Mathematical Journal of Okayama University

Volume 2, *Issue* 2 2008 *Article* 2

MARCH 1953

Notes on Basic Rings

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NOTES ON BASIC RINGS

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Every ring R with minimum condition on left ideals can be completely described by means of a certain semi-primitive subring S and a number of positive integers $f_1, f_2, \cdots\cdots, f_n$. By Brauer (see [6]), S is called the basic ring of R. In this paper, we shall define the basic rings for rings with milder condition and prove that the main properties of basic rings are also valid in our general case (Theorems 1 and 3 below).

1. Let R be an arbitrary ring. An element e of R is called an idempotent if $e^2 = e$. Two idempotents e and f are called *isomorphic* in R if there exist two elements a and b such that $ab = e$ and $ba = f$. We write then $e \cong f$. This is evidently a reflexive, symmetric, and transitive relation, by means of which the idempotents are classified into disjoint classes of isomorphic elements. A class containing an idempotent e is denoted by $C(e)$. We may assume in the above definition that $a \in eRf$ and $b \in fRe$ (see [1]).

Two idempotents e and f are isomorphic if and only if the left ideals Re and Rf are R-isomorphic. Further if $e \cong f$, then two subrings eRe and fRf are isomorphic under the mapping $x \to bxa$ ($x \in eRe$), where $e = ab$, $f = ba$ and $a \in eRf$, $b \in fRe$.

Lemma 1. *Let $e = e_1 + e_2 + \cdots\cdots + e_t$ be a decomposition of an idempotent e into a sum of t orthogonal idempotents e_i. Similarly let $f = f_1 + f_2 + \cdots\cdots + f_t$. If $e_i \cong f_i$ for every i, then $e \cong f$.*

Proof. Since $e_i = p_i q_i$ and $f_i = q_i p_i$ ($p_i \in e_i R f_i$, $q_i \in f_i R e_i$), we have $e = pq$ and $f = qp$, where $p = \sum p_i$ and $q = \sum q_i$.

In what follows we assume that R contains an identity 1. Let

$$(1) \qquad 1 = \sum_{i=1}^{n} \sum_{\alpha=1}^{f(i)} e_{i,\alpha}$$

be a decomposition of the identity into a sum of orthogonal idempotents $e_{i,\alpha}$ such that all $e_{i,\alpha}$ with the same first subscript are isomorphic and no two $e_{i,\alpha}$'s with different subscripts are isomorphic. Since $e_{i,1} \cong e_{i,\alpha}$, there exist two elements $c_{i,1\alpha} \in e_{i,1} R e_{i,\alpha}$ and $c_{i,\alpha 1} \in e_{i,\alpha} R e_{i,1}$ such that

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$$e_{i,1} = c_{i,1\alpha}c_{i,\alpha 1}, \quad e_{i,\alpha} = c_{i,\alpha 1}c_{i,1\alpha}.$$

We may set $c_{i,11} = e_{i,1}$. Further, if we put

$$(2) \qquad c_{i,\alpha\beta} = c_{i,\alpha 1}c_{i,1\beta},$$

then $c_{i,\alpha\beta} \in e_{i,\alpha}Re_{i,\beta}$ and

$$e_{i,\alpha} = c_{i,\alpha\beta}c_{i,\beta\alpha}, \quad e_{i,\beta} = c_{i,\beta\alpha}c_{i,\alpha\beta}.$$

From (2) we have $c_{i,\alpha\alpha} = e_{i,\alpha}$. We see easily that

$$(3) \qquad c_{i,\alpha\beta}c_{j,\kappa\lambda} = \delta_{ij}\delta_{\beta\kappa}c_{i,\alpha\lambda}.$$

