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ON THE CARTAN INVARIANTS OF ALGEBRAS

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1. Let A be an algebra with unit element over an algebraically closed field K and let

$$A = A^* + N \tag{1}$$

be a splitting of A into a direct sum of a semisimple subalgebra A^* and the radical N of A. We shall denote by

$$A^* = A_1^* + A_2^* + \cdots\cdots + A_k^* \tag{2}$$

the unique splitting of A^* into a direct sum of simple invariant subalgebras. Let $e_{i,\alpha\beta}(\alpha, \beta = 1, 2, \cdots\cdots, f(i))$ be a set of matrix units for the simple algebra A_i^*. We denote by $F_1, F_2, \cdots\cdots, F_k$ the distinct irreducible representations of A and we set for a in A

$$F_i(a) = (f_{\alpha\beta}^i(a)). \tag{3}$$

Let

$$e_{i_u,\alpha 1} b_u e_{j_u,1\beta} \qquad \begin{aligned} u &= 1, 2, \cdots\cdots, t \\ \alpha &= 1, 2, \cdots\cdots, f(i_u) \\ \beta &= 1, 2, \cdots\cdots, f(j_u) \end{aligned} \tag{4}$$

be the Cartan basis[1] of A. An element a of A, expressed in terms of the Cartan basis elements will have the form

$$a = \sum_{u,\alpha\beta} h_{\alpha\beta}^u(a) e_{i_u,\alpha 1} b_u e_{j_u,1\beta}. \tag{5}$$

For a fixed u, we arrange the coefficients $h_{\alpha\beta}^u(a)$ in a matrix $H_u(a) = (h_{\alpha\beta}^u(a))$. The additive group $H_u(a)$ is called an elementary module of A. In particular, for $b_i = e_{i,11}$ we have $H_i(a) = F_i(a)$, that is,

$$h_{\alpha\beta}^i(a) = f_{\alpha\beta}^i(a) \qquad (i = 1, 2 \cdots\cdots k). \tag{6}$$

Let $d_1, d_2, \cdots\cdots, d_n$ be a basis (d_s) of A. Then

$$d_s = \sum_{u,\alpha\beta} h_{\alpha\beta}^u(d_s) e_{i_u,\alpha 1} b_u e_{j_u,1\beta} \tag{7}$$

1) See Nesbitt [3], Scott [5].

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or in matrix form

$$(8) \qquad (d_s) = (e_{i_u,\alpha 1} b_u e_{j_u, 1\beta})(h^u_{\alpha\beta}(d_s))$$

(u, α, β row index: s column index). Since (d_s) is a basis of A, $(h^u_{\alpha\beta}(d_s))$ is a non-singular matrix. Hence we have

Lemma 1. *If (d_s) is a basis of A, then $h^u_{\alpha\beta}(d_s)$ $(u = 1, 2, \cdots\cdots, t;$ $\alpha = 1, 2, \cdots\cdots, f(i_u);$ $\beta = 1, 2, \cdots\cdots, f(j_u))$ are linearly independent.*

In particular, we obtain from (6)

Lemma 2. *If (d_s) is a basis of A, then $f^i_{\alpha\beta}(d_s)$ $(i = 1, 2, \cdots\cdots, k;$ $\alpha, \beta = 1, 2, \cdots\cdots, f(i))$ are linearly independent.*

We denote by χ_i the character of F_i. Then $\chi_i(a) = \sum_\alpha f^i_{\alpha\alpha}(a)$. By Lemma 2

Theorem 1. *Let (d_s) be a basis of A. Then $\chi_1(d_s), \chi_2(d_s), \cdots\cdots, \chi_k(d_s)$ are linearly independent.*

Now we can prove the following theorem by a procedure similar to that of Brauer and Nesbitt[1].

Theorem 2. *Let M_1 and M_2 be two representations of A. If both $M_1(d_s)$ and $M_2(d_s)$ have the same characteristic roots for every d_s of a basis (d_s), then M_1 and M_2 have the same irreducible constituents: $M_1 \leftrightarrow M_2$.*

2. In this section we assume that A is an algebra with unit element over an algebraic number field K and that the irreducible representations $Z_1, Z_2, \cdots\cdots, Z_k$ of A in K are absolutely irreducible. Let J be a domain of integrity in the algebra A in the following sense[2]: (1) J is a subring of A; (2) J contains n linearly independent elements of A; (3) the elements of J when expressed by a basis $\varepsilon_1, \varepsilon_2, \cdots\cdots, \varepsilon_n$ of A have the form $\sum a_i \varepsilon_i$ with $a_i = b_i / w$ where w is a fixed denominator in K and b_i are integers of K; (4) J contains the ring $\mathfrak{o}$ of all integers of K. Every ideal $\mathfrak{m}$ of $\mathfrak{o}$ generates the ideal of J which may be denoted by $\mathfrak{m}$ again. Let $\mathfrak{p}$ be a fixed prime ideal of $\mathfrak{o}$. We denote by $\mathfrak{o}^*$ the ring of all $\mathfrak{p}$-integers of K. Then $\mathfrak{o}^*$ and J generate a subring J^* of A. J^* has a basis $\eta_1, \eta_2, \cdots\cdots, \eta_n$ such that every α of J^* can be written uniquely in the form

1) Cf. Brauer and Nesbitt [2], p. 3.
2) See Brauer [1].

