Mathematical Journal of Okayama University

Volume 3, *Issue* 2 1953 *Article* 4

MARCH 1954

Some remarks on homotopy groups of rotation groups

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SOME REMARKS ON HOMOTOPY GROUPS OF ROTATION GROUPS

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The purpose of this note is to remark slightly that the 15-sphere S^{15} has a 8 everywhere independent continuous vector fields, which can be proved by simple relations of homotopy groups $\pi_r(R_n)$ of rotation groups R_n. Additionally, calculations of $\pi_r(R_n)$ is continued from the previous note [5], whose notations are used in this note.

1. Concerning to the composition of two homomorphisms Δ: $\pi_{r+1}(S^n) \to \pi_r(R_n)$, the boundary homomorphism of the fibre bundle $\{R_{n+1}, p, S^n, R_n, R_n\}$, and $p_*: \pi_r(R_n) \to \pi_r(S^{n-1})$, the induced map of the natural projection $p: R_n \to S^{n-1}$, it holds the following relation.

Lemma 1. $E^{n+3} p_* \Delta(\alpha) = 0,$ *if n is odd,*
$= 2E^{n+2}\alpha,$ *if n is even,*
for $\alpha \in \pi_{r+1}(S^n)$, *where* $E^p: \pi_k(S^l) \to \pi_{k+p}(S^{l+p})$ *is the p-fold iteration of suspension homomorphism E.*

It is known that the homomorphism $J: \pi_r(R_n) \to \pi_{r+n}(S^n)$ satisfies the relation $J\Delta(\alpha) = [\alpha, \iota_n]$, where $[\alpha, \beta]$ is Whitehead product, [9, (3.6)]. As $J(\beta)$ is represented by the Hopf construction of the mapping $S^r \times S^{n-1} \to S^{n-1}$ of type $(p_*(\beta), \iota_{n-1})$[1], $H_0(J(\beta)) = (-1)^{(r+1)n}E(p_*(\beta) * \iota_{n-1}) = (-1)^{(r+1)n}E^{n+1}p_* \beta$[2]. On the other hand, if n is even, $E^2H_0[\alpha, \iota_n] = 2(-1)^n E(\alpha * \iota_n) = 2E^{n+2}\alpha$, and, if n is odd, $E^2H_0[\alpha, \iota_n] = 0$. Thus the lemma holds.

We consider the above relation for the case $r = 14$, $n = 8$ and $\alpha = \nu_8' \in \pi_{15}(S^8)$ represented by the Hopf map $S^{15} \to S^8$. Then $\{\nu_8'\} = \infty \subset \pi_{15}(S^8)$ and $E^i\nu_8'$ is a generator of $\pi_{i+15}(S^{i+8}) = 240$ for $i > 0$[3], and hence $E^{n+3}p_* \Delta(\nu_8') = 2E^{n+2}\nu_8'$ is a element of order 120. Therefore $p_*\Delta(\nu_8')$ generates $\pi_{14}(S^7)$, as $\pi_{14}(S^7) = 120$[3]. This shows that $p_*\Delta$: $\pi_{15}(S^8) \to \pi_{14}(S^7)$ is onto.

Let α be an element of $\pi_{14}(R_8)$. By the above property, there exists $\beta \in \pi_{15}(S^8)$ such that $p_*\Delta\beta = p_*\alpha$, and hence $p_*(\alpha - \Delta\beta) = 0$. Therefore, by the exactness of the homotopy sequence: $\pi_{14}(R_7)$

1) Cf. [8], proofs of Corolary 5.14.
2) Cf. [6], (3.6), (3.11) and (2.24), where $H_0: \pi_r(S^n) \to \pi_{r+1}(S^{2n})$ is the generalized Hopf homomorphism.
3) Cf. [2], Théorème 3. The subgroup generated by α is denoted by $\{\alpha\}$.

$\xrightarrow{i'_*} \pi_{14}(R_8) \xrightarrow{p_*} \pi_{14}(S^7)$, there exists an element $\alpha' \in \pi_{14}(R_7)$ such that $i'_*\alpha' = \alpha - \varDelta\beta$. Hence $(ii')_*\alpha' = i_*\alpha - i_*\varDelta\beta = i_*\alpha$ by the exactness of the sequence: $\pi_{15}(S^8) \xrightarrow{\varDelta} \pi_{14}(R_8) \xrightarrow{i_*} \pi_{14}(R_9)$. This shows that any map $f: S^{14} \to R_8$ is homotopic in R_9 to some map of S^{14} into R_7.