Hence the $c_{i,\alpha\beta}$ are matrix units for a fixed i, and so $\{c_{i,\alpha\beta}\}$ is called a set of matrix units corresponding to the decomposition (1). It follows from $e_{i,\alpha}Re_{j,\beta} = c_{i,\alpha 1}Rc_{j,1\beta}$ that

$$(4) \qquad R = \sum_{i,\alpha}\sum_{j,\beta} c_{i,\alpha 1}Rc_{j,1\beta}.$$

If we set $e_{i,1} = e_i$ and $\sum e_i = e$, then eRe is a subring of R with an identity e. We have from (4)

$$(5) \qquad R = \sum_{i,\alpha}\sum_{j,\beta} c_{i,\alpha 1}(eRe)c_{j,1\beta},$$

so that every element a in R is expressed uniquely as

$$(6) \qquad a = \sum_{i,\alpha}\sum_{j,\beta} c_{i,\alpha 1}(c_{i,1\alpha}ac_{j,\beta 1})c_{j,1\beta}.$$

Let R^* be a second ring with an identity 1^* and let

$$(7) \qquad 1^* = \sum_{i=1}^{n}\sum_{\alpha=1}^{f(i)} e^*_{i,\alpha}$$

be a decomposition of the identity 1^* into a sum of orthogonal idempotents $e^*_{i,\alpha}$ such that $e^*_{i,\alpha} \cong e^*_{i,1}$ and $e^*_{i,\alpha} \not\cong e^*_{j,\beta}$ if $i \neq j$. We set $e^*_{i,1} = e^*_i$ and $\sum e^*_i = e^*$. Suppose that two rings eRe and $e^*R^*e^*$ are isomorphic under the mapping $x \to x^\rho$ $(x \in eRe)$ such that $e_i^\rho = e^*_i$. Let $\{c^*_{i,\alpha\beta}\}$ be a set of matrix units corresponding to (7). It is easy to see that

$$(8) \qquad \begin{aligned} a &= \sum_{i,\alpha}\sum_{j,\beta} c_{i,\alpha 1}(c_{i,1\alpha}ac_{j,\beta 1})c_{j,1\beta} \\ &\to a^\varphi = \sum_{i,\alpha}\sum_{j,\beta} c^*_{i,\alpha 1}(c_{i,1\alpha}ac_{j,\beta 1})^\rho c^*_{j,1\beta} \end{aligned}$$

gives the isomorphism between R and R^*. Moreover we have $c^{\varphi}_{i,\alpha\beta} = c^{*}_{i,\alpha\beta}$ and $a^{\varphi} = a^{\rho}$ for every a in eRe.

Now let

$$(9) \qquad 1 = \sum_{i=1}^{n} \sum_{\alpha=1}^{f(i)} e'_{i,\alpha}$$

be a second decomposition of the identity of R into orthogonal idempotents $e'_{i,\alpha}$ such that $e'_{i,\alpha} \cong e_{i,\alpha}$ and let $\{c'_{i,\alpha\beta}\}$ be a set of matrix units corresponding to (9). We set $e'_{i,1} = e'_i$ and $\sum e'_i = e'$.

Lemma 2. *R contains a regular element s which satisfies*

$$e'Re' = s^{-1}(eRe)s.$$

Proof. By Lemma 1, $e_i = p_i q_i$, $e'_i = q_i p_i$ and $e = pq$, $e' = qp$, where $p = \sum p_i$ and $q = \sum q_i$. Since $e \cong e'$, eRe and $e'Re'$ are isomorphic under the mapping $x \to qxp$. Hence we see that the mapping

$$a = \sum_{i,\alpha} \sum_{j,\beta} c_{i,\alpha 1}(c_{i,1\alpha} a c_{j,\beta 1}) c_{j,1\beta}$$
$$\to a^{\varphi} = \sum_{i,\alpha} \sum_{j,\beta} c'_{i,\alpha 1}(q c_{i,1\alpha} a c_{j,\beta 1} p) c'_{i,1\beta}$$

gives an automorphism of R such that $c^{\varphi}_{i,\alpha\beta} = c'_{i,\alpha\beta}$ and $a^{\varphi} = qap$ for every a in eRe. We see that

$$a^{\varphi} = (\sum_{i,\alpha} c'_{i,\alpha 1} q c_{i,1\alpha})\, a\, (\sum_{j,\beta} c_{j,\beta 1} p c'_{j,1\beta}).$$

If we put $s = \sum_{j,\beta} c_{j,\beta 1} p c'_{j,1\beta}$ and $t = \sum_{i,\alpha} c'_{i,\alpha 1} q c_{i,1\alpha}$, then

$$st = \sum_{j,\beta} c_{j,\beta 1} p e'_j q c_{j,1\beta}$$
$$= \sum_{j,\beta} c_{j,\beta 1} c_{j,1\beta} = \sum_{j,\beta} e_{j,\beta} = 1.$$

Similarly $ts = 1$ and so $t = s^{-1}$. Hence $a^{\varphi} = s^{-1}as$. In particular, $qap = s^{-1}as$ for a in eRe and $c'_{i,\alpha\beta} = s^{-1}c_{i,\alpha\beta}s$.

