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ON H -SEPARABLE POLYNOMIALS IN SKEW POLYNOMIAL RINGS OF AUTOMORPHISM TYPE

Dedicated to Professor Manabu Harada on his 60th birthday

SHŪICHI IKEHATA and GEORGE SZETO

In [2], [3] and [4], one of the authors has studied H -separable polynomials in skew polynomial rings. In [4], we have studied H -separable polynomials of prime degree in skew polynomial rings of automorphism type. The present paper is a natural continuation of [4].

Throughout this paper, B will represent a ring with 1, and ρ an automorphism of B . Let $B[X; \rho]$ be the skew polynomial ring in which the multiplication is given by $bX = X\rho(b)$ ($b \in B$). A ring extension S/B is called a separable extension if the S - S -homomorphism of $S \otimes_B S$ onto S defined by $a \otimes b \mapsto ab$ splits, and S/B is called an H -separable extension if $S \otimes_B S$ is S - S -isomorphic to a direct summand of a finite direct sum of copies of S . A monic polynomial f in $B[X; \rho]$ with $fB[X; \rho] = B[X; \rho]f$ is called a separable (resp. H -separable) polynomial if $B[X; \rho]/fB[X; \rho]$ is a separable (resp. H -separable) extension of B . It is well known that every H -separable extension is a separable extension. As to terminologies used in this note, we follow [2].

In [4, Theorem 2], for any prime number p , we have shown that the center Z of B is a Galois extension over Z^ρ with the Galois group $(\rho|Z)$ whose order is p if and only if $B[X; \rho]$ contains an H -separable polynomial of degree p . In this paper, for general m , we shall characterize the condition that Z is a Galois extension over Z^ρ with the Galois group $(\rho|Z)$ whose order is m in terms of H -separable extensions (Theorem 1). Moreover, we shall obtain a sharpening of [4, Theorem 4]. Some more results will be obtained in [5].

We shall use the following conventions:

Z = the center of B .

$V_S(B)$ = the centralizer of B in S for a ring extension S/B .

$B^\rho = \{\alpha \in B \mid \rho(\alpha) = \alpha\}$, $Z^\rho = \{\alpha \in Z \mid \rho(\alpha) = \alpha\}$.

Let f be a monic polynomial in $B[X; \rho]$ with $fB[X; \rho] = B[X; \rho]f$. Then we shall denote $B[x; \rho] = B[X; \rho]/fB[X; \rho]$, where $x = X + B[X; \rho]/fB[X; \rho]$, and $B[x^i; \rho^i] =$ the subring of $B[x; \rho]$ generated by B and x^i .

Recall that if f is an H -separable polynomial in $B[X; \rho]$ of degree m , then $f = X^m - u$, u is invertible in B^ρ and $au = u\rho^m(a)$ ($a \in B$) ([3, Lemma 1]).

First, we shall state the following theorem which is a generalization of [4, Theorem 2].

Theorem 1. *Let $f = X^m - u$ be in $B[X; \rho]$ with $fB[X; \rho] = B[X; \rho]f$. Then the following are equivalent :*

- (a) *u is invertible in B^ρ , and Z/Z^ρ is a G -Galois extension, where G is the group generated by $\rho|Z$ of order m .*
- (b) *$B[x^n; \rho^n]$ is an H -separable extension over B for every divisor n of m .*

Proof. (a) \implies (b). Assume $m = nd$. Then we have

$$B[x^n; \rho^n] \cong B[Y; \rho^n]/(Y^d - u)B[Y; \rho^n],$$

where $\alpha Y = Y\rho^n(\alpha)$ ($\alpha \in B$), and it is clear that $(Y^d - u)B[Y; \rho] = B[Y; \rho](Y^d - u)$. Since Z/Z^ρ is a G -Galois extension, Z/Z^{ρ^n} is $(\rho^n|Z)$ -Galois extension and $(\rho^n|Z)$ is of order d . Hence, by [2, Proposition 1.4] $Y^d - u$ is an H -separable polynomial in $B[Y; \rho^n]$, and so $B[x^n; \rho^n]$ is an H -separable extension B .

To prove the converse, we need the following elementary lemma :

Lemma 2. *If there exist divisors n_1, n_2, \dots, n_r of m such that $m = n_1 n_2 \dots n_r$ and in the tower*

$$Z = Z^{\rho^m} \supset \dots \supset Z^{\rho^{n_1 n_2 \dots n_r}} \supset Z^{\rho^{n_1 n_2 \dots n_r}} \supset \dots \supset Z^{\rho^{n_r}} \supset Z^\rho,$$

each $Z^{\rho^{n_1 n_2 \dots n_r}}/Z^{\rho^{n_1 n_2 \dots n_r}}$ is a $(\rho^{n_1 n_2 \dots n_r}|Z^{\rho^{n_1 n_2 \dots n_r}})$ -Galois extension, where the group $(\rho^{n_1 n_2 \dots n_r}|Z^{\rho^{n_1 n_2 \dots n_r}})$ is of order n_i ($1 \leq i \leq r$), then Z/Z^ρ is a $(\rho|Z)$ -Galois extension, where the group $(\rho|Z)$ is of order m .