It is known that S^{15} admits 7 everywhere independent continuous vector fields, i.e., 7-field, and so the characteristic map $T_{16}: S^{14} \to R_{15}$ of the fibre bundle $\{R_{16}, p, S^{15}, R_{15}, R_{15}\}$ is homotopic in R_{15} to a map of S^{14} into R_8[1]. As shown above, last map is homotopic in R_9 to a map $f_0: S^{14} \to R_7$, and hence T_{16} is homotopic to f_0 in R_{15}. This shows that S^{15} admits a continuous 8-fields[1]. Therefore, by the analogous proofs of [4, 27.12], it folds

Proposition 1[2]. *If $n = 16m + 15$, S^n admits 8 everywhere independent continuous vector fields.*

2. In [5], $\pi_7(R_n)$ is calculated halfway. A. Borel proved that there is a factorization $Spin\,(9)/Spin\,(7) = S^{15}$, and hence $\pi_i(R_9) \approx \pi_i(R_7)$ for $i \leqslant 13$ as $Spin\,(n)$ is a covering group of R_n, [1, Théorèmes 3, 4]. This shows that the case i) of Theorem 1 of [5] is not valid and, therefore, it follows from [5, 3.2, 3.4 and 3.6]:

Proposition 2[3]. $\pi_7(R_5) = \infty = \{\gamma_7\}$, $\pi_7(R_6) = \infty = \{\delta_7\}$, $\pi_7(R_7) = \infty = \{\varepsilon_7\}$, $\pi_7(R_8) = \infty + \infty = \{\varepsilon_7\} + \{\zeta_7\}$ *and* $\pi_7(R_n) = \infty = \{\zeta_7\}$ *for* $n \geqslant 9$, *where the relations* $2\delta_7 = \gamma_7$ *in* $\pi_7(R_6)$, $2\varepsilon_7 = \delta_7$ *in* $\pi_7(R_7)$, *and* $2\varepsilon_7 = \zeta_7$ *in* $\pi_7(R_9)$ *are hold.*

Before continuing the calculation of $\pi_r(R_n)$ for $r \geqslant 9$, we remark slightly at a left distributive law for homotopy groups. In general, if X is any space, $(\alpha_1 + \alpha_2)\circ\beta$ is not equal to $\alpha_1\circ\beta + \alpha_2\circ\beta$ for $\alpha_1, \alpha_2 \in \pi_n(X)$ and $\beta \in \pi_r(S^n)$. However, if X is a topological group G, above two elements are equal. Let $f_1, f_2: S^n \to G$ be representatives of α_1, α_2 and $h: S^r \to S^n$ of β, respectively. As G is a topological group, by [4, 17.6], $f_1 + f_2$ is homotopic to $f_1 \cdot f_2$ where $(f_1 \cdot f_2)(y) = f_1(y) \cdot f_2(y)$ for $y \in S^n$. Hence $(f_1 + f_2)\circ h$ is homotopic to $(f_1 \cdot f_2)\circ h = f_1(h) \cdot f_2(h)$, and the latter is homotopic to $f_1(h) + f_2(h) = f_1\circ h + f_2\circ h$ by the same reason. Therefore we have

Lemma 2. *If G is a topological group,* $(\alpha_1 + \alpha_2)\circ\beta = \alpha_1\circ\beta + \alpha_2\circ\beta$, *for* $\alpha_1, \alpha_2 \in \pi_n(G)$ *and* $\beta \in \pi_r(S^n)$.

1) Cf. [4], 27.12 and 27.6.
2) Yosihiro Saito has constructed practically a 8 independent vector fields over S^{15}.
3) These results agree with those of Serre and Peachter, [2, Lemme 3].

3. For $\pi_9(R_n)$, it holds the following

Proposition 3. *$\pi_9(R_3) = 3 = \{\alpha_9\}$, where $\alpha_9 = \alpha_3 \circ \mu_3 \circ \mu_6$. $\pi_9(R_4) = 3 + 3 = \{\alpha_9\} + \{\beta_9\}$, where $\beta_9 = \beta_3 \circ \mu_3 \circ \mu_6$. $\pi_9(R_5) = 0$. $\pi_9(R_6) = 2 = \{\delta_9\}$, where $p_* \delta_9 = \nu_5 \circ \eta_8$ a generator of $\pi_9(S^5)$. $\pi_9(R_7) = 2 + 2 = \{\delta_9\} + \{\varepsilon_9\}$, where $\varepsilon_9 = \varepsilon_7 \circ \eta_7 \circ \eta_8$ and hence $p_* \varepsilon_9 = 12\nu_6 \in \pi_9(S^6)$. $\pi_9(R_8) = 2 + 2 + 2 = \{\delta_9\} + \{\varepsilon_9\} + \{\zeta_9\}$, where $\zeta_9 = \zeta_7 \circ \eta_7 \circ \eta_8$. $\pi_9(R_9) = 2 + 2 = \{\delta_9\} + \{\zeta_9\}$. $\pi_9(R_{10}) = 2 + \infty = \{\delta_9\} + \{\xi_9\}$, where $p_* \xi_9 = 2\iota_9 \in \pi_9(S^9)$. $\pi_9(R_n) = 2 = \{\delta_9\}$ for $n \geqslant 11$.*

