $T(h_0)$ and $T(h_1)$; hence T preserves homotopy.

Given a continuous functor T of the category of pointed spaces into itself, put $F(X) = T(X)/T(*)$ and $E(X) = \Omega F(\Sigma X)$. (When $T(X) = \widetilde{Q}^M(X)$, $F(X) = F^M(X)$ and $E(X) = E^M(X)$). As $F(*) = *$ and $E(*) = *$, there are natural maps

$$F(X) \xrightarrow{\varepsilon} \Omega F(\Sigma X) = E(X), \quad E(X) \xrightarrow{\varepsilon} \Omega E(\Sigma X)$$

adjoint to $S^1 \wedge F(X) \to F(S^1 \wedge X)$ and $S^1 \wedge E(X) \to E(S^1 \wedge X)$ respectively.

Theorem 2.10. *Suppose T satisfies the following conditions*:

(C1) *If (X, A) is an* NDR *pair then so is $(T(X), T(A))$.*
(C2) *$T(*)$ is contractible.*
(C3) *For any X, Y the natural map $T(X \vee Y) \to T(X) \times T(Y)$ induced by the evident projections is a homotopy equivalence.*
(C4) *For any pointed simplicial space $X_\bullet$ the natural map $|T(X_\bullet)| \to T(|X_\bullet|)$ is a homotopy equivalence.*

Let us regard $T(X)$ as an H-space whose composition law is given by the composite

$$T(X) \times T(X) \xrightarrow{p^{-1}} T(X \vee X) \xrightarrow{m} T(X),$$

where p^{-1} is a homotopy inverse to the equivalence $T(X \vee X) \to T(X) \times T(X)$ and m is induced by the folding map $X \vee X \to X$. Then we have

(1) *the composite $\nu\colon T(X) \to E(X)$ of the natural map $T(X) \to F(X)$ with $F(X) \xrightarrow{\varepsilon} E(X)$ is a group completion,*
(2) *the map $E(X) \xrightarrow{\varepsilon} \Omega E(\Sigma X)$ is a homotopy equivalence, and*
(3) *for every pointed subspace A of X, $E(A) \to E(X) \to E(X \cup CA)$ is a homotopy fibration sequence.*

Proof. Given a pair of pointed maps $f\colon A \to X$, $g\colon A \to Y$, let $Z(f,g)_\bullet$ denote the pointed simplicial space defined as follows: $Z(f,g)_n = X \vee A^{\vee n} \vee Y$, where $A^{\vee n}$ is the n-fold wedge sum of A, and for every $\theta\colon [m] \to [n]$, $\theta^*\colon Z(f,g)_n \to Z(f,g)_m$ is given by $\theta^*(z) = z$ if $z \in X \vee Y$, and

$$\theta^*(\iota_j(a)) = \begin{cases} f(a) & \text{if } j \le \theta(0), \\ \iota_k(a) & \text{if } \theta(k-1) < j \le \theta(k),\ 1 \le k \le m, \\ g(a) & \text{if } \theta(m) < j. \end{cases}$$

Here ι_j denotes the inclusion of A into the j-th summand of $A^{\vee n}$. As every point of $X \vee A^{\vee n} \vee Y$ is the image of a unique point of $X \vee A \vee Y$ under some degeneracy map, the realization of $Z(f,g)_\bullet$ is the quotient of $(X \vee Y) \coprod I \times (X \vee A \vee Y)$ by the identifications, $(t, z) \sim z$, $(0, a) \sim f(a)$, $(1, a) \sim g(a)$,

where $t \in I$, $z \in X \vee Y$ and $a \in A$. Thus $|Z(f,g)_\bullet|$ is homeomorphic to the reduced double mapping cylinder

$$Z(f,g) = X \cup_f (I \times A) \cup_g Y/I \times *.$$

By (C4), there is a natural equivalence

$$|T(Z(f,g)_\bullet)| \to T(Z(f,g)).$$

Moreover, by (C1), $T(Z(f,g)_\bullet)$ is a proper simplicial space (or good simplicial space in the sense of [8]). Hence the natural map $\|T(Z(f,g)_\bullet)\| \to |T(Z(f,g)_\bullet)|$ is an equivalence. Here $\|T(Z(f,g)_\bullet)\|$ denotes the fat realization of $T(Z(f,g)_\bullet)$.

To see that $\nu\colon T(X) \to E(X)$ is a group completion, let f and g be the constant map $X \to *$ and write $Z(f,g)_\bullet = X^{\vee\bullet}$. Then the simplicial space $T(X^{\vee\bullet})$ is the composite of the Γ-space $\mathbf{n} \mapsto \Lambda(\mathbf{n}) = T(X \wedge \mathbf{n}) = T(X^{\vee n})$ with the contravariant functor $\Delta \to \Gamma$, $[n] \mapsto \mathbf{n}$.

But the properties (C2) and (C3) ensure that Λ satisfies the conditions (i) and (ii) of [8, Definition 1.2]. Thus, by the main theorems of [8] (or [6, §3]), the map

$$\Lambda(\mathbf{1}) \to \Omega B\Lambda(\mathbf{1}) = \Omega(|\Lambda|/\Lambda(\mathbf{0}))$$

induced by the natural map $\Delta^1 \times \Lambda(\mathbf{1}) \to |\Lambda|$ is a group completion.

As ν can be written as the composite

$$T(X) = \Lambda(\mathbf{1}) \to \Omega(|\Lambda|/\Lambda(\mathbf{0})) \xrightarrow{\psi} E(X),$$

where ψ is induced by the equivalence $|\Lambda| = |T(X^{\vee\bullet})| \to T(\Sigma X)$, we conclude that ν is a group completion.

The second statement is proved by the similar argument with X replaced by ΣX. Since $\Lambda(\mathbf{1}) = T(\Sigma X)$ has a continuous homotopy inverse induced by the map

$$S^1 \wedge X \to S^1 \wedge X, \quad [t,x] \mapsto [1-t,x],$$

$\Lambda(\mathbf{1}) \to \Omega B\Lambda(\mathbf{1})$ is a homotopy equivalence by Proposition 1.5 of [8]. This implies that $F(\Sigma X) \to E(\Sigma X)$ is an equivalence; hence so is $E(X) \to \Omega E(\Sigma X)$.

To prove the last statement, we need only show that

$$T(A) \to T(X) \to T(X \cup CA) \tag{2.1}$$

is a homotopy fibration sequence if $T(A)$ has a homotopy inverse. But this is equivalent to say that

$$T(A) \xrightarrow{T(j)} T(X \cup ZA) \xrightarrow{T(p)} T(X \cup CA). \tag{2.2}$$

is a homotopy fibration sequence. Here $X \cup ZA = Z(i, 1_A)$ is the reduced mapping cylinder of the inclusion $i\colon A \to X$, j is the inclusion of A into

$\{1\} \times A \subset X \cup ZA$ and $p\colon X \cup ZA \to X \cup CA$ is the map which collapses $\{1\} \times A$ to the basepoint.

Let us write $Z(i, 1_A)_\bullet = X \vee A^{\vee\bullet} \vee A$ and $Z(i, *)_\bullet = X \vee A^{\vee\bullet}$, where $*$ is the constant map $X \to *$. Then j is induced by the inclusion $j_\bullet$ of A as the last summand of $X \vee A^{\vee\bullet} \vee A$ and p is induced by the simplicial map

$$p_\bullet\colon X \vee A^{\vee\bullet} \vee A \to X \vee A^{\vee\bullet}$$

which collapses the last summand A to the basepoint. There is a commutative diagram

$$\begin{tikzcd} T(A) \arrow[r] \arrow[d, equal] & T(X \vee A) \arrow[r] \arrow[d] & T(X) \arrow[d] \\ T(A) \arrow[r, "|T(j_\bullet)|"] & |T(X \vee A^{\vee\bullet} \vee A)| \arrow[r, "|T(p_\bullet)|"] & |T(X \vee A^{\vee\bullet})| \end{tikzcd} \tag{2.3}$$

in which the vertical arrows are the natural maps sending 0-simplexes into realizations.

As $T(p_\bullet)$ is a map of proper simplicial spaces, we can apply the argument of the proof of Proposition 1.5 of [8], with the standard realization instead of the fat one. Thus the right hand square of (2.3) is homotopy-cartesian if $T(A)$ has a homotopy inverse. As $T(A) \to T(X \vee A) \to T(X)$ is a homotopy fibration sequence, we conclude that (2.2) is also a homotopy fibration sequence. This completes the proof of Theorem 2.2. $\square$

Remark 2.11. If we take the non-abelian version of $\widetilde{\mathcal{Q}}^M(X)$ discussed in Remark 1.6 then the corresponding $T(X)$ no longer satisfies (C3). Therefore the commutativity is essential in the construction of $E^M(X)$.

The following variation of Theorem 2.10 is also useful. Again, let T be a continuous functor of the category of pointed spaces into itself.

Theorem 2.12. *Suppose T satisfies the following conditions*:

(C2)′ $T(*) = *$.

