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On the (2,3)-closures of Ideals

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ON THE (2, 3)-CLOSURES OF IDEALS

SUSUMU ODA and KEN-ICHI YOSHIDA

Let R be a Noetherian integral domain and let I be an ideal of R . Assume that the integral closure R' of R in its quotient field is a finite R -module. Let $R[t^{-1}, It]$ denote the generalized Rees ring, where t is an indeterminate. In [9], it is shown that if $R[t^{-1}, It]$ is seminormal then I^* is equal to the relevant component $(I^k)^* := \cup\{I^{i+k} : I^i; i \geq 1\}$ for all $k \in \mathbb{N}$ (where \mathbb{N} denotes the set of integers ≥ 1). But the converse statement are not necessarily valid. So it seems natural to ask when $R[t^{-1}, It]$ is seminormal. In this paper, we define the (2, 3)-closure I' of I in R , which has the following properties :

- (a) I' contains I ,
- (b) I is (2, 3)-closed in R (i.e., $a^2 \in I^2, a^3 \in I^3 (a \in R)$ imply $a \in I$) if and only if $I = I'$,
- (c) I' is (2, 3)-closed in R ,
- (d) I is a reduction of I' , that is, $I(I')^n = (I')^{n+1}$ for all large n ,
- (e) $I \subset I' \subset I_a$, where I_a denotes the integral closure of I .

Our objectives of this paper are to investigate the relations among I, I^* and I' , to study the relations between the seminormality of R and (2, 3)-closedness of divisorial ideals and to determine the seminormalization of the (generalized) Rees ring.

Unless otherwise specified, let R be a *Noetherian domain*, let R' denote the integral closure of R in its quotient field K , let I be an ideal of R and let B be an intermediate ring between R and R' which is a *finite* R -module. Our unexplained technical terms are standard and are seen in [4].

1. Definitions and Basic Properties of (2, 3)-Closures of Ideals. Let t denote an indeterminate, and we call $R[It]$ (resp. $R[t^{-1}, It]$) the *Rees ring* (resp. the *generalized Rees ring*) of R with respect to I . The ring $R[t^{-1}, It]$ is a subring of the torus extension $R_R = R[t^{-1}, t]$ of R . After Mirbagheri and Ratliff [7], we call the ideal $I^* := \cup\{I^{i+1} : I^i; i \geq 1\}$ the *relevant component* of I .

Let ${}^B\Delta_*(I) := \{\alpha \in B; \alpha^2 \in I^2, \alpha^3 \in I^3\}$ and let ${}_B I_*$ denote an R -submodule of B generated by ${}^B\Delta_*(I)$. When $I = R$, we denote ${}_B R_*$, an R -module generated by ${}^B\Delta_*(R) = \{\alpha \in B; \alpha^2 \in R, \alpha^3 \in R\}$. We claim that ${}_B R_*$ is an R -subalgebra of B and ${}_B I_*$ is an ideal of ${}_B R_*$. Indeed, since ${}_B R_*$ is an R -module generated by ${}^B\Delta_*(R)$, for $x, y \in {}^B\Delta_*(R)$, $x + y \in {}_B R_*$ is trivial. Next $(xy)^2 =$

$x^2y^2 \in R$ and $(xy)^3 = x^3y^3 \in R$ imply $xy \in {}^B\Delta_*(R)$. Since for any elements $\alpha = \sum r_i x_i$ and $\beta = \sum t_i y_i$ in ${}_B R_*$ with $x_i, y_i \in {}^B\Delta_*(R)$, $r_i, t_i \in R$, $\alpha\beta$ is expressed as a linear combination of $x_i y_j$ over R , the above argument shows that ${}_B R_*$ is an R -algebra. Since ${}_B I_*$ is an R -module generated by ${}^B\Delta_*(R)$, for $x, y \in {}^B\Delta_*(I)$, $x + y \in {}_B I_*$ is trivial. For $z \in {}^B\Delta_*(R)$, $(zx)^2 = z^2 x^2 \in {}_B I_*$ and $(zx)^3 = z^3 x^3 \in {}_B I_*$. So $zx \in {}^B\Delta_*(I)$. By the same reason as above, we conclude that ${}_B I_*$ is an ideal of ${}_B R_*$. Let ${}^B\Delta_0(I) := {}^B\Delta_*(I)$, ${}_B I_* = {}_B I_0$ and ${}_B R_0 = {}_B R_*$. Once ${}_B R_i$ and ${}_B I_i$ are defined, we put ${}^B\Delta_{i+1}(I) := {}^B\Delta_*({}_B I_i) = \{\alpha \in B; \alpha^2 \in ({}_B I_i)^2, \alpha^3 \in ({}_B I_i)^3\}$ and let ${}_B I_{i+1}$ denote an R -submodule ${}_B({}_B I_i)_*$ of B generated by ${}^B\Delta_{i+1}(I)$. It is clear that ${}^B\Delta_i(I) \subset {}^B\Delta_{i+1}(I)$. Then by induction, we can easily see that ${}_B R_{i+1}$ is an R -subalgebra of B and ${}_B I_{i+1}$ is an ideal of ${}_B R_{i+1}$. Since B is a finite R -module, the ascending chain of R -submodules of B :