(9) $$\alpha = c_1\eta_1 + c_2\eta_2 + \cdots\cdots + c_n\eta_n, \quad c_i \text{ in } \mathfrak{o}^*.$$

The η_i can be chosen in J. The ideal $\mathfrak{p}$ generates an ideal of $\mathfrak{o}^*$ and an ideal of J^*, both of which will be denoted by $\mathfrak{p}^*$. We denote the residue class of an element α of J^* (mod $\mathfrak{p}^*$) by $\bar{\alpha}$. We have

(10) $$\bar{\mathfrak{o}} = \mathfrak{o}^* / \mathfrak{p}^* \cong \mathfrak{o} / \mathfrak{p}; \qquad \bar{A} = J^* / \mathfrak{p}^* \cong J / \mathfrak{p}$$

for the residue class field and residue class algebra. The elements $\bar{\eta}_1, \bar{\eta}_2, \cdots\cdots, \bar{\eta}_n$ form a basis of $\bar{A}$ with regard to $\bar{\mathfrak{o}}$. Let $S(\alpha)$ and $R(\alpha)$ be the left and the right regular representations of A, formed by means of the basis (η_i). Every α of J^* is then represented by matrices $S(\alpha)$ and $R(\alpha)$ with coefficients in $\mathfrak{o}^*$. Hence $\bar{\alpha} \to S(\bar{\alpha})$ and $\bar{\alpha} \to R(\bar{\alpha})$ give the left and the right regular representations of $\bar{A}$, formed by means of the basis $(\bar{\eta}_i)$. We denote by $F_1, F_2, \cdots\cdots, F_m$ the distinct absolutely irreducible representations of $\bar{A}$. Let us assume here that all F_κ lie already in $\bar{\mathfrak{o}}$. Then we have[1]

(11) $$S(\alpha)R'(\beta) \leftrightarrow \sum_{i,j} c_{ij} Z_i(\alpha) \times Z'_j(\beta),$$

(12) $$S(\bar{\alpha})R'(\bar{\beta}) \leftrightarrow \sum_{\kappa,\lambda} c^*_{\kappa\lambda} F_\kappa(\bar{\alpha}) \times F'_\lambda(\bar{\beta})$$

where c_{ij} and $c^*_{\kappa\lambda}$ denote the Cartan invariants of A and $\bar{A}$ respectively. We may assume that Z_i represents the elements of J^* by matrices with coefficients in $\mathfrak{o}^*$. Then $Z_i(\bar{\alpha})$ gives a representation of $\bar{A}$. Let $d_{i\kappa}$ denote the multiplicity of $F_\kappa(\bar{\alpha})$ in $Z_i(\bar{\alpha})$:

(13) $$Z_i(\bar{\alpha}) \leftrightarrow \sum_\kappa d_{i\kappa} F_\kappa(\bar{\alpha}).$$

The $d_{i\kappa}$ are called the decomposition numbers of A.

Theorem 3. *Let c_{ij}, $c^*_{\kappa\lambda}$ be the Cartan invariants of A and $\bar{A}$. Then*

$$c^*_{\kappa\lambda} = \sum_{i,j} d_{i\kappa} c_{ij} d_{j\lambda},$$

where $d_{i\kappa}$ are the decomposition numbers of A.

Proof. From (13) we have

$$\begin{aligned}\sum_{i,j} c_{ij} Z_i(\bar{\alpha}) \times Z'_j(\bar{\beta}) &\leftrightarrow \sum_{i,j} c_{ij} (\sum_\kappa d_{i\kappa} F_\kappa(\bar{\alpha})) \times (\sum_\lambda d_{j\lambda} F'_\lambda(\bar{\beta})) \\ &= \sum_{\kappa,\lambda} (\sum_{i,j} d_{i\kappa} c_{ij} d_{j\lambda}) F_\kappa(\bar{\alpha}) \times F'_\lambda(\bar{\beta}).\end{aligned}$$

1) See Osima [4].

By (11), $S(\bar{\alpha})R'(\bar{\beta})$ and $\sum_{i,j} c_{ij}Z_i(\bar{\alpha}) \times Z'_j(\bar{\beta})$ have the same characteristic roots for every $\bar{\alpha}$ and $\bar{\beta}$. Hence it follows from Theorem 2 that

$$S(\bar{\alpha})R'(\bar{\beta}) \leftrightarrow \sum_{i,j} c_{ij}Z_i(\bar{\alpha}) \times Z'_j(\bar{\beta}). \tag{14}$$

Consequently we have from (12)

$$\sum_{\kappa,\lambda} c^*_{\kappa\lambda}F_\kappa(\bar{\alpha}) \times F'_\lambda(\bar{\beta}) \leftrightarrow \sum_{\kappa,\lambda}(\sum_{i,j} d_{i\kappa}c_{ij}d_{j\lambda})F_\kappa(\bar{\alpha}) \times F'_\lambda(\bar{\beta}),$$

so that we obtain

$$c^*_{\kappa\lambda} = \sum_{i,j} d_{i\kappa}c_{ij}d_{j\lambda}.$$

We set $C = (c_{ij})$, $D = (d_{i\kappa})$ and $C^* = (c^*_{\kappa\lambda})$. Then

$$C^* = D'CD. \tag{15}$$

This shows that if C is a symmetric matrix, then C^* is also symmetric. If A is semisimple, then C is a unit matrix. Hence, from (15) we obtain

$$C^* = D'D. \tag{16}$$

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