For R_3 and R_4, it follows immediately from $\pi_9(S^3) = 3 = \{\mu_3 \circ \mu_6\}$[1]. For the case R_5, in the homotopy sequence of the factorization $R_5/R_4 = S^4 : \pi_9(R_4) \xrightarrow{i_*} \pi_9(R_5) \xrightarrow{p_*} \pi_9(S^4) \xrightarrow{\Delta} \pi_8(R_4)$, $i_* \pi_9(R_4) = i_*(\{\alpha_3 \circ \mu_3 \circ \mu_6\} + \{\beta_3 \circ \mu_3 \circ \mu_6\}) = \{i_*(\alpha_3 \circ \mu_3) \circ \mu_6\} + \{i_*(\beta_3 \circ \mu_3) \circ \mu_6\} = 0$ by [5, 2.6] and Δ is isomorphic onto, and hence $\pi_9(R_5) = 0$. $\pi_9(R_6) = 2$ is followed immediately from $\pi_8(R_5) = \pi_9(R_5) = 0$, and moreover $\delta_9 = \delta_8 \circ \eta_8$. For R_7, $i_* : \pi_9(R_6) \to \pi_9(R_7)$ is isomorphic into as $\pi_{10}(S^6) = 0$, and the image of $p_* : \pi_9(R_7) \to \pi_9(S^6)$ is the subgroup $\{12\nu_6\} = 2$ of $\pi_9(S^6)$ and, moreover, the element $\varepsilon_9 = \varepsilon_7 \circ \eta_7 \circ \eta_8$ has the properties that it is of order two and $p_* \varepsilon_9 = (p_* \varepsilon_7) \circ \eta_7 \circ \eta_8 = \eta_6 \circ \eta_7 \circ \eta_8 = 12\nu_6$. Therefore $\pi_9(R_7) = 2 + 2$, and $\pi_9(R_8) = 2 + 2 + 2$ as R_8 is equivalent to the product $S^7 \times R_7$. $\pi_9(R_9) = 2 + 2$ is followed from the fact that $i_* : \pi_9(R_8) \to \pi_9(R_9)$ is onto and its kernel is equal to $T_{9*}\pi_9(S^7) = \{(-\varepsilon_7 + 2\zeta_7) \circ \eta_7 \circ \eta_8\} = \{\varepsilon_9\}$. In the homotopy sequence: $\pi_9(R_9) \xrightarrow{i_*} \pi_9(R_{10}) \xrightarrow{p_*} \pi_9(S^9) \to \pi_8(R_9) \xrightarrow{i_*^8} \pi_8(R_{10})$, kernel $i_*^9 = T_{10*}\pi_9(S^8) = \{(a\delta_8 + \zeta_8) \circ \eta_8\} = \{a\delta_9 + \zeta_9\}$ where $a = 0$ or 1, and hence image $i_*^9 = 2$. On the other hand, image $p_* = \infty = \{2\iota_9\} \subset \pi_9(S^9)$ as kernel $i_*^8 = 2$, and so $\pi_9(R_{10}) = 2 + \infty$. Moreover we can take as a generator ξ_9 of this infinite cyclic part the element represented by the characteristic map $T_{11} : S^9 \to R_{10}$, because $pT_{11} : S^9 \to S^9$ represents $2\iota_9$, [4, 23.4]. $\pi_9(R_{11}) = 2$ is followed immediately from the fact that the kernel of $i_* : \pi_9(R_{10}) \to \pi_9(R_{11})$ is equal to $T_{11*}\pi_9(S^9) = \{\xi_9\}$.

4. Furthermore, $\pi_r(R_n)$ can be calculated partly for $r = 10$ and 11.

Proposition 4. *$\pi_{10}(R_3) = 15 = \{\alpha_{10}\}$, and $\pi_{10}(R_4) = 15 + 15 = \{\alpha_{10}\} + \{\beta_{10}\}$, where $\alpha_{10} = \alpha_3 \circ \lambda_3^{10}$ and $\beta_{10} = \beta_3 \circ \lambda_3^{10}$*[2]*. $\pi_{10}(R_5) = 15 + 8 = \{\beta_{10}\} + \{\gamma_{10}\}$, where $p_* \gamma_{10} = 3\nu_4 \circ \nu_7 \in \pi_{10}(S^4)$*[2] *and $2\beta_{10} = \alpha_{10}$ in $\pi_{10}(R_5)$. $\pi_{10}(R$*

1) Where μ_3 is a generator of $\pi_6(S^3) = 6$ and $\mu_{3+p} = E^p\mu_3$, cf. [3], Théorème 1.