$$I \subset {}_B I_0 \subset \cdots \subset {}_B I_i \subset \cdots$$

terminates, that is, there exists an integer N such that for all $n \geq N$ ${}_B I_n = {}_B I_{n+1} = \cdots$. Put ${}_B I := {}_B I_n$ for such n . When $I = R$, we employ the notation ${}_B R$ for ${}_B I$. When $B = R'$ (here R' is assumed to be a finite R -module), we use the notation I^*_i for ${}_B I_i$, I^* for ${}_B I$ and R^* for ${}_B R$. Moreover when $B = R$, we denote ${}_B I_i$ by I_i , ${}_B I$ by I' . We denote also $I_{i+1} = (I_i)_*$. Note here that I_i and I' are ideals of R by definition. Thus we obtain the following lemma.

- Lemma 1.1.** (i) ${}_B R_i$ is an R -subalgebra of B for all $i \in N$;
 (ii) ${}_B I_i$ is an ideal of ${}_B R_i$ for all $i \in N$;
 (iii) ${}_B I$ is an ideal of an R -algebra ${}_B R$.

We call ${}_B I$ the (2, 3)-closure of I in B and I' the (2, 3)-closure of I in R . When $I = {}_B I$ (resp. $I = I'$), we say that I is (2, 3)-closed in B (resp. I is (2, 3)-closed in R). Since R is an integral domain, the ideal (0) is (2, 3)-closed. More generally it is obvious that any radical ideal is (2, 3)-closed in R by Proposition 1.4 below.

Hereafter in this section, we treat only the case $B = R$.

Proposition 1.2. *The following statements hold :*

- (i) $(I)' = I'$, i.e., I' is (2, 3)-closed ;
 (ii) for any ideal J of R satisfying $I \subset J \subset I'$, we have $J' = I'$;
 (iii) $(I_0)^n \subset (I^n)_0$ and $(I')^n \subset (I^n)'$ for all $n \in N$.

Proof. (i) By construction, $I' = I_m$ for all large $m \in N$. Hence $(I')_* = (I_m)_* = I_{m+1} = I'$, which implies that $(I)' = I'$.

(ii) follows from (i).

(iii) Let $\alpha = a_1 \cdots a_n$ be an element in R with $a_i \in \Delta_0(I)$. Then $\alpha^2 = a_1^2 \cdots a_n^2 \in I^{2n}$ and $\alpha^3 = a_1^3 \cdots a_n^3 \in I^{3n}$ and hence $\alpha \in (I^n)_0$. Since I_0 is generated by $\Delta_0(I)$, any element in $(I_0)^n$ is a linear combination of products of n elements in $\Delta_0(I)$ over R . Thus by the preceding argument shows that $(I_0)^n \subset (I^n)_0$. Replace I by I_i in this inclusion and we get $((I_i)_0)^n \subset ((I_i^n)_0)$. So $(I_{i+1})^n = ((I_i)_0)^n \subset ((I_i^n)_0) \subset ((I_i^n)')$, that is, $(I_{i+1})^n \subset ((I_i^n)')$. Thus $(I_{i+1})^n \subset ((I_i^n)') \subset (((I_{i-1})^n)') \subset ((I_{i-1})^n)'$ by (i), and consequently we have $(I_{i+1})^n \subset ((I_0^n)') \subset ((I^n)_0)' = (I^n)'$. Since $I_m = I'$ for large m , we have $(I')^n \subset (I^n)'$ for all $n \in \mathbb{N}$.

Proposition 1.3. *Let J be an ideal generated by the set $\{a \in R ; a^k \in I^k \text{ for all large } k \in \mathbb{N}\}$. Then $I' \subset J, I^* \subset J$ and $\sqrt{I} = \sqrt{I'} = \sqrt{I^*} = \sqrt{J}$.*

Proof. Let J be the ideal generated by $\{a \in R ; a^k \in I^k \text{ for all large } k \in \mathbb{N}\}$. Then it is obvious that $I' \subset J$ because for all large $k, k = 2m + 3n$ for some $m, n \in \mathbb{N}$. Take $a \in I^*$. Then $a^k \in (I^*)^k = I^k$ for all large k by [8, (2.1)]. Hence $a \in J$. The second assertion follows from $\sqrt{I} = \sqrt{J}$.

