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ON JORDAN LEFT DERIVATIONS

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Throughout the present paper, $R (\neq 0)$ will represent an associative ring with center Z and X a nonzero left R -module. Following [1], X is called prime if $aRx = 0$ for $a \in R$ and $x \in X$ implies that either $x = 0$ or $aX = 0$. As is well known, R is a prime ring if and only if there exists a nonzero faithful prime left R -module. Following [2], an additive mapping $D : R \rightarrow X$ is called a Jordan left derivation if $D(a^2) = 2aD(a)$ for all $a \in R$.

Now, let X be a faithful prime left R -module and p a prime number. Suppose that R is of characteristic p . Then for any nonzero $a \in R$, $aR(pX) = (pa)RX = 0$, and so $pX = 0$. Conversely, suppose that $px = 0$ for some nonzero $x \in X$. Then, for any $a \in R$, $0 = aR(px) = (pa)Rx$, and so $pa = 0$. Consequently, we see that X is not p -torsionfree, or what is the same, $pX = 0$, if and only if R is of characteristic p .

Our present objective is to improve [2, Theorem 1.2] as follows.

Theorem 1. *Let R be a prime ring of characteristic $\neq 2$, and X a nonzero left R -module. Suppose that X is faithful and prime. If there exists a nonzero Jordan left derivation $D : R \rightarrow X$, then R is commutative.*

In preparation for proving our theorem, we state several lemmas.

Lemma 1. *Suppose that X is faithful and prime. Let $a, b \in R$, and $x \in X$. If (the prime ring) R is of characteristic $\neq 2$ and $arbrx = 0$ for all $r \in R$, then $a = 0$ or $b = 0$ or $x = 0$.*

Proof. Obviously, $0 = a(u+v)b(u+v)x = aubvx + avbux$ for all $u, v \in R$. Replacing v by $rarbr$, we have $0 = aubrarbrx + ararbrbux = ararbrbux$ for all $u, r \in R$. Suppose that $x \neq 0$. Noting that X is faithful and prime, we obtain $ararbrb = 0$. Since R is prime, [5, Theorem] shows that either $a = 0$ or $b = 0$.

Lemma 2. *Let R be a ring of characteristic 3. If $D : R \rightarrow X$ is a Jordan left derivation, then for all $a, b, c \in R$, there holds the following :*

- (1) $D(ab + ba) = 2aD(b) + 2bD(a)$.
- (2) $D(aba) = a^2D(b) - baD(a)$.

$$(3) \quad D(abc + cba) = (ac + ca)D(b) - baD(c) - bcD(a).$$

Proof. See the proof of [2, Proposition 1.1].

Lemma 3. *Let R be a ring of characteristic 3, and $D : R \rightarrow X$ a Jordan left derivation. Suppose that X is faithful and prime. If $D(a) \neq 0$ for some $a \in R$, then $[a, [a, b]]^2 = 0$ for all $b \in R$.*

Proof. Note that X is 2-torsionfree, and see the proof of [2, Lemma A].

The next will play an essential role in the proof of Theorem 1.

Lemma 4. *Let R be a prime ring of characteristic 3. Suppose that X is faithful and prime. If there exists a nonzero Jordan left derivation $D : R \rightarrow X$, then R has no nonzero nilpotent elements (more precisely, R has no nonzero divisors of zero).*

Proof. Suppose, to the contrary, that R contains a nonzero element a with $a^2 = 0$. Then $aD(a) = 0$. Now, by Lemma 2(1) and (2), we obtain $2aD(ba) = 2aD(ba) + 2baD(a) = D(aba) = a^2D(b) - baD(a) = 0$, and so $aD(ba) = 0$. Next by Lemma 2(3),

$$\begin{aligned} D(ab^2a) + D(baba) &= D(ab^2a + baba) \\ &= (aba + ba^2)D(b) - baD(ba) - b^2aD(a) = abaD(b). \end{aligned}$$

Combining this with $D(ab^2a) = 0$ (Lemma 2(2)) and $D(baba) = 2baD(ba) = 0$, we obtain $abaD(b) = 0$. So, linearizing this on b , we obtain

$$(\#) \quad abaD(c) + acaD(b) = 0 \quad \text{for all } b, c \in R.$$

Replacing c by $ac + ca$ in (#), we obtain $abaD(ac + ca) = 0$, and so $abacD(a) = 0$ by Lemma 2(1). Hence $D(a) = 0$ by Lemma 1. Further, $acaD(bab) = aca(b^2D(a) - abD(b)) = 0$ by Lemma 2(2). Now, replacing b by bab in (#), we get $ababaD(c) = ababaD(c) + acaD(bab) = 0$. Hence $aD(c) = 0$ by Lemma 1. Replace c by c^2 to get $acD(c) = 0$. Linearizing this on c , we obtain $abD(c) + acD(b) = 0$. Furthermore, replacing c by ac , we have $abD(ac) = 0$ for all $b \in R$, and so $D(ac) = 0$ by the faithfulness and the primeness of X . Recalling that $D(a) = 0$ and $aD(c) = 0$, we obtain $D(ca) = D(ac + ca) = 0$ by Lemma 2(1), and so $D(cba) = 0$. Combining this with $D(a(bc)) = 0$, we have $acD(b) = D(a(bc) + (cb)a) = 0$ by Lemma 2(1). Hence $D(b) = 0$ for all $b \in R$. But this is a contradiction.

We are now ready to complete the proof of Theorem 1.

Proof of Theorem 1. In view of [2, Theorem 1.2], it suffices to consider the case that R is of characteristic 3. Choose $a \in R$ such that $D(a) \neq 0$. Then $[a, [a, b]] = 0$ for all $b \in R$, by Lemmas 3 and 4, and so [4, Theorem 1] shows that a is in Z . Thus $R = Z \cup \{a \in R \mid D(a) = 0\}$. Since D is nonzero, we conclude that $R = Z$, by Brauer's trick.

Finally in connection with Theorem 1, we shall improve [6, Theorem 2] as follows:

Theorem 2. *Let R be a prime ring of characteristic $\neq 2$. If there exists a nonzero derivation $D : R \rightarrow R$ such that $[a, [a, D(a)]] \in Z$ for all $a \in R$, then R is commutative.*

Proof. In view of [6, Theorem 2], it suffices to consider the case that R is of characteristic 3. Then, for any $a \in R$,

$$\begin{aligned} D(a^3) &= a^2D(a) + aD(a)a + D(a)a^2 = a^2D(a) - 2aD(a)a + D(a)a^2 \\ &= [a, [a, D(a)]] \in Z, \end{aligned}$$

and so $D(a^{3 \cdot 3}) = 3a^{3 \cdot 2}D(a^3) = 0$. Hence R is commutative by [3, Theorem 2].

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