2) By [3, Théorème 1], $\pi_{10}(S^3) = 15 = \{\lambda_3^{10}\}$, $\pi_{11}(S^3) = 2 = \{\lambda_3^{11}\}$ and $\pi_{10}(S^4) = 3 + 24 = \{\mu_4 \circ \mu_7\} + \{\nu_4 \circ \nu_7\}$.

$= 15 + 8 + 2 = \{\beta_{10}\} + \{\gamma_{10}\} + \{\delta_{10}\}$, *where* $\delta_{10} = \delta_8 \circ \eta_8 \circ \eta_9$ *and hence* $p_* \delta_{10} = \nu_5 \circ \eta_8 \circ \eta_9 \in \pi_{10}(S^5)$. $\pi_{10}(R_7) = A + 8 + 2 = \{\bar{\beta}_{10}\} + \{\gamma_{10}\} + \{\delta_{10}\}$[1], *where* $A = 3$ *or* 0. $\pi_{10}(R_8) = A + 8 + 2 + 24 = \{\bar{\beta}_{10}\} + \{\gamma_{10}\} + \{\delta_{10}\} + \{\zeta_{10}\}$, *where* $\zeta_{10} = \zeta_7 \circ \nu_7$. $\pi_{10}(R_9) = A + 2 + 8 = \{\bar{\beta}_{10}\} + \{\delta_{10}\} + \{\bar{\zeta}_{10}\}$, $\pi_{10}(R_{11}) = A + 2 + 4$ $\{\bar{\beta}_{10}\} + \{\delta_{10}\} + \{\bar{\bar{\zeta}}_{10}\}$, $\pi_{10}(R_{11}) = A + 2 + 2 = \{\bar{\beta}_{10}\} + \{\delta_{10}\} + \{\bar{\bar{\bar{\zeta}}}_{10}\}$, *and* $\pi_{10}(R_n) = A + 2 = \{\bar{\beta}_{10}\} + \{\delta_{10}\}$ *for* $n \geqslant 12$.

Proposition 5. $\pi_{11}(R_3) = 2 = \{\alpha_{11}\}$ *and* $\pi_{11}(R_4) = 2 + 2 = \{\alpha_{11}\} + \{\beta_{11}\}$, *where* $\alpha_{11} = \alpha_3 \circ \lambda_3^{11}$ *and* $\beta_{11} = \beta_3 \circ \lambda_3^{11}$[1]. $\pi_{11}(R_5) = 2 = \{\beta_{11}\}$ *and* $\pi_{11}(R_6)$ *is equal to* i) $2 + 2 = \{\beta_{11}\} + \{\delta_{11}\}$ *or* ii) $4 = \{\delta_{11}\}$, *where* $\delta_{11} = \delta_8 \circ \nu_8$ *and hence* $p_* \delta_{11} = \nu_5 \circ \nu_8$ *a generator of* $\pi_{11}(S^5)$. $\pi_{11}(R_7) = B + 2 + \infty = \{\bar{\beta}_{11}\} + \{\bar{\delta}_{11}\} + \{\varepsilon_{11}\}$, *where* B *is equal to* i) 2 *or* ii) 0 *and* $p_* \varepsilon_{11} = a_0[\iota_6, \iota_6] \in \pi_{11}(S^6)$ *and* $a_0 = 5$ *or* 15 *according to* $A = 3$ *or* 0 *respectively.* $\pi_{11}(R_7) = \pi_{11}(R_n)$ *for* $8 \leqslant n \leqslant 11$ *and* $n \geqslant 13$, *and* $\pi_{11}(R_{12}) = B + 2 + \infty + \infty = \{\bar{\beta}_{11}\} + \{\bar{\delta}_{11}\} + \{\varepsilon_{11}\} + \{\chi_{11}\}$, *where* $p_* \chi_{11} = 2\iota_{11} \in \pi_{11}(S^{11})$.