Proposition 1.4. *The following statements are equivalent :*

- (a) $I = I'$ i.e., I is (2, 3)-closed in R ;
- (b) $\Delta_0(I) \subset I$;
- (c) $I_0 = I$;
- (d) $a^2 \in I^2, a^3 \in I^3 (a \in R)$ imply $a \in I$.

Proof. (b) \iff (c) is trivial because I_0 is generated by $\Delta_0(I)$ over R .

(a) \implies (c) is trivial because $I \subset I_0 \subset I'$.

(c) \implies (d) I_0 is an ideal generated by $\Delta_0(I)$. Hence $a^2 \in I^2, a^3 \in I^3 (a \in R)$ imply $a \in \Delta_0(I) \subset I_0 \subset I$.

(d) \implies (c): We have only to show that $\Delta_0(I) \subset I$ since I_0 is an ideal generated by $\Delta_0(I)$. But this is given by the condition in (d).

(c) \implies (a): $I_0 = I$ yields that $I_i = I$ for all i by the definition. So $I' = I_i = I$.

Recall that the *integral closure* I_a of I in R is the set of elements x in R that satisfy an equation of the form $x^n + a_1x^{n-1} + \cdots + a_n = 0$, where $a_i \in I_i$ for $i = 1, \cdots, n$. Also recall that I is said to be normal in case $(I^i)_a = I^i$ for all $i \geq 1$. It is shown in [7, p.34] that $J \subset I \implies J_a \subset I_a$ and $(I_a)_a = I_a$.

In the following theorem, we summarize some basic properties of (2, 3)-closures of ideals.

Theorem 1.5. *The following statements hold :*

- (i) $I' \subset I_a$;
- (ii) $(I_P)' = (I')_P$ for any $P \in \text{Spec}(R)$;
- (iii) I is a reduction of I' , that is, $I(I')^n = (I')^{n+1}$ for all large n ;
- (iv) I is (2, 3)-closed in R if and only if so is I_m in R_m for each maximal ideal m of R .

Proof. (i) It is clear that $I_i \subset (I_{i-1})_a$ by the definition. Hence $I_n \subset (I_{n-1})_a \subset (I_{n-2}) \subset \cdots \subset I_a$ by the preceding paragraph. Since $I' = I_n$ for all large n , we have $I' \subset I_a$.

(ii) We prove $(I_i)_P = (I_P)_i$ by induction on i . Take $a/s \in \Delta_i(I_P) \subset R_P$ ($a \in R, s \in R \setminus \mathfrak{p}$). Then $(a/s)^2 \in ((I_P)_{i-1})^2 = ((I_{i-1})_P)^2$ and $(a/s)^3 \in ((I_P)_{i-1})^3 = ((I_{i-1})_P)^3$; so that $a/s \in \Delta_i(I)R_P$. Thus $(I_P)_i = (I_i)_P$. The converse is similar. Since $I_n = I'$ for all large n , we have our conclusion.

(iii) We need to prove I_{i-1} is a reduction of I_i . For this we have only to show that I is a reduction of I_0 . Put $I_0 = (a_1, \cdots, a_r)R$ with $a_i \in \Delta_0(I)$ and let $J = (a_1^2, \cdots, a_r^2)R$. Then J is a reduction of $(I_0)^2$ by [8, (2.8.2)], that is, $J(I_0^2)^n = (I_0^2)^{n+1}$ for all large n . Since $J \subset I^2$, we have $I(I_0)^{2n+1} = I_0^{2n+2}$ for all large n . Thus $I(I_0)^m = I_0^{m+1}$ for all large m , which shows that I is a reduction of I_0 . Similarly we can prove I_{i-1} is a reduction of I_i for all i , and since $I_n = I'$ for a large n , I is a reduction of I' .

(iv) is clear by (ii).

Proposition 1.6. *Any intersection of ideals which are (2, 3)-closed in R is also (2, 3)-closed in R .*

Proof. We have only to show that $\bigcap J_i$ is (2, 3)-closed in R for (2, 3)-closed ideals J_i . Since $(\bigcap J_i)_0$ is generated by $\Delta_0(\bigcap J_i)$, we need only to prove $\Delta_0(\bigcap J_i) \subset \bigcap J_i$. Take $\alpha \in \Delta_0(\bigcap J_i)$. Then $\alpha^2 \in (\bigcap J_i)^2$ and $\alpha^3 \in (\bigcap J_i)^3$. Since $\bigcap J_i \subset J_i$ implies $(\bigcap J_i)^2 \subset \bigcap J_i^2$ and $(\bigcap J_i)^3 \subset \bigcap J_i^3$, we have $\alpha^2 \in \bigcap J_i^2$ and $\alpha^3 \in \bigcap J_i^3$. Since J_i is (2, 3)-closed in R , $\alpha \in J_i$, that is, $\alpha \in \bigcap J_i$. Thus $(\bigcap J_i)_0 = \bigcap J_i$. By Proposition 1.4, $\bigcap J_i$ is (2, 3)-closed in R .

