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SOME REMARKS ON RADICAL IDEALS

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Let R be a (non-commutative) ring. As is well known, every ideal A in R decides a uniquely determined ideal \bar{A} called the *radical* of A , which is defined as the intersection of all (minimal) prime divisors of A . Clearly, the operation: $A \rightarrow \bar{A}$ defined in the set \mathfrak{I} consisting of all ideals in R possesses the following properties:

- 1) $A \subseteq \bar{A}$,
- 2) $\bar{\bar{A}} = \bar{A}$,
- 3) $A \subseteq B$ implies $\bar{A} \subseteq \bar{B}$, where $A, B \in \mathfrak{I}$.

In this note, we consider several properties of ideals A with $\bar{A} = A$, which are called *radical ideals* by N. H. McCoy [3]¹⁾.

Theorem 1. *Let C be an ideal in R . Then the following conditions are equivalent to each other:*

- a) C is a radical ideal.
- b) If an ideal A is nilpotent modulo C , then A is contained in C .
- c) $AB \subseteq C$ implies $A \cap B \subseteq C$, where A, B are ideals.
- d) C is an intersection of some prime ideals.

Proof. Cor. 4 to Th. 2 of [2] shows that the intersection of all the prime ideals of a ring is $\{0\}$ if and only if the zero ideal is the only nilpotent ideal of the ring. Hence, the equivalence between a) and b) may be easily seen.

Let C be a radical ideal. Then, for each prime divisor P of C , $AB \subseteq C$ implies $P \supseteq A$ or B . Hence $A \cap B \subseteq P$, accordingly, $A \cap B \subseteq C$. This shows that a) implies c). Conversely, C be not a radical ideal. Then, by b), there exists an ideal $A \not\subseteq C$ such that $A^2 \subseteq C$. Thus, c) does not hold.

The equivalence of d) to a) is also easy.

In general, let $A \rightarrow A^*$ be a operation defined in \mathfrak{I} which satisfies the following axioms:

- 1') $A \subseteq A^*$,
- 2') $(A^*)^* = A^*$,
- 3') $B \subseteq A$ implies $B^* \subseteq A^*$, where A, B are in \mathfrak{I} .

For arbitrary ideals A, B in R we define the join $A \cup B$ as the ideal

1) Numbers in brackets refer to the references cited at the end of this note. The term "ideal" will mean a two-sided ideal.

$(A + B)^*$, and the meet $A \cap B$ as the intersection of A and B . We set $\mathfrak{C} = \{A \in \mathfrak{S} \mid A^* = A\}$.

Lemma. \mathfrak{C} forms a distributive lattice with respect to the above-defined join and meet if one of the following (trivially equivalent) conditions is satisfied:

- a) For any $C \in \mathfrak{C}$, $(AB)^* \subseteq C$ implies $A^* \cap B^* \subseteq C$, where $A, B \in \mathfrak{S}$.
- b) $(AB)^* = A^* \cap B^*$, where $A, B \in \mathfrak{S}$.

Proof. As our operation $*$ satisfies the axioms 1'), 2') and 3') we have $(A^* + B^*)^* = (A + B)^*$. Let A, B and C be in \mathfrak{C} , then $(A \cap B) \cup (A \cap C) = ((A \cap B) + (A \cap C))^* = ((AB)^* + (AC)^*)^* = (AB + AC)^* = (A(B + C))^* = A \cap (B + C)^* = A \cap (B \cup C)$.

Theorem 2¹⁾. The set consisting of all radical ideals in a ring R forms a distributive lattice, where the join and meet are defined as in the above lemma.

Proof. Let C be a radical ideal. Then $\overline{(AB)} \subseteq C$ implies $AB \subseteq C$. For any prime divisor P of C , $AB \subseteq C$ implies A or $B \subseteq P$, accordingly \bar{A} or $\bar{B} \subseteq P$. Hence, we have $\bar{A} \cap \bar{B} \subseteq P$, that is, $\bar{A} \cap \bar{B} \subseteq C$. This fact shows that our operation: $A \rightarrow \bar{A}$ satisfies the condition a) of the lemma.

Next, we prove the following theorem which has been proved in the commutative case by S. Mori [4, Satz 1]:

Theorem 3. The maximum condition is satisfied for radical ideals in R if and only if the following conditions are satisfied:

- 1) The maximum condition is satisfied for prime ideals in R .
- 2) Every radical ideal is represented as the intersection of a finite number of prime ideals in R .

Proof. Necessity: As 1) is a special case of our assumption it is desired only to prove 2).

Let $C = \cap P_\sigma$ be a radical ideal, where P_σ is a minimal prime divisor of C . Here, we may assume that C is not a prime ideal. Then, there exist two ideals A, B not contained in C such that $AB \subseteq C$. We can easily see that $R \supset C_1 = CB^{-1(2)} = \cap (P_\sigma B^{-1}) = \cap P_\sigma \supset C$, where $\{P_\sigma\}$ is a subset of $\{P_\sigma\}$. Clearly C_1 is a radical ideal.

1) Theorem 2 has been proved for commutative rings by J.-C. Herz [1].