Evidently two idempotents e and $t^{-1}et$ are isomorphic. The converse does not hold generally. But we have from Lemma 2

Lemma 3. *Let $1 = \sum_{i=1}^{m} f_i$ be a decomposition of the identity of R into a sum of orthogonal idempotents f_i. If $f_j \cong f_k$, then there exists a regular element t such that $f_k = t^{-1}f_j t$.*

2. In this section we assume that a ring R with an identity satisfies the following condition:

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(*) *R is decomposed into a direct sum of a finite number of indecomposable left ideals and this decomposition is unique up to R-isomorphism.*

For instance, if R is a ring with minimum condition on left ideals and with an identity, then R satisfies the condition (*).

If R satisfies the condition (*), the identity is decomposed into a sum of a finite number of orthogonal *primitive* idempotents:

$$1 = \sum_{i=1}^{n} \sum_{\alpha=1}^{f(i)} e_{i,\alpha}, \tag{10}$$

where $e_{i,\alpha} \cong e_{i,1}$ and $e_{i,\alpha} \not\cong e_{j,\beta}$ if $i \neq j$, and this decomposition is unique up to isomorphism. We put $e_{i,1} = e_i$.

Lemma 4. *The number of classes $C(e)$ of isomorphic idempotents is finite.*

Proof. Let g be an arbitrary idempotent. Then $1 = g + (1-g)$, where g and $1-g$ are orthogonal. Hence g is decomposed into a sum of orthogonal primitive idempotents: $g = \sum g_j$ and this decomposition is unique up to isomorphism. Moreover every g_j is isomorphic to one of e_i. We denote by s_i the number of g_j which are isomorphic to e_i. Then $0 \leqq s_i \leqq f(i)$. We say that an idempotent g is of type $(s_1, s_2, \cdots\cdots, s_n)$, or simply of type (s_i). Then we see that two isomorphic idempotents are of same type. The converse is also true. Hence a $(1 \cdot 1)$ correspondence between the classes $C(g)$ and the systems of positive integers $(s_1, s_2, \cdots\cdots, s_n)$, $(0 \leqq s_i \leqq f(i))$ is established. Thus the number of classes $C(g)$ is equal to $\prod_i (1 + f(i))$.

Let $C(e)$ and $C(f)$ be of type (s_i) and (t_i), and let $C(g)$ and $C(h)$ be of type $(\max(s_i, t_i))$ and $(\min(s_i, t_i))$. If we define the join and the meet of $C(e)$ and $C(f)$ by

$$C(e) \cup C(f) = C(g), \quad C(e) \cap C(f) = C(h),$$

then the classes of isomorphic idempotents form a lattice L and the structure of L is determined by a system of positive integers $(f(1), f(2), \cdots\cdots, f(n))$.

Let e be an idempotent of type $(1, 1, \cdots\cdots, 1)$. $R^0 = eRe$ is a subring of R with an identity e. R^0 is called the *basic ring* of R. The concept of basic ring first appeared in Nakayama [4] p. 617 (see [5] p. 335 too). The basic ring R^0 is uniquely determined up to isomorphism.