Proposition 1.7. *Let b be an element in R . Then*

- (a) *if I is (2, 3)-closed, then $I : {}_R b := \{a \in R ; ab \in I\}$ is (2, 3)-closed ;*
- (b) *if the ideal bI is (2, 3)-closed in R , then I is (2, 3)-closed in R .*

Proof. (a) First we prove that $(I : {}_R b)^2 \subset I^2 : {}_R b^2$ and $(I : {}_R b)^3 \subset I^3 : {}_R b^3$. Let $I : {}_R b = (x_1, \cdots, x_r)$. Then $(I : {}_R b)^2$ (resp. $(I : {}_R b)^3$) is generated by $\{x_i x_j\}$

(resp. $\{x_i x_j x_k\}$). So we have only to show that $x_i x_j \in I^2 :_R b^2$ and $x_i x_j x_k \in I^3 :_R b^3$. Since $x_i x_j b^2 = (x_i b)(x_j b) \in I \cdot I = I^2$ and $x_i x_j x_k b^3 = (x_i b)(x_j b)(x_k b) \in I^3$. Thus $x_i x_j \in I^2 :_R b^2$ and $x_i x_j x_k \in I^3 :_R b^3$. Next take $\alpha \in (I :_R b)_0$. Then we may assume that α belongs to $\Delta_0(I :_R b)$ since $(I :_R b)_0$ is generated by $\Delta_0(I :_R b)$. So $\alpha^2 \in (I :_R b)^2$ and $\alpha^3 \in (I :_R b)^3$. By the previous argument, we have $\alpha^2 \in I^2 :_R b^2$ and $\alpha^3 \in I^3 :_R b^3$, and hence $\alpha^2 b^2 \in I^2$ and $\alpha^3 b^3 \in I^3$. Thus $\alpha b \in I$ because I is (2, 3)-closed in R , which implies that $\alpha \in I :_R b$. Therefore $(I :_R b)_0 = I :_R b$. By Proposition 1.4, we conclude that $I :_R b$ is (2, 3)-closed in R .

(b) This is clear because $(bI :_R b) = I$ and (a).

Corollary 1.7.1. *If I is (2, 3)-closed in R , then any isolated primary component q of I is (2, 3)-closed in R .*

Proof. Let $I = q_1 \cap \cdots \cap q_r$ be an irredundant primary decomposition of I . We may assume that $q = q_1$. Put $p = \sqrt{q}$. Then there exists an element $x \in q_2 \cap \cdots \cap q_r \setminus p$. Thus $I :_R x = (q_1 :_R x) = \cap \cdots \cap (q_r :_R x) = q_1 :_R x$. Since $x \notin p$, we have $q_1 :_R x = q_1$. Hence $q = q_1 = I :_R x$ is (2, 3)-closed in R by Proposition 1.7.

Corollary 1.7.2. *Assume that I has no embedded prime divisors. Let $I = q_1 \cap \cdots \cap q_r$ be an irredundant primary decomposition. Then I is (2, 3)-closed in R if and only if q_i is (2, 3)-closed in R for all $i = 1, \dots, r$.*

Proof. This follows from Proposition 1.6 and Corollary 1.7.1.

2. Seminormal Domains and (2, 3)-Closed Ideals. When R is (2, 3)-closed in B , that is, $a^2, a^3 \in R$ for $a \in B$ implies that $a \in R$, we say that R is (2, 3)-closed in B . When R is (2, 3)-closed in R' , we say that R is *seminormal*.

The following proposition asserts that the converse statement of Proposition 1.6(b) holds if R is seminormal.

Proposition 2.1. *If R is seminormal and I is (2, 3)-closed in R , then bI is (2, 3)-closed in R for any $b \in R$.*

Proof. Take $\alpha \in \Delta_0(bI)$. Then $\alpha^2 \in (bI)^2$ and $\alpha^3 \in (bI)^3$. So $(\alpha/b)^2 \in I^2 \subset R$ and $(\alpha/b)^3 \in I^3 \subset R$. Since R is seminormal, α/b belongs to R . Hence $\alpha \in bI$, which implies that bI is (2, 3)-closed in R .

Corollary 2.1.1. *Assume that R is seminormal. If I is (2, 3)-closed in R ,*

any ideal J of R which is R -isomorphic to I is $(2, 3)$ -closed in R .