(C3) *For any X, Y, the natural map $T(X \vee Y) \to T(X) \times T(Y)$ induced by the projections is a homotopy equivalence.*

(C4)′ *For any pointed simplicial space $X_\bullet$, the natural map $\|T(X_\bullet)\| \to T(\|X_\bullet\|)$ is a homotopy equivalence.*

Let $E(X) = \Omega T(\Sigma X)$ for any pointed space X. Then we have

(1) *the map $T(X) \xrightarrow{\varepsilon} E(X)$, adjoint to the natural map $S^1 \wedge T(X) \to T(S^1 \wedge X)$, is a group completion,*

(2) *$E(X) = \Omega T(\Sigma X) \xrightarrow{\Omega\varepsilon} \Omega E(\Sigma X)$ is a homotopy equivalence, and*

(3) *for every pointed subspace A of X, $E(A) \to E(X) \to E(X \cup CA)$ is a homotopy fibration sequence.*

Proof. By arguing as in the proof of Theorem 2.10, but replacing the standard realization with the fat one, we see that the natural map $T(X) \to \Omega\|T(X^{\vee\bullet})\|$ is a group completion for any X, and is an equivalence if $T(X)$ has a continuous homotopy inverse. As the composite

$$\Omega\|T(X^{\vee\bullet})\| \to \Omega T(\|X^{\vee\bullet}\|) \to \Omega T(|X^{\vee\bullet}|) = E(X)$$

is an equivalence, we see that the first and the second properties hold. The last property is proved similarly. □

3. Classical examples

3.1. **The stable homotopy theory.** In [8] Segal showed that for any space (without basepoint) X there exists a Γ-space $B\Sigma_X$ such that the associated spectrum $\mathbf{B}(B\Sigma_X)$ is equivalent to the suspension spectrum $\mathbf{S}(X_+)$.

By definition, $B\Sigma_X$ is the Γ-space associated with the symmetric monoidal category equivalent to $\mathcal{Q}^{\{1\}}(X)$. Hence there is a natural homotopy equivalence

$$B\Sigma_X(\mathbf{1}) = |\mathcal{Q}^{\{1\}}(X)| \xrightarrow{\simeq} \widetilde{Q}^{\{1\}}(X_+)$$

induced by the inclusion $\mathcal{Q}^{\{1\}}(X) \to \widetilde{\mathcal{Q}}^{\{1\}}(X_+)$. It follows by Proposition 3.6 of [8] that $E^{S^0}(X_+) = E^{\{1\}}(X_+)$ has the weak homotopy type of $\Omega^\infty\Sigma^\infty(X_+)$. (The case $X_+ = S^0$ is the Barratt-Priddy-Quillen theorem).

More generally, we can prove

Theorem 3.1. *Let M be a pointed space having the homotopy type of a CW-complex. If we regard M as the set of generators for* $\mathrm{SP}(M,*)$ *then there is a natural isomorphism of generalized homology theories*

$$\pi_\bullet E^M(X) \cong \pi_\bullet^S(X \wedge M)$$

defined on the category of pointed spaces having the homotopy type of a CW-complex.

Proof. Let $T(X) = \widetilde{Q}^M(X)/\widetilde{Q}^{\{0\}}(X)$ and $\widehat{E}^M(X) = \Omega(T(\Sigma X)/T(*))$. As $\widetilde{Q}^{\{0\}}(X)$ is contractible, the natural map $\widetilde{Q}^M(X) \to T(X)$ is an equivalence; hence so is $E^M(X) \to \widehat{E}^M(X)$. Let $\psi\colon X \wedge M \to \widehat{E}^M(X)$ be the natural map induced by the composite

$$X \times M \xrightarrow{\subset} \mathrm{Ob}\,\widetilde{\mathcal{Q}}^M(X) \to |\widetilde{\mathcal{Q}}^M(X)| = \widetilde{Q}^M(X) \to T(X) \to \widehat{E}^M(X)$$

Then there is a diagram of natural maps

$$\Omega^\infty\Sigma^\infty(X \wedge M) \xrightarrow{\Omega^\infty\psi} \Omega^\infty\widehat{E}^M(\Sigma^\infty X) \xleftarrow{\varepsilon^\infty} \widehat{E}^M(X) \xleftarrow{\simeq} E^M(X)$$

in which $\Omega^\infty\psi$ is the limit of

$$\Omega^n\psi\colon \Omega^n\Sigma^n(X \wedge M) = \Omega^n(\Sigma^n X \wedge M) \to \Omega^n\widehat{E}^M(\Sigma^n X)$$

and ε^∞ is the natural map

$$\widehat{E}^M(X) \to \lim_n \Omega^n \widehat{E}^M(\Sigma^n X) = \Omega^\infty \widehat{E}^M(\Sigma^\infty X).$$

As ε^∞ is a weak equivalence, we obtain a natural transformation

$$\tau\colon \pi_\bullet^s(X \wedge M) = \pi_\bullet \Omega^\infty \Sigma^\infty (X \wedge M) \to \pi_\bullet E^M(X).$$

Observe that $\widetilde{Q}^M(X)$ satisfies the properties (C1)–(C4) with respect to M as well as X. Hence $\pi_\bullet E^M(X) \cong \pi_\bullet \widehat{E}^M(X)$ defines a homology theory in either variable X or M. Therefore, to prove that τ is an isomorphism we need only show the case $X = M = S^0$. But this follows from the classical Barratt-Priddy-Quillen theorem. (See Proposition 3.5 of [8]). $\square$

Note that if X is pathwise connected then so is $\widetilde{C}^M(\mathbf{R}^\infty \ltimes X)$. Therefore $\widetilde{C}^M(\mathbf{R}^\infty \ltimes X)$ has the weak homotopy type of $E^M(X) \simeq \Omega^\infty \Sigma^\infty (X \wedge M)$ if X has the homotopy type of a connected finite CW-complex. But this also follows from the next theorem together with the fact that the inclusion $C^{X \wedge M}(\mathbf{R}^\infty) \to \widetilde{C}^M(\mathbf{R}^\infty \ltimes X)$ is a homotopy equivalence if X admits an embedding into $\mathbf{R}^\infty$. (Compare Proposition 4.4).

Theorem 3.2. *The natural map $C^X(\mathbf{R}^\infty) \to \Omega C^{\Sigma X}(\mathbf{R}^\infty)$ is a group completion and there is a natural isomorphism of homology theories*

$$\pi_\bullet \Omega C^{\Sigma X}(\mathbf{R}^\infty) \cong \pi_\bullet^S X.$$

Proof. Let $T(X) = C^X(\mathbf{R}^\infty)$. We first show that T satisfies the conditions (C2)$'$, (C3) and (C4)$'$ of Theorem 2.12. It is obvious that (C2)$'$ holds. To see that (C3) holds, choose a linear isometry $l\colon \mathbf{R}^\infty \times \mathbf{R}^\infty \to \mathbf{R}^\infty$. Then there is a composite

$$C^X(\mathbf{R}^\infty) \times C^Y(\mathbf{R}^\infty) \xrightarrow{(i_*, j_*)} C^{X \vee Y}(\mathbf{R}^\infty)^2 \xrightarrow{\oplus_l} C^{X \vee Y}(\mathbf{R}^\infty)$$

where i_* and j_* are induced by the inclusions X and Y into $X \vee Y$, and $\oplus_l$ takes $(\{(v_i, p_i)\}, \{(w_j, q_j)\})$ to $\{(l(0, v_i), p_i)\} \cup \{(l(e, w_j), q_j)\}$ for some $e \neq 0$. This gives a homotopy inverse to $T(X \vee Y) \to T(X) \times T(Y)$, for the space of linear isometries of $\mathbf{R}^\infty$ is contractible. Finally, (C4)$'$ is equivalent to the condition that the natural map

$$|\tau C^{X_\bullet}(\mathbf{R}^\infty)| \to |C^{\tau X_\bullet}(\mathbf{R}^\infty)| \cong C^{|\tau X_\bullet|}(\mathbf{R}^\infty)$$

is a homotopy equivalence, where $\tau X_\bullet$ denotes the proper simplicial space associated with $X_\bullet$. (See [8, Appendix A]). But for every $n \geq 0$, the composite

$$C^{X_n}(\mathbf{R}^\infty) \xrightarrow{\simeq} \tau_n C^{X_\bullet}(\mathbf{R}^\infty) \to C^{\tau_n X_\bullet}(\mathbf{R}^\infty)$$

is an equivalence induced by $X_n \xrightarrow{\simeq} \tau_n X_\bullet$. Hence $|\tau C^{X_\bullet}(\mathbf{R}^\infty)| \to |C^{\tau X_\bullet}(\mathbf{R}^\infty)|$ is a homotopy equivalence by Theorem A.4 of [5].