From (5) we have

(11) $$R = \sum_{i,\alpha} \sum_{j,\beta} c_{i,\alpha 1} R^0 c_{j,1\beta}.$$

By the results obtained in section 1, we have immediately

Theorem 1. *The structure of every ring R which satisfies the condition* (*) *is completely determined by its basic ring R^0 and a system of positive integers $(f(1), f(2), \cdots\cdots, f(n))$.*

Theorem 2. *Let R be a ring which satisfies the condition* (*). *Suppose that*

$$1 = \sum_{i=1}^{n} \sum_{\alpha=1}^{f(i)} e_{i,\alpha} = \sum_{i=1}^{n} \sum_{\alpha=1}^{f(i)} e'_{i,\alpha}$$

are two decompositions of the identity into a sum of orthogonal primitive idempotents such that $e_{i,\alpha} \cong e_{i,1}$, $e_{i,\alpha} \cong e_{j,\beta}$ and $e_{i,\alpha} \cong e'_{i,\alpha}$. If $\{c_{i,\alpha\beta}\}$ and $\{c'_{i,\alpha\beta}\}$ are two systems of matrix units corresponding to above decompositions of the identity, then there exists a regular element s such that $c'_{i,\alpha\beta} = s^{-1}c_{i,\alpha\beta}s$.

Corollary 1. *Let $a \to a^\sigma$ be an automorphism of R and let $e_i^\sigma \cong e_{\pi(i)}$, where $\pi(1), \pi(2), \cdots\cdots, \pi(n)$ is a permutation of $1, 2, \cdots\cdots, n$. There exists a regular element s such that $c^\sigma_{i,\alpha\beta} = s^{-1}c_{\pi(i),\alpha\beta}s$.*

Corollary 2. *If two idempotents f and g are isomorphic, then there exists a regular element s such that $g = s^{-1}fs$.*

Theorem 3. *The basic ring R^0 of R is uniquely determined up to inner automorphism.*

3. We consider a ring with minimum condition on left ideals and with an identity.

Lemma 5. *Let $\mathfrak{z}$ be a two-sided ideal of R. If $e \cong f$ and $e \in \mathfrak{z}$, then $f \in \mathfrak{z}$.*

Proof. Since $e = ab \in \mathfrak{z}$, $f = bea \in \mathfrak{z}$.

If $\mathfrak{z}$ is a two-sided ideal of R, then $\mathfrak{z}^0 = R^0 \cap \mathfrak{z} = e\mathfrak{z}e$ is a two-sided ideal of R^0 and $R\mathfrak{z}^0R = \mathfrak{z}$, while if $\mathfrak{z}^0$ is a two-sided ideal of R^0, then $\mathfrak{z} = R\mathfrak{z}^0R$ is a two-sided ideal of R and $R^0 \cap \mathfrak{z} = \mathfrak{z}^0$, as can easily seen from (6). Hence $\mathfrak{z} \rightleftarrows \mathfrak{z}^0 = R^0 \cap \mathfrak{z}$ gives a (1-1) correspondence between two-sided ideals of R and R^0.

Lemma 6. *Let $\mathfrak{z}$ be a two-sided ideal of R. Then $(R/\mathfrak{z})^0 \cong R^0/\mathfrak{z}^0$,* where $\mathfrak{z}^0 = R^0 \cap \mathfrak{z}$.

Let N be the radical of R. We denote by $\bar{R}$ the residue class ring of R modulo N. Then

$$(12) \qquad \bar{R} = R/N = \sum_i \sum_\alpha \bar{R}\bar{e}_{i,\alpha},$$

where every $\bar{R}\bar{e}_{i,\alpha}$ is a simple left ideal of $\bar{R}$. The radical of R is $N^0 = R^0 \cap N = eNe$ and

$$(13) \qquad (R/N)^0 \cong R^0/N^0.$$

The basic ring R^0 is a semi-primitive ring, since the residue class ring R^0/N^0 is a direct sum of division rings.

It is well known that two idempotents f and g are isomorphic in R if and only if $\bar{f}$ and $\bar{g}$ are isomorphic in $\bar{R}$.

We say that two indecomposable left ideals Rf and Rg belong to the same block if there is a sequence of indecomposable left ideals $Rf = Rf_1, Rf_2, \cdots\cdots, Rf_t = Rg$ such that $f_i^2 = f_i$ and each Rf_i has a composition factor R-isomorphic to one of the composition factors of Rf_{i+1}. When Rf and Rg belong to the same block, we say also that f and g belong to the same block and write $f \sim g$. This is a reflexive, symmetric, and transitive relation, by means of which the primitive idempotents are classified into disjoint classes. Evidently if two primitive idempotents are isomorphic, then they belong to the same block.