Proof. Since $\text{Hom}_R(I, J) \subset \text{Hom}_R(I, J)_R \otimes K = \text{Hom}_K(K, K) = K$ (where K denotes the field of fractions of R), any R -isomorphism of I into J is the multiplication of an element α in K , that is, $\alpha I = J$. Put $\alpha = c/d$ with $c, d \in R$. Then $cI = dJ$. Since I is $(2, 3)$ -closed in R , $cI = dJ$ is $(2, 3)$ -closed in R by Proposition 2.1. Hence J is $(2, 3)$ -closed in R by Proposition 1.7(b).

Remark. It is known and is not hard to see that $R = R'$ if and only if any principal ideal of R is integrally closed, i.e., $(bR)_a = bR$ for all $b \in R$.

The next proposition shows that the similar argument in the above Remark is valid for $(2, 3)$ -closedness.

Proposition 2.2. *The following statements are equivalent :*

- (a) R is seminormal ;
- (b) $(aR)' = aR$ for any non-unit a in R ;
- (c) $(aR)' = aR$ for any element a in R ;
- (d) $(aR)'$ is a principal ideal for any element a of R .

Proof. Note first that $aR = (aR)' \iff aR = (aR)_0$ by Proposition 1.4.

(a) \implies (b): Since R itself is $(2, 3)$ -closed in R , aR is $(2, 3)$ -closed in R by Proposition 2.1.

(b) \implies (a): Take $\alpha \in R'$ with $\alpha^2 \in R$ and $\alpha^3 \in R$. Put $\alpha = b/a$ with $a, b \in R$. If a is a unit in R , then $\alpha \in R$. Suppose a is not a unit in R . Then $b^2 \in a^2R$ and $b^3 \in a^3R$, so that $b \in aR$ by the assumption. Hence $\alpha = b/a \in R$. We conclude that R is seminormal.

(c) \iff (b) and (b) \implies (d) are trivial.

(d) \implies (b): By Theorem 1.5, we may assume that R is a local domain with the maximal ideal m . Let $(aR)' = bR$ and let x_1, \dots, x_r be elements in $\Delta_0(aR)$ which generates $(aR)'$ over R . Then there exist x_i such that $(aR)' = x_iR$. Indeed, take $x_i \in (aR)' \setminus (aR)'m$. Then the image x_i in $(aR)' / (aR)'m \simeq bR/bm \simeq R/m$ is a basis over the field R/m . From this, $x_iR + bm = bR$ and hence $x_iR = bR = (aR)'$ by Nakayama's lemma. Put $x = x_i$. Since $a \in bR = xR$, we have $a = xc$ for some $c \in R$. Since $x \in \Delta_0(aR)$, we have $x^2 = a^2r \in a^2R$ for some $r \in R$. Thus $x^2 = x^2c^2r$ implies that c is a unit in R . Thus $(aR)' = xR = aR$.

Relating to Proposition 1.7 and Propositions 2.1 and 2.2, we refer to the example raised in [9].

Example. Let $R = k[X^2, X^3]$ be a subring of a polynomial ring $k[X]$ over a field k and let I be an ideal X^2R . It is easy to see that R is not seminormal and that $(X^3)^2 \in I^2$ and $(X^3)^3 \in I^3$; hence $X^3 \in I'$ but X^3 does not belong to I . Since $I^* = I$, we have $I^* \neq I'$. Moreover it is clear that $(I^n)' \neq I^n$ for all $n \in \mathbb{N}$ because $X^{2n+1} \notin I^n$ but $X^{2n+1} \in (I^n)'$.

Corollary 2.2.1. *Assume that I is invertible and R is seminormal. Then $(I^n)' = I^n$ for all $n \in \mathbb{N}$.*

Proof. Note that R is seminormal if and only if R_P is seminormal for each $P \in \text{Spec}(R)$ and that I being invertible implies that I is locally principal. So our conclusion follows from Theorem 1.5(ii) and Proposition 2.2.

Corollary 2.2.2. *Assume that R is a Dedekind domain. Then any ideal of R is (2, 3)-closed in R .*

We denote by K the field of fractions of R , and $R : {}_R\alpha$ denotes the denominator ideal $\{a \in R ; a\alpha \in R\}$ of $\alpha \in K$.

Lemma 2.3. *Assume that R is seminormal. Then for any element α in K , the ideal $R : {}_R\alpha$ is (2, 3)-closed in R .*

Proof. Put $\alpha = c/d$ with $c, d \in R$ ($d \neq 0$). Then $R : {}_R\alpha = dR : {}_RcR$. Since R is seminormal, dR is (2, 3)-closed in R by Proposition 2.2. Hence by Proposition 1.6, $R : {}_R\alpha = dR : {}_RcR$ is (2, 3)-closed in R .