It follows by Theorem 2.12 that the natural map $C^X(\mathbf{R}^\infty) \to \Omega C^{\Sigma X}(\mathbf{R}^\infty)$ is a group completion, and the correspondence $X \mapsto \pi_\bullet \Omega C^{\Sigma X}(\mathbf{R}^\infty)$ defines a generalized homology theory. Moreover, there is a natural transformation of homology theories $\nu\colon \pi_\bullet \Omega C^{\Sigma X}(\mathbf{R}^\infty) \to \pi_\bullet^S X$ induced by the composite

$$\Omega C^{\Sigma X}(\mathbf{R}^\infty) \xrightarrow{\simeq} \Omega^\infty C^{\Sigma^\infty X}(\mathbf{R}^\infty) \xleftarrow{\Omega^\infty \psi} \Omega^\infty \Sigma^\infty X,$$

where ψ is the natural map $X \to C^X(\mathbf{R}^\infty)$ taking x to $(0, x)$. That ν is an isomorphism for $X = S^0$ follows from the Barratt-Priddy-Quillen theorem. (Compare Section 3 of [7]). $\square$

3.2. **The ordinary homology theory.** Now suppose M is a topological abelian monoid. Then $\widetilde{C}^M(X) = \widetilde{B}^M(X)$ is a functor of X, and there is a natural map

$$E^M(X) \to \Omega \widetilde{C}^M(\Sigma X)$$

induced by the functor $\widetilde{\mathcal{Q}}^M(X) \to \widetilde{C}^M(X)$ which takes a morphism $(\theta^* y, a) \to (y, \theta_* a)$ to the class $[\theta^* y, a] = [y, \theta_* a]$.

Given a graded group $G_\bullet$, let $\widetilde{H}_\bullet(X; G_\bullet)$ denote the ordinary homology of X with coefficients in $G_\bullet$.

The following generalizes the results of Dold-Thom [1].

Theorem 3.3. *Let M be an arbitrary topological abelian monoid. Then the natural map $\widetilde{C}^M(X) \to \Omega \widetilde{C}^M(\Sigma X)$ is a group completion, and there are natural isomorphisms of homology theories*

$$\pi_\bullet E^M(X) \cong \pi_\bullet \Omega \widetilde{C}^M(\Sigma X) \cong \widetilde{H}_\bullet(X; \pi_\bullet G),$$

where G is the group completion of M, i.e. $G = \Omega BM$. In particular, we have

$$\pi_n \widetilde{C}^M(X) \cong \widetilde{H}_n(X; \pi_\bullet G) = \sum\nolimits_r \widetilde{H}_{n-r}(X; \pi_r G)$$

if M is grouplike or if X is connected.

Proof. We first show that the functor $\widetilde{C}^M$ satisfies the conditions (C2)$'$, (C3) and (C4)$'$ of Theorem 2.12. It is obvious that (C2)$'$ holds. (C3) follows from the fact that the composite

$$\widetilde{C}^M(X) \times \widetilde{C}^M(Y) \xrightarrow{(i_*, j_*)} \widetilde{C}^M(X \vee Y)^2 \xrightarrow{\oplus} \widetilde{C}^M(X \vee Y)$$

is the inverse to $\widetilde{C}^M(X \vee Y) \to \widetilde{C}^M(X) \times \widetilde{C}^M(Y)$. Here i_* and j_* are induced by the inclusions $X \to X \vee Y$ and $Y \to X \vee Y$ respectively, and $\oplus$ is the multiplication given by $[x, a] \oplus [y, b] = [x, y, a, b]$. (C4)$'$ is equivalent to say that

$$|\tau \widetilde{C}^M(X_\bullet)| \to |\widetilde{C}^M(\tau X_\bullet)| \cong \widetilde{C}^M(|\tau X_\bullet|)$$

is a homotopy equivalence, and is proved by the argument similar to that of Theorem 3.2.

By Theorem 2.12, $\widetilde{C}^M(X) \to \Omega\widetilde{C}^M(\Sigma X)$ is a group completion and the correspondence $X \mapsto \pi_\bullet\Omega\widetilde{C}^M(\Sigma X)$ defines a generalized homology theory which is represented by the Ω-spectrum $\{\Omega\widetilde{C}^M(S^{n+1})\}$. However, as $\widetilde{C}^M(S^{n+1})$ are topological abelian monoids, $\{\Omega\widetilde{C}^M(S^{n+1})\}$ is equivalent to the generalized Eilenberg-MacLane spectrum $\prod_n K(\pi_n G, n)$. (Cf. Satz 7.1 of [1]). Thus there is a natural isomorphism of homology theories

$$\pi_\bullet\Omega\widetilde{C}^M(\Sigma X) \cong \widetilde{H}_\bullet(X; \pi_\bullet G).$$

On the other hand, as $\widetilde{\mathcal{Q}}^M(S^0) \to \widetilde{C}^M(S^0) = M$ has a right adjoint $M \to \widetilde{\mathcal{Q}}^M(S^0)$, $a \mapsto (1, a)$, the induced map $\widetilde{Q}^M(S^0) \to \widetilde{C}^M(S^0)$ is a homotopy equivalence. Therefore $E^M(S^0) \to \Omega\widetilde{C}^M(\Sigma S^0)$ is a weak equivalence, and so is $E^M(X) \to \Omega\widetilde{C}^M(\Sigma X)$ for any pointed CW-complex X. This completes the proof of the theorem. □

3.3. **Theories associated with subsets of $\mathbf{N}^m$.** As we stated in Examples 1.2 and 1.5, the labeled configuration space associated with a subset of $\mathbf{N}^m$ is interesting in that it is related to the topology of the space of rational algebraic maps. The following proposition together with (P4) implies that if M is sufficiently large, $\pi_\bullet\widetilde{C}^M(\mathbf{R}^n \ltimes X)$ converges to $\widetilde{H}_\bullet(X; \mathbf{Z}^m)$ as $n \to \infty$.

Proposition 3.4. *Let M be a subset of $\mathbf{N}^m$. Suppose M contains all the subsets of the form $\{0\}^{k-1} \times \mathbf{N} \times \{0\}^{m-k}$, $1 \le k \le m$. Then there is a natural isomorphism of homology theories*

$$\pi_\bullet E^M(X) \cong \widetilde{H}_\bullet(X; \mathbf{Z}^m).$$

Proof. Let $M' = \bigcup_{k=1}^m \{0\}^{k-1} \times \mathbf{N} \times \{0\}^{m-k}$. We shall show that the H-map

$$\widetilde{Q}^{M'}(S^0) \to \widetilde{Q}^M(S^0)$$

induced by the inclusion $M' \subset M$ is a homotopy equivalence. This implies, in particular, $E^{M'}(S^0) \simeq E^{\mathbf{N}^m}(S^0)$. But $E^{\mathbf{N}^m}(S^0) \simeq \mathbf{Z}^m$ as we have already seen. Hence $E^M(S^0) \simeq \mathbf{Z}^m$ for any M containing M'.

As $\widetilde{Q}^M(S^0)$ is homotopy equivalence to $Q^M(\{1\}) = |\mathcal{Q}^M(\{1\})|$, we need only show that the inclusion $I\colon Q^{M'}(\{1\}) \to Q^M(\{1\})$ has a homotopy inverse. For brevity, let us write $\mathcal{Q}^M(\{1\}) = \mathcal{Q}^M$. The elements of $\mathcal{Q}^M$ are identified with finite sequences of the elements of M. Let η be the functor $\mathcal{Q}^M \to \mathcal{Q}^{M'}$ given by

$$\eta(a_1, \dots, a_p) = (a_1^1 e_1, \dots, a_1^m e_m, \dots, a_p^1 e_1, \dots, a_p^m e_m) \in (M')^{mp},$$

where $a_j = (a_j^1, \dots, a_j^m)$, $1 \le j \le p$, and $e_k = (0, \dots, 1, \dots, 0) \in \{0\}^{k-1} \times \mathbf{N} \times \{0\}^{m-k}$. Then its realization $J\colon Q^M(\{1\}) \to Q^{M'}(\{1\})$ gives a homotopy inverse to I. In fact, if $\mathcal{Q}_0^{M'}$ denotes the full subcategory of $\mathcal{Q}^{M'}$ consisting of those objects $(a_1, \dots, a_p)$ such that $a_k \ne 0$ for $1 \le k \le p$ then the

inclusion $\mathcal{Q}_0^{M'} \to \mathcal{Q}^{M'}$ has the evident retraction $\rho\colon \mathcal{Q}^{M'} \to \mathcal{Q}_0^{M'}$; hence $|\mathcal{Q}_0^{M'}| \simeq |\mathcal{Q}^{M'}| = Q^{M'}(\{1\})$. But it is easily verified that $\rho\eta$ gives a left adjoint to the composite $\mathcal{Q}_0^{M'} \to \mathcal{Q}^{M'} \to \mathcal{Q}^{M}$. Therefore $JI \simeq \mathrm{Id}$. Similarly we can show that $IJ \simeq \mathrm{Id}$. □

4. The relation between $\widetilde{C}^M$ and E^M

If X is a pointed space, we write $\mathbf{R}^n \ltimes X = \mathbf{R}^n \times X/\mathbf{R}^n \times \{*\}$. Let $\mathcal{L}(n)$ be the space of linear isometries $(\mathbf{R}^\infty)^n \to \mathbf{R}^\infty$. For each $l \in \mathcal{L}(n)$, consider the composite

$$\widetilde{B}^M(\mathbf{R}^\infty \ltimes X)^n \xrightarrow{\prod_{j=1}^n i_j \ltimes 1_*} \widetilde{B}^M((\mathbf{R}^\infty)^n \ltimes X)^n$$
$$\xrightarrow{(l\ltimes 1)_*^n} \widetilde{B}^M(\mathbf{R}^\infty \ltimes X)^n \xrightarrow{\oplus} \widetilde{B}^M(\mathbf{R}^\infty \ltimes X),$$

where i_j is the inclusion of $\mathbf{R}^\infty$ into the j-th component of $(\mathbf{R}^\infty)^n$ and $\oplus$ is the juxtaposition map. Clearly this restricts to a continuous map

$$\widetilde{C}^M(\mathbf{R}^\infty \ltimes X)^n \to \widetilde{C}^M(\mathbf{R}^\infty \ltimes X).$$

Thus we obtain a linear isometries operad action on $\widetilde{C}^M(\mathbf{R}^\infty \ltimes X)$, hence a group completion of $\widetilde{C}^M(\mathbf{R}^\infty \ltimes X)$ into an infinite loop space.