Proof. By [1, Theorem 1.3], there exist elements $\alpha_k^{(i)}, \beta_k^{(i)} \in Z^{\rho^{n_1 n_2 \dots n_r}}$ such that

$$\sum_k \alpha_k^{(i)} \rho^{\nu n_1 n_2 \dots n_r} (\beta_k^{(i)}) = \delta_{0, \nu} \quad (0 \leq \nu \leq n_i - 1, 1 \leq i \leq r).$$

Then we can easily verify that

$$\sum_{k_1, k_2, \dots, k_r} \alpha_{k_1}^{(1)} \alpha_{k_2}^{(2)} \dots \alpha_{k_r}^{(r)} \rho^\nu (\beta_{k_r}^{(r)} \beta_{k_{r-1}}^{(r-1)} \dots \beta_{k_1}^{(1)}) = \delta_{0, \nu} \quad (0 \leq \nu \leq m - 1).$$

Therefore we have the assertion by [1, Theorem 1.3] again.

Now, we come back to prove Theorem 1. (b) \implies (a).

Assume $m = p_1^{e_1} p_2^{e_2} \dots p_r^{e_r}$, where each p_i are different prime numbers and $e_i > 0$. We define the sequence n_1, n_2, \dots, n_r of divisors of m as follows :

$$n_i = \begin{cases} p_1 & (1 \leq i \leq e_1) \\ p_2 & (1+e_1 \leq i \leq e_1+e_2) \\ \dots\dots\dots & \\ p_k & (1+e_1+e_2+\dots+e_{k-1} \leq i \leq e_1+e_2+\dots+e_k), \\ & r = e_1+e_2+\dots+e_k \text{ and } 1 \leq i \leq r. \end{cases}$$

Then $m = n_1 n_2 \dots n_r$. We shall prove that in the tower

$$Z = Z^{\rho^m} \supset \dots \supset Z^{\rho^{n_1 n_2 \dots n_r}} \supset Z^{\rho^{n_1 n_2 \dots n_r}} \supset \dots \supset Z^{\rho^{n_r}} \supset Z^{\rho},$$

each $Z^{\rho^{n_1 n_2 \dots n_r}} / Z^{\rho^{n_1 n_2 \dots n_r}}$ is a $(\rho^{n_1 n_2 \dots n_r} | Z^{\rho^{n_1 n_2 \dots n_r}})$ -Galois extension of order n_i ($1 \leq i \leq r$).

We put here $s = n_{i+1} n_{i+2} \dots n_r$ and $t = n_i n_{i+1} \dots n_r$. Then $t = sn_i$, and we may assume

$$t = p_j^{v_j+1} p_{j+1}^{e_{j+1}} \dots p_k^{e_k} \text{ and } s = p_j^{v_j} p_{j+1}^{e_{j+1}} \dots p_k^{e_k}, \text{ so } t = sp_j.$$

Now we have

$$B[x^s; \rho^s] \supset B[x^t; \rho^t] = B[x^{sp_j}; \rho^{sp_j}] \supset B.$$

Since $B[x^t; \rho^t] \cong B[Y; \rho^t] / (Y^q - u)B[Y; \rho^t]$, where $m = qt$, $Y^q - u$ is an H -separable polynomial in $B[Y; \rho^t]$. Naturally, we can extend ρ^s to the automorphism $\tilde{\rho}^s$ of $B[x^{sp_j}; \rho^{sp_j}]$. Consider the skew polynomial ring $B[x^{sp_j}; \rho^{sp_j}][T; \tilde{\rho}^s]$, where $\alpha T = T\tilde{\rho}^s(\alpha)$ ($\alpha \in B[x^{sp_j}; \rho^{sp_j}]$). Then we have the following $B[x^{sp_j}; \rho^{sp_j}]$ -ring isomorphism

$$B[x^s; \rho^s] \cong B[x^{sp_j}; \rho^{sp_j}][T; \tilde{\rho}^s] / (T^{p_j} - x^{sp_j})B[x^{sp_j}; \rho^{sp_j}][T; \tilde{\rho}^s].$$