An R -submodule J of K is called *fractional* if $rJ \subset R$ for some $r \in R \setminus \{0\}$. Any ideal of R is a fractional ideal of R . We say that a fractional ideal J of R is *divisorial* if $R : {}_R(R : {}_R J) = J$. It is known that J is divisorial if and only if J is the intersection of principal fractional ideals of R . For $\alpha \in K$, the denominator ideal $R : {}_R\alpha$ is divisorial ideal of R . Indeed, it is obvious that $R : {}_R\alpha = \alpha^{-1}R \cap R$ if $\alpha \neq 0$ and $R : {}_R\alpha = R$ if $\alpha = 0$.

We now extend Proposition 2.2 as follows.

Theorem 2.4. *The following statements are equivalent :*

- (i) R is seminormal ;
- (ii) any principal ideal aR ($a \in R$) is (2, 3)-closed in R ;
- (iii) any denominator ideal $R : {}_R\alpha$ ($\alpha \in K$) is (2, 3)-closed in R ;
- (iv) any divisorial ideal of R is (2, 3)-closed in R .

Proof. (ii) \implies (i) is shown in Proposition 2.2.

(iv) \implies (iii) \implies (ii) is clear because any principal ideal aR ($a \neq 0$) is $R : {}_R a^{-1}R$ and $R : {}_R a$ is divisorial for any $a \in K$ by the preceding argument.

(i) \implies (iv): Let I be a divisorial ideal of R ($I \subset R$). Then $I = \bigcap (\alpha^{-1}R \cap R)$ for some α 's $\in K$. We may assume that $I \neq (0)$ because (0) is (2, 3)-closed in R . Since $\alpha^{-1}R \cap R = R : {}_R \alpha$, $\alpha^{-1}R \cap R$ is (2, 3)-closed in R by Lemma 2.3. Hence $I = \bigcap (\alpha^{-1}R \cap R)$ is (2, 3)-closed in R by Proposition 1.5.

Proposition 2.5. *Let A be an integral domain containing R and let I be an ideal of R . Assume that A is faithfully flat over R . If IA is (2, 3)-closed in A , then I is (2, 3)-closed in R .*

Proof. We have only to show that $I_0 = I$ by Proposition 1.4. For this we must show that $\Delta_0(I) \subset I$. Take $\alpha \in \Delta_0(I)$. Then $\alpha^2 \in I^2 \subset (IA)^2$ and $\alpha^3 \in I^3 \subset (IA)^3$ and hence $\alpha \in IA$ because IA is (2, 3)-closed in A . Since A is faithfully flat over R , $\alpha \in IA \cap R = I$.

We close this section by showing what happen when $(I^k)' = I^k$ for some $k \in N$.

Proposition 2.6. *Assume that $I^k = (I^k)'$ for some $k \in N$. Then*

- (i) $(I')^n = (I^*)^n = I^n$ for all large n .
- (ii) $I' \subset I^*$.

Proof. (i) By Theorem 1.5(iii), I is a reduction of I' . So for any $i \in N$ with $1 \leq i \leq k$ we have $I^i(I')^r = (I')^{r+i}$ for all large r . For a large m , consider the case $r = mk$. Then $(I')^{km+i} \subset I^i(I')^{km} \subset I^i((I^k)')^m = I^i(I^k)^m = I^{km+i}$, where we use Proposition 1.2 in the second inclusion. Since $I \subset I'$ implies $I^{km+i} \subset (I')^{km+i}$, we have $(I')^{km+i} = I^{km+i}$ ($1 \leq i \leq k$) for all large m . Thus $(I')^n = I^n$ for all large n .

(ii) Since $(I')^n = (I^*)^n$ for all large $n \in N$ by (i), we have $I' \subset I^*$ by [8, (2.1)].

Corollary 2.6.1. *If I^k is (2, 3)-closed, i.e., $I^k = (I^k)'$ for some $k \in N$, then $I' \subset I^*$.*

Proof. By Proposition 1.2, we have $(I')^k \subset (I^k)'$. So $I^k \subset (I')^k \subset (I^k)' = I^*$, and consequently $I' = I^*$ by Proposition 2.6(ii).

3. Generalized Rees Rings and (2, 3)-Closed Ideals. Throughout this sec-

tion, we assume that the integral closure R' of R is a finite R -module.