The following theorem is nothing but (P4) of the main theorem.

Theorem 4.1. *If X is a pointed finite CW-complex then $E^M(X)$ has the weak homotopy type of a group completion of $\widetilde{C}^M(\mathbf{R}^\infty \ltimes X)$.*

From now on we assume, by replacing M by $\overline{M}$ if necessary, that M contains 0.

To establish a relationship between $E^M(X)$ and $\widetilde{C}^M(\mathbf{R}^\infty \ltimes X)$, we introduce a pointed map

$$\Phi\colon \widetilde{Q}^{|S_\bullet M|}(|S_\bullet X|) \to |S_\bullet \widetilde{C}^M(\mathbf{R}^\infty \ltimes X)|,$$

where $S_\bullet X = \{S_n X\}$ is the total singular complex of X and $\widetilde{Q}^{|S_\bullet M|}(|S_\bullet X|)$ is the realization of the diagonal of the bisimplicial set

$$([m],[n]) \mapsto N_n \widetilde{\mathcal{Q}}^{S_m M}(S_m X) = S_m(N_n \widetilde{\mathcal{Q}}^M(X)).$$

As the evaluation maps $|S_\bullet(N_n\widetilde{\mathcal{Q}}^M(X))| \to N_n\widetilde{\mathcal{Q}}^M(X)$ are weak equivalences, the induced map $\widetilde{Q}^{|S_\bullet M|}(|S_\bullet X|) \to \widetilde{Q}^M(X)$ is a homology equivalence. (Cf. Theorem A.4 of [5]). But an H-map between grouplike admissible H-spaces is a weak homotopy equivalence if and only if it is a homology equivalence. (Cf. [5, Remark 1.5]). Hence

Lemma 4.2. *The natural map $\rho\colon E^{|S_\bullet M|}(|S_\bullet X|) \to E^M(X)$ induced by the evaluation maps $|S_\bullet M| \to M$ and $|S_\bullet X| \to X$ is a weak homotopy equivalence.*

Let Δ^n be the standard n-simplex with vertices $e_0, e_1, \dots, e_n$. Let δ_j be the inclusion of Δ^{n-1} as the j-th face of Δ^n and σ_j the j-th projection $\Delta^{n+1} \to \Delta^n$. Write $\partial_j \Delta^n = \delta_j(\Delta^{n-1})$ and $\partial \Delta^n = \bigcup_{j=0}^n \partial_j \Delta^n$.

Lemma 4.3. *There is a system of continuous maps $R(\theta, i)\colon \Delta^n \to \mathbf{R}^\infty$, defined for every sequence of pointed maps $\theta = (\mathbf{p_0} \xrightarrow{\theta_1} \mathbf{p_1} \xrightarrow{\theta_2} \cdots \xrightarrow{\theta_n} \mathbf{p_n})$ and every $i \in \mathbf{p_0}$, enjoying the following properties*:

(1) $R(\theta, 0) \equiv 0$,
(2) *for any* $s \in \Delta^n - \partial_0 \Delta^n$, $R(\theta, i)(s) \neq R(\theta, j)(s)$ *if* $i \neq j$,
(3) $R(\theta, i)\sigma_j = R(s_j\theta, i)$, $R(\theta, i)\delta_0 = R(d_0\theta, \theta_1(i))$ *and* $R(\theta, i)\delta_j = R(d_j\theta, i)$ *if* $j > 0$.

Here we denote $s_j\theta = (\dots, \theta_j, 1, \theta_{j+1}, \dots)$ *for* $0 \le j \le n$, *and*

$$d_j\theta = \begin{cases} (\theta_2, \dots, \theta_n) & \text{if } j = 0 \\ (\dots, \theta_{j-1}, \theta_{j+1}\theta_j, \theta_{j+2}, \dots) & \text{if } 1 \le j \le n-1 \\ (\theta_1, \dots, \theta_{n-1}) & \text{if } j = n. \end{cases}$$

Proof. For every sequence $\theta = (\mathbf{p_0} \to \mathbf{p_1} \to \cdots \to \mathbf{p_n})$ let $|\theta|$ be a copy of $\mathbf{p_0}$ and let $S = \bigvee_\theta |\theta|$ be the union of all $|\theta|$. We may identify $\mathbf{R}^\infty$ with the linear space $\mathbf{R}\langle S\rangle$ spanned by S.

The construction of $R(\theta, i)$ proceeds by induction on the length n of θ. For $n = 0$ we define $R(\theta, i)$ by $R(\theta, i)(e_0) = i \in |\theta| \subset \mathbf{R}\langle S\rangle$. Assume that we have assigned $R(\theta', i')$ for every sequence θ' of length $< n$. Assume further that the image of such $R(\theta', i')$ is contained in $\mathbf{R}\langle S_{n-1}\rangle$, where S_{n-1} is the subset of S consisting of all $|\theta|$ such that the length of θ is less than n.

Let $\theta = (\theta_1, \theta_2, \dots, \theta_n)$ be a sequence of length n. Let b_n be the barycenter of Δ^n, so that every element of $\Delta^n = b_n * \partial\Delta^n$ can be written in the form $(1-t)b_n + t\delta_j x$ for some j and $x \in \Delta^{n-1}$. Then we define $R(\theta, i)$ as follows:

(1) if $\theta = s_j\theta'$ for some θ' of length $n-1$ then

$$R(\theta, i)(x) = R(\theta', i)(\delta_j(x))$$

for any $x \in \Delta^n$,

(2) if $\theta \neq s_j\theta'$ for all θ' then

$$R(\theta, i)((1-t)b_n + t\delta_j x) = (1-t)i_\theta + tR(d_j\theta, i')(x),$$

where $i_\theta \in |\theta|$ corresponds to i under the bijection $|\theta| \cong \mathbf{p_0}$ and $i' = \theta_1(i)$ if $j = 0$, $i' = i$ otherwise.

As $i_\theta \in S_n - S_{n-1}$, one easily verifies that $R(\theta, i)$ enjoys the required properties and that the image of $R(\theta, i)$ is contained in $\mathbf{R}\langle S_n\rangle$. Thus the proof proceeds by induction on n. $\square$

Let $D(X)_\bullet$ be the diagonal of the bisimplicial set

$$([m],[n]) \mapsto N_n \widetilde{\mathcal{Q}}^{S_m M}(S_m X).$$

Then each element f of $D(X)_n$ is of the form

$$(x^0, a^0) \xrightarrow{\theta_1} (x^1, a^1) \xrightarrow{\theta_2} \cdots \xrightarrow{\theta_n} (x^n, a^n),$$

where $(x^i, a^i) \in S_n X^{p_i} \times S_n M^{p_i}$, $x^{i-1} = {\theta_i}^* x^i$ and $a^i = \theta_{i*} a^{i-1}$. Let us write

$$\theta = (\theta_1, \theta_2, \ldots, \theta_n),\ x^0 = (x_1, \ldots, x_{p_0}),\ a^0 = (a_1, \ldots, a_{p_0})$$

and define $\Phi_n(f)\colon \Delta^n \to \widetilde{B}^M(\mathbf{R}^\infty \ltimes X)$ by

$$\Phi_n(f)(s) = [(R(\theta,1)(s), x_1(s)), \ldots, (R(\theta,p_0)(s), x_{p_0}(s)), a_1(s), \ldots, a_{p_0}(s)].$$

By using the properties of $R(\theta, i)$, one easily verifies that $\operatorname{Im}\Phi_n(f) \subset \widetilde{C}^M(\mathbf{R}^\infty \ltimes X)$. Thus we obtain a simplicial map

$$\Phi_\bullet\colon D(X)_\bullet \to S_\bullet \widetilde{C}^M(\mathbf{R}^\infty \ltimes X).$$

Let Φ denote its realization. Then Φ factors as

$$\widetilde{Q}^{|S_\bullet M|}(|S_\bullet X|) \xrightarrow{\Phi'} |S_\bullet \widetilde{C}^M(\mathbf{R}^\infty \ltimes X)'| \xrightarrow{\subset} |S_\bullet \widetilde{C}^M(\mathbf{R}^\infty \ltimes X)|,$$

where $\widetilde{C}^M(\mathbf{R}^\infty \ltimes X)'$ is the subset of $\widetilde{C}^M(\mathbf{R}^\infty \ltimes X)$ consisting of those $\{((v_i, x_i), a_i)\}$ such that $v_1, \ldots, v_p$ are mutually distinct in $\mathbf{R}^\infty$. Observe that if $M = \{1\}$ then $\widetilde{C}^{\{1\}}(\mathbf{R}^\infty \ltimes X)' = C^X(\mathbf{R}^\infty)$.