2) $CB^{-1}[B^{-1}C]$ is defined as the totality of elements x such that $xB \subseteq C[Bx \subseteq C]$. The following properties of quotients are easily verified (see [5]):

- (1) $(AB^{-1})C^{-1} = A(CB)^{-1}$.
- (2) $(\cap A_\sigma)B^{-1} = \cap A_\sigma B^{-1}$.

(3) For any prime ideal P , $PA^{-1} = P$ or R itself in accordance with $A \not\subseteq P$ or $A \subseteq P$ respectively.

If C_1 is not prime, then we repeat the above procedure for C_1 instead of C and obtain $C_2 = CB_1^{-1}$, and so on. From our assumption, our procedures must terminate after a finite number of steps. Hence we have a prime ideal $(R \supset) C_n = CB_{n-1}^{-1}$, where B_{n-1} is not contained in C_n as C is a radical ideal. We now consider all different prime ideals $P'_\mu (\subset R)$ of the form: $P'_\mu = CU'_\mu^{-1}$. As C is a radical ideal U'_μ is not contained in P'_μ , and P'_μ is a minimal prime divisor of C . If $\mu \neq \lambda$, then $U'_\mu \subseteq P'_\lambda$. For, if not, $U'_\mu \not\subseteq P'_\lambda$ and $P'_\mu U'_\mu \subseteq C \subseteq P'_\lambda$ imply that $P'_\mu \subseteq P'_\lambda$. As P'_λ is a minimal prime divisor of C , $P'_\mu = P'_\lambda$, contradicting with our assumption. The fact proved now shows that the representation $D_1 = \cap P'_\mu$ is irredundant. The maximum condition for radical ideals implies that the set $\{P'_\mu\}$ is finite.

By the same manner as in the above, we can see that the set of all different prime ideals $P''_\nu = U''_\nu^{-1}C$ with $U''_\nu \not\subseteq P''_\nu$ is finite, and we set $D_2 = \cap P''_\nu$.

Clearly, $D = D_1 \cap D_2$ contains C . We shall prove now that $D = C$. If $D \supset C$, as $U''_\nu D \subseteq C$ and $U''_\nu \not\subseteq C$, we can construct a prime ideal $(R \supset) P' = CU'^{-1}$ with $U' \not\subseteq P'$, $\subseteq D$ by using the same argument as in the first part of the proof. Hence, for some μ , $P' = P'_\mu$. On the other hand, $U' \subseteq D$ implies that $U' \subseteq P'_\mu = P'$. This is a contradiction.

Sufficiency: We assume now the conditions 1) and 2). Let $C_1 \subset C_2 \subset \dots$ be an infinite ascending chain of radical ideals, where C_i has a short representation $P_{i_1} \cap \dots \cap P_{i_{n_i}}$ with its minimal prime divisors P_{i_j} . Then each P_{i_j} ($j = 1, \dots, n_i$) is a divisor of some $P_{i-1, k}$ ($k = 1, \dots, n_{i-1}$). Now, we call an ascending chain $(R \not\subseteq) Q_1 \subseteq Q_2 \subseteq \dots$ of which each Q_i is some P_{i_j} or R itself a branch of the chain $C_1 \subset C_2 \subset \dots$, and Q_1 the starting point of the branch. In case $Q_1 \subseteq Q_2 \subseteq \dots \subseteq Q_n \subset Q_{n+1} = R$ we say that the length of the branch is n . In the other case, it is infinite. A prime divisor P_{i_j} of C_i is called trivial if P_{i_j} is some P_{hk} for each $h > i$. If the lengths of all branches with the starting point P_{i_j} of the sub-chain $C_i \subset C_{i+1} \subset \dots$ are bounded, we say that P_{i_j} is finite, and in the other case it is infinite.

Here, without loss of generality, we may assume that each P_{i_j} is non-trivial. As $C_1 \subset C_2 \subset \dots$ is infinite, there exists at least one infinite P_{r_s} for each r .

1) If an ideal is represented as the intersection of a finite number of prime ideals, then there exists a unique short representation [5, Theorem 3].

Let P_{1,t_1} be infinite. Then there exists a branch $Q_1 = \dots = Q_{m_1-1} \subset Q_{m_1} \subset \dots$ with the starting point P_{1,t_1} such that $Q_{m_1} \neq R$. As P_{m_1,t_1} is a minimal prime divisor of C_{m_1} , for each branch $Q'_1 \subset Q'_2 \subset \dots$ with the starting point $Q'_1 = P_{1,t_1}$, we have $Q'_1 \subset Q'_{m_1}$. Hence, there exists at least one infinite P_{m_1,t_2} properly containing P_{1,t_1} . We can repeat the above argument for P_{m_1,t_2} instead of P_{1,t_1} and obtain an infinite P_{m_2,t_3} properly containing P_{m_1,t_2} , and so on. Thus, we obtain an infinite ascending chain of prime ideals $P_{1,t_1} \subset P_{m_1,t_2} \subset P_{m_2,t_3} \subset \dots$. But this is a contradiction.

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