Let $C = \sum_{n \in \mathbb{N}} C_n$ be a graded domain with integral closure C' in the domain $S^{-1}C$, where S denotes the set of all non-zero homogeneous elements in C . Then the integral closure C' is a graded domain $\sum_{n \in \mathbb{N}} C'_n$. After D. F. Anderson [1], we say that C is *almost seminormal* if whenever $x^2, x^3 \in C$ for homogeneous $x \in C'$ with $\deg x > 0$, then $x \in C$. It is known that the canonical homomorphism $\text{Pic}(C_0) \rightarrow \text{Pic}(C)$ is an isomorphism if and only if C is almost seminormal (cf. [1]). We also say that a \mathbb{Z} -graded domain $L = \sum_{n \in \mathbb{Z}} L_n$ is almost seminormal if whenever $x^2, x^3 \in L$ for homogeneous $x \in L'$ with $\deg x \neq 0$, then $x \in L$, where $L' = \sum_{n \in \mathbb{Z}} L'_n$ is its integral closure in the domain $T^{-1}L$ with T the set of all non-zero homogeneous elements in L . It is shown that C (resp. L) is seminormal if and only if any homogeneous element $\alpha \in C'$ (resp. L') with $\alpha^2, \alpha^3 \in C$ (resp. L) belongs to C (resp. L) (cf. [2]).

Proposition 3.1. *The generalized Rees ring $R[t^{-1}, It]$ is (2, 3)-closed in the torus extension $R_R = R[t, t^{-1}]$ if and only if $(I^n)' = I^n$ for all $n \in \mathbb{N}$.*

Proof. In order to prove that $R[t^{-1}, It]$ is (2, 3)-closed in the torus extension $T_R = R[t, t^{-1}]$, we have only to show that any homogeneous element x in $R[t, t^{-1}]$ with $x^2, x^3 \in R[t^{-1}, It]$ belongs to $R[t^{-1}, It]$. Take a homogeneous element x in $R[t, t^{-1}]$ with $x^2, x^3 \in R[t^{-1}, It]$ whose degree is s . Then $x^2 \in I^{2s}t^{2s}$ and $x^3 \in I^{3s}t^{3s}$. Put $x = yt$ with $y \in R$. Since $y^2 \in I^{2s}$ and $y^3 \in I^{3s}$, $y \in (I^s)_0 \subset (I^s)' = I^s$. Hence $x = yt \in It \subset R[t^{-1}, It]$. Conversely take $y \in \Delta_i(I^n)$ for a fixed large i such that $(I^n)_i = (I^n)'$. Then $(yt^n)^2 \in ((I^n)_{i-1})^2 t^{2n}$, $(yt^n)^3 \in ((I^n)_{i-1})^3 t^{3n}$. By induction we may assume that $(I^n)_{i-1} t^n \subset R[t^{-1}, It]$. Hence $yt^n \in R[t^{-1}, It]$, which implies that $y \in I^n$. Thus $(I^n)' \subset I^n$.

Remark. It is not hard to see that the statement in Proposition 3.1 is valid for the Rees ring $R[It]$ in the polynomial ring $R[t]$.

By use of [2,Th.3], we know the following: the Rees ring $R[It]$ is seminormal if and only if R is seminormal and $R[It]$ is almost seminormal.

Proposition 3.2. *The following statements are equivalent:*

- (a) *The generalized Rees ring $R[t^{-1}, It]$ is almost seminormal;*
- (b) *$R[t^{-1}, It]$ is seminormal;*
- (c) *R is seminormal and $(I^n)' = I^n$ for all $n \in \mathbb{N}$.*

Proof. (a) \iff (c) follows from the fact that we have only to consider all

homogeneous elements in the integral closure of $R[t^{-1}, It]$ as remarked above.

(a) \iff (b): We need to prove that R is seminormal. Since any homogeneous component of negative degree is the form Rt^{-s} ($s > 0$), almost seminormality of $R[t^{-1}, It]$ implies that R is (2, 3)-closed in R' . The converse implication is shown by the similar argument to that in the proof of Proposition 3.1.

The following Corollary is established for the Rees ring $R[It]$ in [2].

Corollary 3.2.1. *If R is seminormal and I is an invertible ideal of R , then the generalized Rees ring $R[t^{-1}, It]$ is seminormal.*

Proof. This follows from Corollary 2.2.1 and Theorem 3.2.

The seminormalization of R in B was defined by Traverso to be

$${}_B^+R = \{x \in B; x/1 \in R_P + J(B_P) \text{ for all } P \in \text{Spec}(R)\},$$

where J denote the Jacobson radical. Equivalently, ${}_B^+R$ is the largest subring C of B containing R such that (i) $\text{Spec}(C) \rightarrow \text{Spec}(R)$ is injective and (ii) for all $Q \in \text{Spec}(C)$ the canonical map of residue class fields $k(Q \cap R) \rightarrow k(Q)$ is an isomorphism. R is called seminormal in B if $R = {}_B^+R$, and R is called seminormal if it is seminormal in its integral closure R' . It is known that R is seminormal in this sense if and only if R is (2, 3)-closed in R' (cf.[3]). So our definition of seminormality by use of (2, 3)-closedness is equivalent to the one defined here.