Proposition 4.4. *If X admits an embedding into $\mathbf{R}^\infty$ then the inclusion*

$$\widetilde{C}^M(\mathbf{R}^\infty \ltimes X)' \to \widetilde{C}^M(\mathbf{R}^\infty \ltimes X)$$

is a homotopy equivalence.

Proof. Let i be an embedding of X into $\mathbf{R}^\infty$ and define a homotopy $h\colon I \times X \to \mathbf{R}^\infty$ by $h(t,x) = (1-t)i(x)$. If we write $h_t(x) = h(t,x)$ then $h_0 = i$ and h_1 is the constant map with value 0. Let l be a linear isometry $\mathbf{R}^\infty \times \mathbf{R}^\infty \to \mathbf{R}^\infty$. Then there is a homotopy

$$H\colon I_+ \wedge \widetilde{B}^M(\mathbf{R}^\infty \ltimes X) \to \widetilde{B}^M(\mathbf{R}^\infty \ltimes X)$$

such that H_t is induced by the composite

$$\mathbf{R}^\infty \times X \xrightarrow{1\times\text{diag.}} \mathbf{R}^\infty \times X \times X \xrightarrow{1\times h_t\times 1} \mathbf{R}^\infty \times \mathbf{R}^\infty \times X \xrightarrow{l\times 1} \mathbf{R}^\infty \times X.$$

One easily verifies that

(1) H restricts to $I_+ \wedge \widetilde{C}^M(\mathbf{R}^\infty \ltimes X) \to \widetilde{C}^M(\mathbf{R}^\infty \ltimes X)$ and also to $I_+ \wedge \widetilde{C}^M(\mathbf{R}^\infty \ltimes X)' \to \widetilde{C}^M(\mathbf{R}^\infty \ltimes X)'$,
(2) the image of $f_0 = H_0|\widetilde{C}^M(\mathbf{R}^\infty \ltimes X)$ is in $\widetilde{C}^M(\mathbf{R}^\infty \ltimes X)'$, and
(3) $f_1 = H_1|\widetilde{C}^M(\mathbf{R}^\infty \ltimes X)$ is induced by $l' \times 1\colon \mathbf{R}^\infty \times X \to \mathbf{R}^\infty \times X$, where l' is the composite $\mathbf{R}^\infty \cong \mathbf{R}^\infty \times \{0\} \subset \mathbf{R}^\infty \times \mathbf{R}^\infty \xrightarrow{l} \mathbf{R}^\infty$.

As the space of linear isometries of $\mathbf{R}^\infty$ is contractible, l' is isotopic to the identity through linear isometries. Thus we have $f_0 \simeq f_1 \simeq 1$, showing that f_0 is a homotopy inverse to the inclusion $\widetilde{C}^M(\mathbf{R}^\infty \ltimes X)' \to \widetilde{C}^M(\mathbf{R}^\infty \ltimes X)$. $\square$

Clearly the correspondence $X \mapsto \widetilde{C}^M(\mathbf{R}^\infty \ltimes X)'$ is a continuous functor of X. Let $T(X) = \widetilde{C}^M(\mathbf{R}^\infty \ltimes X)'$ and $E(X) = \Omega T(\Sigma X)$. Then

Proposition 4.5. *The natural map $\nu\colon T(X) \to E(X)$ is a group completion and the correspondence $X \mapsto \pi_\bullet E(X)$ defines a generalized homology theory on the category of pointed spaces.*

Proof. It suffices to show that T satisfies the conditions (C2)$'$, (C3) and (C4)$'$ of Theorem 2.12. But this follows from the argument similar to that used in the proof of Theorem 3.2. $\square$

It follows, by Propositions 4.4 and 4.5, that $E(X)$ has the weak homotopy type of a group completion of $\widetilde{C}^M(\mathbf{R}^\infty \ltimes X)$. Moreover, there is a diagram

$$E^M(X) \xleftarrow{\rho} E^{|S_\bullet M|}(|S_\bullet X|) \xrightarrow{\psi} E(X),$$

in which ψ is induced by the composite

$$\widetilde{Q}^{|S_\bullet M|}(|S_\bullet X|) \xrightarrow{\Phi'} |S_\bullet \widetilde{C}^M(\mathbf{R}^\infty \ltimes X)'| \to \widetilde{C}^M(\mathbf{R}^\infty \ltimes X)' = T(X).$$

Now Theorem 4.1 follows from

Proposition 4.6. *If X is a pointed finite CW-complex then*

$$\psi\colon E^{|S_\bullet M|}(|S_\bullet X|) \to E(X)$$

is a weak equivalence.

Beware that $E^{|S_\bullet M|}(|S_\bullet X|)$ is not continuous in X. Nevertheless, the correspondence $X \mapsto \pi_\bullet E^{|S_\bullet M|}(|S_\bullet X|)$ defines a generalized homology theory on pointed finite CW-complexes, if its suspension isomorphism is defined to be the composite

$$\begin{aligned}\pi_n E^{|S_\bullet M|}(|S_\bullet X|) &\xrightarrow{\varepsilon_*} \pi_n \operatorname{Map}_0(|S_\bullet S^1|, E^{|S_\bullet M|}(|S_\bullet \Sigma X|)) \\ &\xrightarrow{\iota_*^{-1}} \pi_n \operatorname{Map}_0(S^1, E^{|S_\bullet M|}(|S_\bullet \Sigma X|)) = \pi_{n+1} E^{|S_\bullet M|}(|S_\bullet \Sigma X|).\end{aligned}$$

Here ε is the adjoint to $|S_\bullet S^1| \wedge E^{|S_\bullet M|}(|S_\bullet X|) \xrightarrow{\mu} E^{|S_\bullet M|}(|S_\bullet (S^1 \wedge X)|)$ and ι is induced by the equivalence $|S_\bullet S^1| \to S^1$. Clearly

$$\psi_*\colon \pi_\bullet E^{|S_\bullet M|}(|S_\bullet X|) \to \pi_\bullet E(X)$$

is a natural transformation of homology theories.

Thus, to prove Proposition 4.6, we need only show the case $X = S^0$. But this is a consequence of

Proposition 4.7. *The map*

$$\Phi' \colon \widetilde{Q}^{|S_\bullet M|}(S^0) \to |S_\bullet \widetilde{C}^M(\mathbf{R}^\infty \ltimes S^0)'| = |S_\bullet C^M(\mathbf{R}^\infty)|$$

is a homology equivalence.

We give a proof of this proposition in the next section.

5. Proof of Proposition 4.7

We shall prove the case M is discrete. The general case follows from this, for Φ' is obtained as the realization of the map of proper simplicial spaces

$$\widetilde{Q}^{S_n M}(S^0) \to |S_\bullet C^{S_n M}(\mathbf{R}^\infty)|, \quad n \geq 0.$$

As in the preceding section, we assume M contains 0.

Definition 5.1. A continuous map g of a simplex $\sigma = |p_0 p_1 \cdots p_n|$ into $C^M(\mathbf{R}^\infty)$ is called proper if $n = 0$ or if there are maps $v_1, \ldots, v_p$ of σ into $\mathbf{R}^\infty$ and elements $a_1, \ldots, a_p$ of M satisfying the following conditions:

(1) $g(s) = [v_1(s), \ldots, v_p(s), a_1, \ldots, a_p] \in C^M(\mathbf{R}^\infty)$ for all $s \in \sigma$,
(2) if $v_i(s) = v_j(s)$ holds for some $s \in \sigma - \partial_0\sigma$, where $\partial_0\sigma = |p_1 p_2 \cdots p_n|$, then $v_i = v_j$ all over σ,
(3) the restriction of g to $\partial_0\sigma$ is proper with respect to the restrictions of the maps $v_1, \ldots, v_p$ to $\partial_0\sigma$ and labels $a_1, \ldots, a_p$.

Observe that if $g \colon \sigma \to C^M(\mathbf{R}^\infty)$ is proper then so is its restriction to each face

$$\partial_i\sigma = |p_0 \cdots p_{i-1} p_{i+1} \cdots p_n|, \quad 0 \leq i \leq n.$$

Moreover, its restriction to each simplex of $\operatorname{Sd}\sigma$ is also proper. Here $\operatorname{Sd}\sigma$ denotes the barycentric subdivision of σ; a simplex of of which is of the form $|b_{\sigma_0} b_{\sigma_1} \cdots b_{\sigma_r}|$, where $\sigma_0, \sigma_1, \ldots, \sigma_r$ are faces of σ such that $\sigma_0 > \sigma_1 > \cdots > \sigma_r$ and b_{σ_i} is the barycenter of σ_i, $0 \leq i \leq r$.