Since $B[x^s; \rho^s]$ and $B[x^{sp_j}; \rho^{sp_j}]$ are H -separable extension over B , it follows from [9, Proposition 2.2] that $T^{p_j} - x^{sp_j}$ is an H -separable polynomial in $B[x^{sp_j}; \rho^{sp_j}][T; \tilde{\rho}^s]$. We shall show that the center of $B[x^{sp_j}; \rho^{sp_j}] = Z^{\rho^{sp_j}}$. In fact, the center of $B[x^{sp_j}; \rho^{sp_j}] \supseteq Z^{\rho^{sp_j}}$ is clear and for any $y = \sum_{\nu=0}^{q-1} (x^{sp_j})^\nu d_\nu$ in the center of $B[x^{sp_j}; \rho^{sp_j}]$, we have

$$(\rho^{sp_j})^\nu(b)d_\nu = d_\nu b \text{ (} b \in B \text{) and } \rho^{sp_j}(d_\nu) = d_\nu \text{ (} 0 \leq \nu \leq q-1 \text{)}.$$

Since $Y^q - u$ is an H -separable polynomial in $B[Y; \rho^t] = B[Y; \rho^{sp_j}]$, it follows from [3, Lemma 1(1)] that $d_\nu = 0$ ($1 \leq \nu \leq q-1$). Hence $y = d_0 \in Z^{\rho^{sp_j}}$. Since $T^{p_j} - x^{sp_j}$ is H -separable in $B[x^{sp_j}; \rho^{sp_j}][T; \tilde{\rho}^s]$ and p_j is a prime number, $Z^{\rho^{sp_j}} / Z^{\rho^s}$ is a $(\rho^s | Z^{\rho^{sp_j}})$ -Galois extension of order p_j by [4, Theorem 2]. Thus the assertion follows from Lemma 2.

In the proof of [4, Theorem 4] we have proved the following: Let $f = X^{p^e} - u$ be a separable polynomial in $B[X; \rho]$. If p is a prime number, and p is contained in the Jacobson radical $J(B)$ of B , then Z/Z^p is a $(\rho|Z)$ -Galois extension, and the group $(\rho|Z)$ is of order p . We shall generalize this result as follows:

Theorem 3. *Let $f = X^m - u$ be in $B[X; \rho]$ with $fB[X; \rho] = B[X; \rho]f$, and $m = \ell p^e$, $(\ell, p) = 1$. Assume that p is a prime number, and p is contained in the Jacobson radical $J(B)$ of B .*

- (1) *If f is a separable polynomial in $B[X; \rho]$, then Z/Z^{p^e} is a $(\rho^e|Z)$ -Galois extension, and the group $(\rho^e|Z)$ is of order p^e .*
- (2) *If f is an H -separable polynomial in $B[X; \rho]$ and ℓ is a prime number, then Z/Z^p is a $(\rho|Z)$ -Galois extension and the group $(\rho|Z)$ is of order m .*

Proof. (1) Since f is a separable polynomial in $B[X; \rho]$, it follows from [6, Theorem 3.1] that there exists an element $c \in Z$ such that

$$c + \rho(c) + \rho^2(c) + \cdots + \rho^{m-1}(c) = 1.$$

We put here

$$d = c + \rho(c) + \cdots + \rho^{\ell-1}(c).$$

Then we have

$$d + \rho^\ell(d) + (\rho^\ell)^2(d) + \cdots + (\rho^\ell)^{p^e-1}(d) = 1.$$

We consider the polynomial $g = Y^{p^e} - u \in B[Y; \rho^\ell]$. Then g is a separable polynomial in $B[Y; \rho^\ell]$ by [6, Theorem 3.1] again. Since $p \in J(B)$, it follows from the proof of [4, Theorem 4] that Z/Z^{p^e} is a $(\rho^e|Z)$ -Galois extension and the group $(\rho^e|Z)$ is of order p^e .

(2) We have

$$B[x; \rho] \supset B[x^\ell; \rho^\ell] \cong B[Y; \rho^\ell]/(Y^{p^e} - u)B[Y; \rho^\ell] \supset B.$$

As was shown in (1), $Y^{p^e} - u$ is an H -separable polynomial in $B[Y; \rho^\ell]$. Since $B[x; \rho]$ is H -separable over B , it follows from [9, Proposition 2.2] that $B[x; \rho]$ is H -separable over $B[x^\ell; \rho^\ell]$. Naturally, we can extend ρ to the automorphism $\bar{\rho}$ of $B[x^\ell; \rho^\ell]$. Consider the skew polynomial ring $B[x^\ell; \rho^\ell][T; \bar{\rho}]$, where $\alpha T = T\bar{\rho}(\alpha)$ ($\alpha \in B[x^\ell; \rho^\ell]$). Since

$$B[x; \rho] \cong B[x^\ell; \rho^\ell][T; \bar{\rho}]/(T^\ell - x^\ell)B[x^\ell; \rho^\ell][T; \bar{\rho}],$$