Lemma 3.3. *Let ${}_B^+R$ denote the seminormalization of R in B . Then ${}_B R \subset {}_B^+R$.*

Proof. By induction on i , we shall show the following ;

(1) The canonical map $\text{Spec}({}_B R_{i-1}) \rightarrow \text{Spec}({}_B R_{i-2})$ is injective (where ${}_B R_{-1} := R$);

(2) For $P \in \text{Spec}({}_B R_{i-1})$, the canonical homomorphism of residue class fields $k(P \cap {}_B R_{i-2}) \rightarrow k(P)$ is an isomorphism.

For this, we have only to prove the following special case :

(1') The canonical map $\text{Spec}({}_B R_0) \rightarrow \text{Spec}(R)$ is injective ;

(2') For $P \in \text{Spec}({}_B R_0)$, the canonical homomorphism of residue class fields $k(P \cap R) \rightarrow k(P)$ is an isomorphism.

(1'): We may assume that $R \neq {}_B R_0$. Suppose $P \cap R = Q \cap R$ for $P \not\subset Q \in \text{Spec}({}_B R_0)$. There exists $\alpha \in P$, $\alpha \notin Q$ and $\alpha^n \in R$ (Indeed, we can take such α in ${}^B \mathcal{A}_0(R)$). Hence $\alpha^n \in P \cap R = Q \cap R$; so that $\alpha \in Q$, contradiction. Thus $P \subset Q$. Similarly we get $Q \subset P$.

(2'): Since $R/P \cap R \rightarrow {}_B R_0/P$ is injective, the map $k(P \cap R) \rightarrow k(P)$ is injective. Take a non-zero element $\alpha' \in ({}_B R_0/P)_P$. We may assume that a preimage α of α' in ${}_B R_0$ is an element in ${}^B \Delta_0(R)$. Then $\alpha^n \in R$ for all large n . Since $\alpha \notin P$ implies $\alpha^n \notin P \cap R$ for all n . Thus $\alpha' = \alpha'^{n+1}/\alpha'^n \in (R/P \cap R)_{P \cap R}$. Thus $({}_B R_0/P)_P \subset (R/P \cap R)_{P \cap R}$, which yields that $k(P) = k(P \cap R)$. Thus after repeating the above argument, we conclude that ${}_B R$ satisfies the condition (i) and (ii) mentioned above. Since ${}_B^+ R$ is the largest subring of R' satisfying the same conditions (i) and (ii), we obtain that $R \subset {}_B R \subset {}_B^+ R$.

Traverso [10] shows that ${}_B^+ R$ has no proper subrings containing R and seminormal in B .

Proposition 3.4. *${}_B R$ is the seminormalization of R in B , that is, ${}_B R = {}_B^+ R$.*

Proof. By the above remark by Traverso, we have only to show that ${}_B R$ is seminormal in B because ${}_B R \subset {}_B^+ R$ by Lemma 3.3. By the definition, there exists $n \in N$ such that ${}_B R_n = {}_B R_{n+1} = \dots = {}_B R$. If $\alpha^2, \alpha^3 \in {}_B R = {}_B R_n$ for $\alpha \in B$, then $\alpha \in {}^B \Delta_{n+1}(R)$. So $\alpha \in {}_B R_{n+1} = {}_B R$, as was to be shown.

We close this paper by determining the seminormalization of $R[t^{-1}, It]$ and $R[It]$.

Theorem 3.5. *Let R be a Noetherian domain and let I be an ideal of R . Assume that R' is a finite R -module. Then the seminormalization of $R[It]$ (resp. $R[t^{-1}, It]$) in the integral closure $R'[t, t^{-1}]$ (resp. in $R'[t]$) is $R^*[t^{-1}, \{(I^i)^* t^i; i > 0\}]$ (resp. $R^*[\{(I^i)^* t^i; i > 0\}]$).*

Proof. It is clear that the integral closure of $R[t^{-1}, It]$ (resp. $R[It]$) is $R'[t, t^{-1}]$ (resp. $R'[t]$). By Lemma 3.3, $R^*[t^{-1}, \{(I^i)^* t^i; i > 0\}]$ (resp. $R^*[\{(I^i)^* t^i; i > 0\}]$) is contained in the seminormalization of $R[t^{-1}, It]$ (resp. $R[It]$) in $R'[t, t^{-1}]$ (resp. $R'[t]$). By the same way as in the proof of Proposition 1.2, we can see that $(I^i)^*$ is (2, 3)-closed in R' for all $i \in N$. So $R^*[t^{-1}, \{(I^i)^* t^i; i > 0\}]$ (resp. $R^*[\{(I^i)^* t^i; i > 0\}]$) is (2, 3)-closed in $R'[t, t^{-1}]$ (resp. $R'[t]$). Thus by the Traverso's remark mentioned above, we get our conclusion.

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