For each $n \geq 0$, let $S_n C^M(\mathbf{R}^\infty)''$ denote the set of proper maps from Δ^n to $C^M(\mathbf{R}^\infty)$. Then $S_\bullet C^M(\mathbf{R}^\infty)'' = \{S_n C^M(\mathbf{R}^\infty)''\}$ is a subcomplex of $S_\bullet C^M(\mathbf{R}^\infty)$ and Φ' factors as

$$\widetilde{Q}^M(S^0) \xrightarrow{\overline{\Phi}'} |S_\bullet C^M(\mathbf{R}^\infty)''| \xrightarrow{I} |S_\bullet C^M(\mathbf{R}^\infty)|,$$

where I is the inclusion map. Let Φ'' denote the composite

$$Q^M(\{1\}) \xrightarrow{\sim} \widetilde{Q}^M(S^0) \xrightarrow{\overline{\Phi}'} |S_\bullet C^M(\mathbf{R}^\infty)''|,$$

where $Q^M(\{1\}) = |\mathcal{Q}^M(\{1\})|$. Then Proposition 4.7 is a consequence of the following two lemmas.

Lemma 5.2. $\Phi''\colon Q^M(\{1\}) \to |S_\bullet C^M(\mathbf{R}^\infty)''|$ *is a homotopy equivalence.*

Lemma 5.3. $I\colon |S_\bullet C^M(\mathbf{R}^\infty)''| \to |S_\bullet C^M(\mathbf{R}^\infty)|$ *is a homology equivalence.*

Proof of Lemma 5.2. To construct a homotopy inverse to Φ'', we introduce a simplicial subdivision of Δ^n. For every $n \geq 0$, let $[[n]]$ denote the partially ordered set $\{(i,j) \mid 0 \leq i \leq j \leq n\}$, where $(i,j) \leq (i',j')$ if and only if both $i \geq i'$ and $j \leq j'$ hold. (Thus we have $(i,j) \leq (0,n)$ for any $(i,j) \in [[n]]$). Let us regard $[[n]]$ as a category in which a morphism from (i,j) to (i',j') is the relation $(i,j) \leq (i',j')$. Then there is a piecewise-linear homeomorphism ϕ_n from the realization of $[[n]]$ to Δ^n such that $\phi_n(j,j) = e_j$ and $\phi_n(i,j) = \frac{1}{2}(e_i + e_j)$ (i.e. the middle point of $|e_i e_j|$) if $i < j$. Given a linearly ordered subset $J = \{(i_0,j_0) < \cdots < (i_k,j_k)\}$ of $[[n]]$, let Δ_J denote the simplex spanned by the vertices $\phi_n(i_0,j_0), \ldots, \phi_n(i_k,j_k)$. Then the collection $\{\Delta_J\}$ defines a simplicial subdivision of Δ^n having 2^n n-simplexes corresponding to those $J = \{(i_0,j_0) < \cdots < (i_n,j_n)\}$ such that $(i_0,j_0) = (i,i)$ for some i and $(i_n,j_n) = (0,n)$. This subdivision coincides with the edgewise subdivision defined in [7, Appendix 1].

Let $g\colon \Delta^n \to C^M(\mathbf{R}^\infty)$ be a proper simplex. Then there are maps $v_1, \ldots, v_p$ of Δ^n into $\mathbf{R}^\infty$ and elements $a_1, \ldots, a_p$ of M such that the following holds:

$$g(s) = [v_1(s), \ldots, v_p(s), a_1, \ldots, a_p], \quad s \in \Delta^n$$

For each j, let $(x_1^{j,j}, \ldots, x_{p(j,j)}^{j,j}, a_1^{j,j}, \ldots, a_{p(j,j)}^{j,j})$ be the unique reduced representative of $g(e_j)$ such that $x_1^{j,j} < \cdots < x_{p(j,j)}^{j,j}$ with respect to the lexicographical order in $\mathbf{R}^\infty$. Then we have

$$a_k^{j,j} = \textstyle\sum_{m \in \Lambda(j,j;k)} a_m,$$

where $\Lambda(j,j;k)$ denotes the set of those integers m such that $v_m(e_j) = x_k^{j,j}$.

More generally, for any pair (i,j) such that $i \leq j$, let

$$\textstyle\bigcup_m \{v_m(e_j)\} = \{x_1^{i,j}, \ldots, x_{p(i,j)}^{i,j}\}, \quad x_1^{i,j} < \cdots < x_{p(i,j)}^{i,j},$$

where m runs through the integers such that $v_m(e_i) = x_k^{i,i}$ for some k, $1 \leq k \leq p(i,i)$. Obviously $\{x_1^{i,j}, \ldots, x_{p(i,j)}^{i,j}\}$ is a subset of $\{x_1^{i',j}, \ldots, x_{p(i',j)}^{i',j}\}$ if $i' < i \leq j$. Let

$$a_k^{i,j} = \textstyle\sum_{m \in \Lambda(i,j;k)} a_m,$$

where $\Lambda(i,j;k) = \{m \mid v_m(e_j) = x_k^{i,j}\}$. Again, $(x_1^{j,j}, \ldots, x_{p(j,j)}^{j,j}, a_1^{j,j}, \ldots, a_{p(j,j)}^{j,j})$ is a (possibly non-reduced) representative of $g(e_j)$.

For any triple (i,i',j) with $i' < i \leq j$, there is an injection

$$\iota_{i>i',j}\colon \{1, \ldots, p(i,j)\} \to \{1, \ldots, p(i',j)\}$$

such that $\iota_{i>i',j}{}^*(x_1^{i',j},\dots,x_{p(i',j)}^{i',j}) = (x_1^{i,j},\dots,x_{p(i,j)}^{i,j})$ holds. On the other hand, if $i \le j < j'$, there is a surjection

$$\pi_{i,j<j'}\colon \{1,\dots,p(i,j)\} \to \{1,\dots,p(i,j')\}$$

defined by choosing any m such that $v_m(e_j) = x_k^{i,j}$ and putting $\pi_{i,j<j'}(k) = l$ if $v_m(e_{j'}) = x_l^{i,j'}$. Clearly we have

$$\begin{aligned}\pi_{i,j'<j''}\,\pi_{i,j<j'} &= \pi_{i,j<j''}\\ \iota_{i'>i'',j}\,\iota_{i>i',j} &= \iota_{i>i'',j}\\ \pi_{i',j<j'}\,\iota_{i>i',j} &= \iota_{i>i',j'}\,\pi_{i,j<j'}.\end{aligned}$$

Moreover, if we write $a^{i,j} = (a_1^{i,j},\dots,a_{p(i,j)}^{i,j})$ then we have

$$\pi_{i,j<j'}{}_*a^{i,j} = a^{i,j'}, \quad \iota_{i>i',j}{}_*a^{i,j} = a^{i',j}.$$

Hence there is a functor $\tilde{g}\colon [[n]] \to \mathcal{Q}^M(\text{point})$ such that $\tilde{g}(i,j) = a^{i,j}$ and

$$\tilde{g}((i,j)<(i,j')) = (a^{i,j} \xrightarrow{\pi_{i,j<j'}} a^{i,j'}), \quad \tilde{g}((i,j)<(i',j)) = (a^{i,j} \xrightarrow{\iota_{i>i',j}} a^{i',j}).$$

Now we define $\Theta\colon |S_\bullet C^M(\mathbf{R}^\infty)''| \to |\mathcal{Q}^M(\text{point})|$ as follows. Suppose $[g,s]$ is a point of $|S_\bullet C^M(\mathbf{R}^\infty)''|$, where $g \in S_nC^M(\mathbf{R}^\infty)''$ and $s \in \Delta^n$. Choose a linearly ordered subset J of $[[n]]$ such that $s \in \operatorname{Int}\Delta_J$. Let ξ_J be the affine homeomorphism $\Delta_J \cong \Delta^j$ which preserves the order of the vertices, and let $\tilde{g}_J$ be the composite

$$[j] \xrightarrow{\cong} J \xrightarrow{\tilde{g}|J} \mathcal{Q}^M(\text{point}).$$

Then $\Theta([g,s])$ is defined to be the class of $(\tilde{g}_J, \xi_J(s)) \in N_j\mathcal{Q}^M(\text{point}) \times \Delta^j$.