We follow proofs briefly. For $r = 10$ and 11, as $E : \pi_r(S^3) \to \pi_{r+1}(S^4)$ is isomorphic onto, the kernel of $i_* : \pi_r(R_4) \to \pi_r(R_5)$ is equal to $T_{5*} \pi_r(S^3) = \{(-\alpha_3 + 2\beta_3) \circ \lambda_3^r\} = -\alpha_r + 2\beta_r$ by Lemma 2. For $\pi_{10}(R_7)$, it can be shown that $\pi_{11}(W_{11}) = \infty + 2$ whose infinite cyclic part is isomorphic onto $\pi_{11}(S^6)$ by the induced map of natural projection, where $W_{11} = R_7 / R_5$ is the vector bundle over S^6, and hence the image of $\Delta : \pi_{11}(S^6) \to \pi_{10}(R_6)$ is equal to the image of $i_* \Delta : \pi_{11}(W_{11}) \to \pi_{10}(R_5) \to \pi_{10}(R_6)$. It is known that the latter subgroup contains 5-cyclic group [3, Proposition 17.3], and therefore $\pi_{10}(R_7) = A + 8 + 2$. $i_* : \pi_{10}(R_8) \to \pi_{10}(R_9)$ is onto and its kernel is equal to $T_{9*} : \pi_{10}(S^7) = \{(-\varepsilon_7 + 2\zeta_7) \circ \nu_7\} = \{-\bar{\gamma}_{10} + a_1 \delta_{10} + 2\zeta_{10}\}$. $i_* : \pi_{10}(R_9) \to \pi_{10}(R_{10})$ is also onto and its kernel is equal to $a\delta_{10} + 4\bar{\zeta}_{10}$, where $a = 0$ or 1, and hence $\pi_{10}(R_{10}) = A + 2 + 4$ or $A + 8$ corresponding to $a = 0$ or 1 respectively. The kernel of $i_* : \pi_{10}(R_{10}) \to \pi_{10}(R_{11})$ is equal 2 or 0; and the kernel of $i_* : \pi_{10}(R_{11}) \to \pi_{10}(R_{12})$ is generated by the element represented by T_{12} which is homotopic to $T_3'' : S^{10} \to R_8$, and T_3'' satisfies the property that $pT_3'' : S^{10} \to S^7$ is the suspension of the map $S^7 \to S^4$ with Hopf invariant 1[3], and hence T_{12} represents the image of $\zeta_{10} + a_2 \delta_{10} + a_3 \beta_{10}$, where $a_2 = 0$ or 1 and $a_3 = 0$ or 1 or 2. On the other

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1) Cf. footonote 2) of p. 131.

2) We denote by $\bar{\alpha}$ the element $i_* \alpha$, where $i_* : \pi_r(R_n) \to \pi_r(R_{n+1})$.

3) $T''_{2k+1} : S^{n-1} \to R_{8k}$ $(n = 8k + 3)$ is same to $\phi_0 \mid S^{n-1}$ being homotopic to $H(w)$ of [7, §3, 1], and $H(w)$ represents the suspension of an element of $\pi_7(S^4)$ whose Hopf invariant is odd. In proofs of latter fact, if $k = 2$, it can easily seen that $H(w)$ represents the suspension of an element of Hopf invariant 1.

hand, as the element $\chi_{11} \in \pi_{11}(R_{12})$ represented by T_{13} satisfies $p_* \chi_{11} = 2\iota_{11} \in \pi_{11}(S^{11})$, the kernel of $i_* : \pi_{10}(R_{11}) \to \pi_{10}(R_{12})$ has at most order two. These properties show that $a = a_3 = 0$ and Prop. 4.

From Lemma 1, it follows that the composition of $\pi_{13}(S^6) \xrightarrow{\Delta} \pi_{12}(R_6) \xrightarrow{p_*} \pi_{12}(S^5)$ is onto, and hence $i_* : \pi_{11}(R_5) \to \pi_{11}(R_6)$ is isomorphic into. The kernel of $i_* : \pi_{11}(R_6) \to \pi_{11}(R_7)$ is equal to $\{\delta_5 \circ \nu_5 \circ \nu_8\} = 2\delta_{11} =$ i) 0 or ii) $\{\beta_{11}\}$. $i_* : \pi_{11}(R_n) \to \pi_{11}(R_{n+1})$ is isomorphic into, as its kernel is equal to $\zeta_8 \circ \gamma_8 = 0$ for $n = 9$, $\xi_9 \circ \eta_9 \circ \eta_{10} = 2\bar{\bar{\zeta}}_{10} \circ \eta_{10} = 0$ for $n = 10$, and as the element represented by $T_7' : S^{12} \to R_{12}$ is transformed to $\eta_{11} \in \pi_{12}(S^{11})$ by $p_* : \pi_{12}(R_{12}) \to \pi_{12}(S^{11})$ for $n = 11$.

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