Let $R = R_1 + R_2 + \cdots\cdots + R_s$ be the direct decomposition of R into indecomposable two-sided ideals. It was shown that two primitive idempotents f and g belong to the same block if and only if they belong to the same two-sided component ([3], see [2] p. 74 too).

Theorem 4. *Let R^* be the residue class ring of R modulo N^2. Two primitive idempotents e and f belong to the same block if and only if e^* and f^* in R^* belong to the same block.*

Proof. Since the "if" part is obvious, we shall prove the "only if" part. Let E_i be the identity of R_i. The E_i are the primitive idempotents of the center Z of R. We have

$$R^* = R/N^2 = R^*E_1^* + R^*E_2^* + \cdots\cdots + R^*E_s^*,$$

where $R^*E_i^*$ are the two-sided ideals of R^*. If we can show that every $R^*E_i^*$ is indecomposable, then our proof is complete. Suppose that one of E_i, say, E_1 is decomposed into a sum of orthogonal idempotents f_1 and f_2 such that f_1^* and f_2^* lie in the center of R^*. Then we have $f_1^*R^*f_2^* = f_2^*R^*f_1^* = 0$, so that $f_1Rf_2, f_2Rf_1 \subset N^2$. Hence

$$f_1Rf_2 = f_1N^2f_2, \quad f_2Rf_1 = f_2N^2f_1.$$

Now we have for $n_1, n_2 \in N$

$$\begin{aligned} f_1n_1n_2f_2 &= \sum_i f_1n_1E_in_2f_2 \\ &= f_1n_1f_1n_2f_2 + f_1n_1f_2n_2f_2 \in f_1N^3f_2. \end{aligned}$$

This impries

$$f_1Rf_2 = f_1Nf_2 = \cdots\cdots = f_1N^\rho f_2 = 0,$$

since $N^\rho = 0$. Similarly $f_2Rf_1 = 0$. Consequently $Rf_1 = f_1R$ and $Rf_2 = f_2R$ are the two-sided ideals and $RE_1 = Rf_1 + Rf_2$. This contradicts to our assumption.

Theorem 5. *Let $R = R_1 + R_2 + \cdots\cdots + R_s$ be the direct decomposition of R into indecomposable two-sided ideals. Then*

$$R/N^2 = R_1/(R_1 \cap N^2) + \cdots\cdots + R_s/(R_s \cap N^2)$$

is the direct decomposition of R/N^2 into indecomposable two-sided ideals.

Let V be an R-space. We assume that V satisfies 1. $v = v$ for $v \in V$. Then we have $V = RV = \sum_{i,\alpha} c_{i,\alpha 1}V$. If V is an R-space, then eV is an R^0-space and $ReV = V$, while if V_0 is an R^0-space, then $V = \sum_{i,\alpha} c_{i,\alpha 1}V_0$ is an R-space under the assumption $a(bv) = (ab)v$ for $a, b \in R$ and $v \in V$. Further we see that $eV = V_0$. Hence $V \rightleftarrows eV$ gives a (1 - 1) correspondence between the R-spaces and the R^0-spaces.

Lemma 7. *Let U be an R-subspace of an R-space V, and let V/U be simple and $\cong \bar{R}\bar{e}_i$. Then $eV/eU \cong \bar{R}^0\bar{e}_i$.*

Proof. It follows from $V = ReV$ and $U = ReU$ that eU is a proper subspace of eV. Since $e_jV = e_jU$ for every $j \neq i$, we have $eV/eU \cong \bar{R}^0\bar{e}_i$.

Theorem 6. *Let V be an R-space satisfying the minimum condition and let*

$$V = V_1 \supset V_2 \supset \cdots\cdots \supset V_m \supset (0)$$

be a composition series for V. Suppose that $V_k/V_{k+1} \cong \bar{R}\bar{e}_i$. Then

$$eV = eV_1 \supset eV_2 \supset \cdots\cdots \supset eV_m \supset (0)$$

is a composition series of the R^0-space eV, and $eV_\lambda / eV_{\lambda+1} \cong \overline{R}^0 \bar{e}_i$.

We shall study in a forthcoming paper the connection between a ring and its basic ring. This problem was studied in [6] in the case when R is an algebra over an algebraically closed field.

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