$T^\ell - x^\ell$ is an H -separable polynomial in $B[x^\ell; \rho^\ell][T; \bar{\rho}]$. We shall show that $V_{B[x^\ell; \rho^\ell]}(B[x^\ell; \rho^\ell]) = Z^{\rho^\ell}$. $V_{B[x^\ell; \rho^\ell]}(B[x^\ell; \rho^\ell]) \supseteq Z^{\rho^\ell}$ is clear. On the other hand, for any $y = \sum_{\nu=0}^{p^e-1} (x^\ell)^\nu \alpha_\nu \in V_{B[x^\ell; \rho^\ell]}(B[x^\ell; \rho^\ell])$, we obtain

$$(\rho^\ell)^\nu(b)\alpha_\nu = \alpha_\nu b \quad (b \in B) \quad \text{and} \quad \rho^\ell(\alpha_\nu) = \alpha_\nu \quad (0 \leq \nu \leq p^e - 1).$$

Since $Y^{p^e} - u$ is an H -separable polynomial in $B[Y; \rho^\ell]$, it follows from [3, Lemma 1(1)] that $\alpha_\nu = 0$ ($1 \leq \nu \leq p^e - 1$). Hence $y = \alpha_0 \in Z^{\rho^\ell}$, and so $V_{B[x^\ell; \rho^\ell]}(B[x^\ell; \rho^\ell]) = Z^{\rho^\ell}$. Since ℓ is a prime number, it follows from [4, Theorem 2] that Z^{ρ^ℓ}/Z^ρ is a $(\rho|Z^{\rho^\ell})$ -Galois extension, and the group $(\rho|Z^{\rho^\ell})$ is of order ℓ . By (1), Z/Z^{ρ^ℓ} is a $(\rho^\ell|Z)$ -Galois extension, and the group $(\rho^\ell|Z)$ is of order p^e . Then the assertion follows from Lemma 2.

Combining Theorem 3 and [2, Proposition 1.4] we have the following which is a generalization of [4, Theorem 4].

Corollary 4. *Let $f = X^m - u$ be in $B[X; \rho]$ with $fB[X; \rho] = B[X; \rho]f$, and $m = \ell p^e$, $(\ell, p) = 1$. Assume that p is a prime number, and p is contained in the Jacobson radical $J(B)$ of B . If f is a separable polynomial in $B[X; \rho]$, then $g = Y^{p^e} - u$ is an H -separable polynomial in $B[Y; \rho^\ell]$.*

The following is a sharpening of [4, Theorem 4], which corresponds to the results of Nagahara [7, Theorems 1 and 2].

Corollary 5. *Let $f = X^m - u$ be in $B[X; \rho]$ with $fB[X; \rho] = B[X; \rho]f$, and $m = \ell p^e$, $(\ell, p) = 1$. Assume that p is a prime number, and p is contained in the Jacobson radical $J(B)$ of B . Then the following are equivalent :*

- (a) *u is invertible in B^ρ , and Z/Z^ρ is a G -Galois extension, where G is the group generated by $\rho|Z$ of order m .*
- (b) *$B[x^n; \rho^n]$ is an H -separable extension over B for every divisor n of m .*
- (c) *$B[x; \rho]$ is a separable extension over B , $B[x; \rho]$ is an H -separable extension over $B[x^\ell; \rho^\ell]$ and $B[x^r; \rho^r]$ is an H -separable extension over B for every divisor r ($1 < r < \ell$) of ℓ .*
- (d) *$B[x; \rho]$ is a separable extension over B and $B[x^r; \rho^r]$ is an H -separable extension over $B[x^\ell; \rho^\ell]$ for every divisor r ($1 \leq r \leq \ell$) of ℓ .*

Proof. (a) \iff (b) was shown in Theorem 1.

(b) \implies (c). Since both $B[x; \rho]$ and $B[x^\ell; \rho^\ell]$ are H -separable extension over B , it follows from [9, Proposition 2.2] that $B[x; \rho]$ is an H -separable extension over $B[x^\ell; \rho^\ell]$.

(c) \implies (d). Since f is a separable polynomial in $B[X; \rho]$, as was shown in the proof of Theorem 3(1), $B[x^\ell; \rho^\ell]$ is an H -separable extension over B . Hence by [9, Proposition 2.2], $B[x^r; \rho^r]$ is an H -separable extension over $B[x^\ell; \rho^\ell]$ ($1 \leq r \leq \ell$).

(d) \implies (a). It follows from Theorem 1, Theorem 3(1) and careful reading of the proof of Theorem 3(2).

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