We will show that Θ is a homotopy inverse to Φ''. Given

$$f = (a^0 \xrightarrow{\theta_1} \cdots \xrightarrow{\theta_n} a^n) \in N_n\mathcal{Q}^M(\text{point}),$$

let $\tilde{f}$ denote the functor $[[n]] \to \mathcal{Q}^M(\text{point})$ which takes (i,j) to a^j and such that

$$\tilde{f}((i,j)<(i,j')) = \theta_{j'}\cdots\theta_{j+1}\colon a^j \to a^{j'}, \quad \tilde{f}((i,j)<(i',j)) = \mathrm{Id}\colon a^j \to a^j.$$

Let F be the self-map of $|\mathcal{Q}^M(\text{point})|$ given by

$$F([f,s]) = [\tilde{f}_J, \xi_J(s)] \quad \text{if } s \in \operatorname{Int}\Delta_J.$$

Then there is a homotopy $\Theta\Phi'' \simeq F$ induced by the evident natural transformation $\widetilde{\Phi_n(f)} \to \tilde{f}$. There is also a homotopy $F \simeq \mathrm{Id}$ given by $h_t([f,s]) = F([f,h_t'(s)])$, where h_t' is a self-map of Δ^n such that

$$h_t'\left(\frac{1}{2}(e_i+e_j)\right) = \frac{1-t}{2}e_i + \frac{1+t}{2}e_j$$

and is linear on each Δ_J. Then h_1' maps Δ^n onto Δ_{J_0}, where $J_0 = \{(0,0) < (0,1) < \cdots < (0,n)\}$, and $\xi_{J_0} h_1'$ is the identity of Δ^n. As $\tilde{f}_{J_0} = f$, h_1 is the identity of $|\mathcal{Q}^M(\text{point})|$. Therefore $\Theta\Phi'' \simeq \mathrm{Id}$.

On the other hand, we can inductively construct a homotopy $\Phi''\Theta \simeq \mathrm{Id}$ by using the general position argument as in the proof of Lemma 4.3. The verification will be left to the reader. □

To prove Lemma 5.3, we need the following lemma.

Lemma 5.4. *Let $X = |K|$ and $Y = |T|$ be polyhedra and let $f\colon X \to W$ be a continuous map of X into an open subset W of Y. Suppose that K is a finite complex and there is a subcomplex L of K such that the restriction of f to $A = |L|$ is simplicial with respect to L and T. Suppose further that W is the union of the open stars $O(p,T)$, where p runs through the vertices contained in W. Then there exists a subdivision K' of K, leaving L unaltered, and a simplicial map $g\colon X \to W$, with respect to K' and T, such that $g(x) = f(x)$ for $x \in A$ and*

$$f \simeq g\colon X \to W \operatorname{rel} A.$$

Proof. By taking the barycentric subdivision of K relative to L, we may assume that L is a full subcomplex of K. Let $\bar{L} = \{\sigma \in K \mid \sigma \cap |L| = \emptyset\}$ and let $K_1 = \mathrm{Sd}_{L\cup\bar{L}} K$ be the barycentric subdivision of K relative to $L \cup \bar{L}$. Then there is a simplicial map $h\colon X \to X$, with respect to K_1, such that

(1) if $v \in L \cup \bar{L}$ then $h(v) = v$,
(2) if $v \notin L \cup \bar{L}$, so that v is the barycenter of some simplex σ such that $\sigma \notin L$ and $\sigma \cap |L| \neq \emptyset$, then $h(v)$ is a vertex of σ that belongs to L.

As we have $O(v, K_1) \subset O(h(v), K_1)$ for every $v \in K_1$, h is a simplicial approximation of the identity and

$$1 \simeq h\colon X \to X \operatorname{rel} A. \tag{5.1}$$

Now let $K_2 = \mathrm{Sd}_L K_1$. Then $h(N(v,K_2)) \subset h(O(v,K_1)) = O(v,L)$ holds for $v \in L$. Here $N(v,K_2)$ denotes the (closed) star of v in K_2. As $f|A$ is simplicial with respect to L and T, we have $fh(N(v,K_2)) \subset f(O(v,L)) \subset O(f(v),T)$ or, equivalently,

$$N(v,K_2) \subset (fh)^{-1}(O(f(v),T)). \tag{5.2}$$

Let $\mathcal{U}$ be a covering of X consisting of those open sets

$$U(p) = (fh)^{-1}(O(p,T)),$$

where p runs through the vertices contained in W. Then, by the auxiliary theorem of [11, p. 318], there is a subdivision K' of K_2, leaving L unaltered,

enjoying the property that for every vertex v of K' there is a vertex $g(v)$ contained in W such that

$$N(v, K') \subset U(g(v)) = (fh)^{-1}(O(g(v), T)). \tag{5.3}$$

Here we may assume, by (5.2), that $g(v) = f(v)$ if $v \in L$.

Evidently the correspondence $v \mapsto g(v)$ determines a simplicial approximation $g\colon X \to W$ of fh with respect to K' and T. Moreover, as $g|A = f|A = fh|A$, we have

$$fh \simeq g\colon X \to W \operatorname{rel} A. \tag{5.4}$$

It follows that $f \simeq g\colon X \to W \operatorname{rel} A$, where g is simplicial with respect to a subdivision K', leaving L unaltered, and T. □

We are now ready to prove Lemma 5.3.

Proof of Lemma 5.3. For $n \geq 0$, let $\mathcal{S}_n C^M(\mathbf{R}^\infty)''$ be the abelian group generated by all proper maps $\Delta^n \to C^M(\mathbf{R}^\infty)$. Then $\mathcal{S}_\bullet C^M(\mathbf{R}^\infty)'' = \{\mathcal{S}_n C^M(\mathbf{R}^\infty)''\}$ is a subcomplex of the singular chain complex $\mathcal{S}_\bullet C^M(\mathbf{R}^\infty)$ of $C^M(\mathbf{R}^\infty)$. We will show that the inclusion

$$I_\bullet\colon \mathcal{S}_\bullet C^M(\mathbf{R}^\infty)'' \to \mathcal{S}_\bullet C^M(\mathbf{R}^\infty)$$

is a chain homotopy equivalence. This, of course, implies the lemma.

For this purpose, we introduce a triangulation of $B^M(\mathbf{R}^\infty)$. Let $X_\bullet$ be the ordered simplicial set associated with the standard triangulation of $\mathbf{R}$, with the integers as vertices. For $N \geq 0$, let $X_\bullet^N$ be the N-fold cartesian product of $X_\bullet$. Then $X_\bullet^N$ is isomorphic to the ordered simplicial set associated with the product triangulation of $\mathbf{R}^N$; hence $|X_\bullet^N| \cong \mathbf{R}^N$. Moreover, if θ is a map of finite sets then $\theta^*\colon (\mathbf{R}^N)^q \to (\mathbf{R}^N)^p$ is induced by a simplicial map $(X_\bullet^N)^q \to (X_\bullet^N)^p$. Clearly the equivalence relation on $\coprod_{p\geq 0} (X_\bullet^N)^p \times M^p$ generated by the relations $(\theta^* x, a) \sim (x, \theta_* a)$ is compatible with the simplicial structure. Thus we obtain a simplicial set $P_\bullet$ such that $|P_\bullet| \cong B^M(\mathbf{R}^N)$ holds.

As $P_\bullet$ is a regular simplicial set, its realization is a regular CW-complex. Hence we can triangulate $B^M(\mathbf{R}^N) = |P_\bullet|$ by successively taking the elementary subdivision $b_\sigma * (\partial\sigma)'$ of every closed cell $\overline{\sigma}$ of positive dimension. Here b_σ is the barycenter of σ and $(\partial\sigma)'$ is the (already constructed) subdivision of $\partial\sigma$. This triangulation is compatible with the natural map $B^M(\mathbf{R}^N) \to B^M(\mathbf{R}^{N+1})$ induced by the inclusion $\mathbf{R}^N \subset \mathbf{R}^{N+1}$. Thus we obtain a triangulation T of $B^M(\mathbf{R}^\infty)$. It is readily seen that $C^M(\mathbf{R}^\infty)$ is the union of open stars $O(p, T)$, where p runs through the vertices contained in $C^M(\mathbf{R}^\infty)$.

To each $f\colon \Delta^n \to C^M(\mathbf{R}^\infty)$ we assign a homotopy $\tilde{f}\colon I\times\Delta^n \to C^M(\mathbf{R}^\infty)$, with $\tilde{f}_0 = f$, and simplicial subdivisions K_f of Δ^n and L_f of $I\times\Delta^n$ enjoying the following properties:

(1) $(\widetilde{d_i f})_t = d_i(\tilde{f}_t)$ for $0 \le i \le n$ and $t \in I$, where $d_i f = f\delta_i\colon \Delta^{n-1} \to C^M(\mathbf{R}^\infty)$,
(2) $K_f|\partial_i\Delta^n = \{\sigma \in K_f \mid \sigma \subset \partial_i\Delta^n\} = K_{d_i f}$,
(3) $L_f|\partial(I\times\Delta^n) = \{0\}\times\Delta^n \cup \{1\}\times K_f \cup \bigcup_{0\le i\le n}(1\times\delta_i)L_{d_i f}$,
(4) $\tilde{f}_1$ is simplicial with respect to K_f and T, and its restriction to σ is proper for every $\sigma \in \operatorname{Sd} K_f$,
(5) if $f \in S_n C^M(\mathbf{R}^\infty)''$ then the restriction of $\tilde{f}$ to $|\tau|$ is proper for every $\tau \in \operatorname{Sd}_{\{0\}\times\Delta^n} L_f$.

The construction is by induction on n. For $n = 0$, we take as $\tilde{f}$ a path in $C^M(\mathbf{R}^\infty)$ joining f to a vertex p of T such that $f \in O(p,T)$. Suppose that we have constructed $\tilde{f}'$, $K_{f'}$ and $L_{f'}$ for every $f'\colon \Delta^k \to C^M(\mathbf{R}^\infty)$ with $k < n$. Then for each $f\colon \Delta^n \to C^M(\mathbf{R}^\infty)$ and each i with $0 \le i \le n$ there are a homotopy $\widetilde{d_i f}$ and subdivisions $K_{d_i f}$ of $\partial_i\Delta^n$ and $L_{d_i f}$ of $I\times\Delta^{n-1}$ enjoying the properties mentioned above. Let $\widetilde{df}\colon I\times\partial\Delta^n \to C^M(\mathbf{R}^\infty)$ be the homotopy determined by the homotopies $\widetilde{d_i f}$. Then, by the homotopy extension property, there is a homotopy $F\colon I\times\Delta^n \to C^M(\mathbf{R}^\infty)$ such that $F_0 = f$ and $F|I\times\partial\Delta^n = \widetilde{df}$.

Let L be the subdivision of $\partial\Delta^n$ determined by the subdivisions $K_{d_i f}$ and let $K = b_n * L$ be the elementary subdivision of Δ^n arising from L. Then L is a subcomplex of K and $F_1|\partial\Delta^n = \widetilde{df}_1$ is simplicial with respect to L and T. Therefore we obtain, by Lemma 5.4, a subdivision K_f of K, leaving L unaltered, and a simplicial map $g\colon \Delta^n \to C^M(\mathbf{R}^\infty)$ with respect to K_f and T such that the following holds:

$$g|\partial\Delta^n = F_1|\partial\Delta^n = \widetilde{df}_1, \quad F_1 \simeq g \operatorname{rel} \partial\Delta^n.$$

Combining these two homotopies $f = F_0 \simeq F_1$ and $F_1 \simeq g$ rel $\partial\Delta^n$, we obtain a homotopy $\tilde{f}$ with $\tilde{f}_0 = f$ and $\tilde{f}_1 = g$.

We will show that the restriction of g to σ is proper for every $\sigma \in \operatorname{Sd} K_f$. By definition, σ is of the form $|b_{\sigma_0} b_{\sigma_1} \cdots b_{\sigma_r}|$, where $\sigma_1, \ldots, \sigma_r$ are simplexes of K_f such that $\sigma_0 > \sigma_1 > \cdots > \sigma_r$. As g is simplicial with respect to K_f and T, its restriction to σ is an affine map of σ onto the (possibly degenerate) simplex $\tau = |b_{\tau_0} b_{\tau_1} \cdots b_{\tau_r}|$, where $\tau_i = g(\sigma_i) \in T$. Such a map is certainly proper by the construction of T.

We also define L_f to be the elementary subdivision $b * L'$, where $b = (\frac{1}{2}, b_n) \in I\times\Delta^n$ and

$$L' = \{0\}\times\Delta^n \cup \{1\}\times K_f \cup \textstyle\bigcup_i (1\times\delta_i)L_{d_i f}.$$

If $f \in S_n C^M(\mathbf{R}^\infty)''$ then, by using the general position argument as in the proof of Lemma 4.3, we can choose such an $\tilde{f}$ that its restriction to each simplex of $\mathrm{Sd}_{\{0\}\times\Delta^n} L_f$ is proper. Thus $\tilde{f}$, together with K_f and L_f, enjoys all the required properties.

Now let J_n be a homomorphism $\mathcal{S}_n C^M(\mathbf{R}^\infty) \to \mathcal{S}_n C^M(\mathbf{R}^\infty)''$ which takes each generator $f\colon \Delta^n \to C^M(\mathbf{R}^\infty)$ to the chain

$$[\tilde{f}_1] = \sum\nolimits_{\sigma\in\mathrm{Sd}\,K_f^n} \varepsilon(\sigma,\Delta^n)(\tilde{f}_1 \circ i_\sigma) \in \mathcal{S}_n C^M(\mathbf{R}^\infty)'',$$

where σ runs through the n-dimensional simplexes of $\mathrm{Sd}\,K_f$, i_σ is the affine homeomorphism $\Delta^n \cong \sigma$ which preserves the order of vertices, and $\varepsilon(\sigma,\Delta^n)$ is either 1 or -1 according as the orientation of σ induced by i_σ is compatible with the standard orientation of Δ^n or not. By the properties of $\tilde{f}_1$ and K_f, $\{J_n\}$ defines a chain map

$$J_\bullet\colon \mathcal{S}_\bullet C^M(\mathbf{R}^\infty) \to \mathcal{S}_\bullet C^M(\mathbf{R}^\infty)''.$$

Similarly let h_n be a homomorphism $\mathcal{S}_n C^M(\mathbf{R}^\infty) \to \mathcal{S}_{n+1} C^M(\mathbf{R}^\infty)$ which takes a generator f to

$$[\tilde{f}] = \sum\nolimits_{\tau\in\mathrm{Sd}_{\{0\}\times\Delta^n} L_f^{n+1}} \varepsilon(\tau, I\times\Delta^n)(\tilde{f}\circ i_\tau) \in \mathcal{S}_{n+1} C^M(\mathbf{R}^\infty),$$

where τ runs through the $(n+1)$-dimensional simplexes of $\mathrm{Sd}_{\{0\}\times\Delta^n} L_f$, i_τ is the affine homeomorphism $\Delta^{n+1} \cong |\tau|$ which preserves the order of vertices, and $\varepsilon(\tau, I\times\Delta^n)$ is either 1 or -1 according as the orientation of $|\tau|$ induced by i_τ is compatible with the orientation of $I\times\Delta^n$ or not. Here $I\times\Delta^n$ is so oriented that we have

$$\partial(I\times\Delta^n) = -(\{0\}\times\Delta^n) + (\{1\}\times\Delta^n) - \sum\nolimits_{i=0}^{n} (-1)^i (I\times\partial_i\Delta^n).$$

Then we have

$$\begin{aligned}
\partial_{n+1} h_n(f) &= \partial_{n+1}\left(\sum\nolimits_{\tau\in\mathrm{Sd}_{\{0\}\times\Delta^n} L_f^{n+1}} \varepsilon(\tau, I\times\Delta^n)(\tilde{f}\circ i_\tau)\right)\\
&= \sum\nolimits_{\tau\in\mathrm{Sd}_{\{0\}\times\Delta^n} L_f^{n+1}} \sum\nolimits_{i=0}^{n+1} (-1)^i \varepsilon(\tau, I\times\Delta^n)(\tilde{f}\circ i_\tau\circ\delta_i)\\
&= \sum\nolimits_{\tau\in\Lambda_f^{n+1}} \varepsilon(\tau, I\times\Delta^n)(\tilde{f}\circ i_{\partial_0\tau}),
\end{aligned}$$

where Λ_f^{n+1} denotes the set of those $\tau \in \mathrm{Sd}_{\{0\}\times\Delta^n} L_f^{n+1}$ such that $\partial_0\tau$ is contained in $\partial(I\times\Delta^n)$ and $i_{\partial_0\tau} = i_\tau\circ\delta_0$. (By the construction of $\mathrm{Sd}_{\{0\}\times\Delta^n} L_f$, $\partial_i\tau$ cannot be contained in $\partial(I\times\Delta^n)$ unless $i=0$).

But we have

$$\varepsilon(\tau, I \times \Delta^n)(\tilde{f} \circ i_{\partial_0 \tau}) = \begin{cases} -f & \text{if } \partial_0 \tau = \{0\} \times \Delta^n \\ \varepsilon(\sigma, \Delta^n)(\tilde{f}_1 \circ i_\sigma) & \text{if } \partial_0 \tau = \{1\} \times \sigma, \\ & \quad \sigma \in \operatorname{Sd} K_f^n \\ -(-1)^i \varepsilon(\tau', I \times \Delta^{n-1})(\widetilde{d_i f} \circ i_{\tau'}) & \text{if } \partial_0 \tau = (1 \times \delta_i)\tau', \\ & \quad \tau' \in \operatorname{Sd}_{\{0\} \times \Delta^n} L_{d_i f}. \end{cases}$$

Therefore

$$\partial_{n+1} h_n(f) + h_{n-1} \partial_n(f) = J_n(f) - f$$

holds for any generator f of $\mathcal{S}_n C^M(\mathbf{R}^\infty)$. This, of course, implies that $h = \{h_n\}$ defines a chain homotopy $I_\bullet J_\bullet \simeq \mathrm{Id}$.

On the other hand, if f belongs to $\mathcal{S}_n C^M(\mathbf{R}^\infty)''$ then $[\tilde{f}] \in \mathcal{S}_{n+1} C^M(\mathbf{R}^\infty)''$. Hence there are homomorphisms $k_n \colon \mathcal{S}_n C^M(\mathbf{R}^\infty)'' \to \mathcal{S}_{n+1} C^M(\mathbf{R}^\infty)''$ defining a chain homotopy $J_\bullet I_\bullet \simeq \mathrm{Id}$.

This completes the proof of Lemma 5.3. □

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