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K-SEMIMETRIZABILITIES AND C-STRATIFIABILITIES OF SPACES

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1. Introduction and definitions

In 1966, Arhangel'skiĭ [1] introduced the concepts of symmetrizable spaces and he showed that a T_2-space is metrizable if, and only if, it has a compatible symmetric d satisfying condition (A): $d(F, K)>0$ for any disjoint closed subset F and compact subset K. Also, Arhangel'skiĭ gave the class of spaces with a compatible symmetric d satisfying condition (K): $d(H, K)>0$ for any disjoint compact subsets H and K, and he conjectured that every symmetrizable space has a compatible symmetric satisfying condition (K). After that, in 1975, Martin [26] presented the question on whether every regular semimetrizable space is K-semimetrizable (i.e. it has a compatible semimetric satisfying condition (K)) , or if every Moore space is K-semimetrizable. In 1979, Burke [6] gave a negative answer that there exists a separable Moore space which is not K-semimetrizable.

Lee [22] defined the class of c-stratifiable spaces which contains the classes of spaces with a regular G_δ-diagonal and of γ, T_2-spaces. He proved that a space X is K-semimetrizable if, and only if, X is c-stratifiable semimetrizable if, and only if, X is regular c-stratifiable, first countable and β. On the other hand, in [31], we introduced the concepts of strong α-ness and showed that every strongly α, wM-space is metrizable. The properties of strongly α-spaces were also studied in the same paper.

In this note, we study the relations among c-stratifiable spaces, strongly α-spaces, K-semimetrizable spaces, developable spaces and Nagata spaces, and the conditions for spaces to be K-semimetrizable or full K-semimetrizable.

We prove that a space X is K-semimetrizable if, and only if, it is a c-stratifiable q, β-space. We also show that a space X is full K-semimetrizable if, and only if, it is a $w\theta$, β-space with a regular G_δ-diagonal, which is a slight generalization of [32; Theorem 2]. We also show that a space X is Nagata if, and only if, it is K-semimetrizable wcc if, and only if, it is regular semimetrizable wcc. Moreover, for metrizations of wM-spaces, we have that every wM-space with a G^*_δ-diagonal is metrizable.

In §2, we study the relations between c-stratifiable spaces and strongly α-spaces. Also, we consider the conditions for spaces to be strongly α or c-stratifiable. In particular, we show that in the realm of c-stratifiable spaces, wN-spaces are Nagata, q-spaces are first countable, wcc-spaces are k-semistratifiable and $w\Delta$-spaces are developable.

In §3, we study the class of K-semimetrizable spaces. First, we show that a space X is K-semimetrizable if, and only if, it is c-stratifiable q, β. Secondly, we prove that in the class of pseudocompact spaces or locally connected rim-compact spaces, developable K-semimetrizable spaces are equivalent to c-stratifiable β-spaces (or K-semimetrizable spaces), and every metacompact p-space with a G_δ-diagonal is a K-semimetrizable Moore space.

In §4, for the class of $w\theta$, wcc-spaces which contains the class of wM-spaces, we show that every $w\theta$, wcc-space with a G^*_δ-diagonal is metrizable and every c-stratifiable $w\theta$, wcc-space is metrizable.

Throughout this paper, we assume that all spaces are T_1, but paracompactness is assumed to be T_2. We denote a sequence $\{x_n|n \in \mathbb{N}\}$ by $\{x_n\}$ and the set of natural numbers by $\mathbb{N}$. Finally, we refer the reader to [9] for undefined terms.

Definition 1.1. A g-*function* on a space X with a topology $\mathcal{T}$ is a map $g : \mathbb{N}\times X \longrightarrow \mathcal{T}$ such that $g(n, x) = g_n(x)$ is an open neighbourhood of x for every $x \in X$ and each $n \in \mathbb{N}$ and we denote the map g by $(\{g_n(x)\}|x \in X)$. For a subset A of X, we put $g_n(A) = \cup\{g_n(x)|x \in A\}$.

A point p in X is called a *cluster point* of a sequence $\{x_n\} \subset X$ if any open neighbourhood of p contains x_n for infinitely many n's.

For a space X, we now consider the following conditions on a g-function $(\{g_n(x)\}|x \in X)$.

(A) If $g_n(x) \cap g_n(x_n) \neq \emptyset$ $(n \geq 1)$, then x is a cluster point of $\{x_n\}$.

(B) If $g_n(x) \cap g_n(x_n) \neq \emptyset$ $(n \geq 1)$, then $\{x_n\}$ has a cluster point.

(C) If $x \in g_n(x_n)$ $(n \geq 1)$, then x is a cluster point of $\{x_n\}$.

(D) If $x \in g_n(x_n)$ $(n \geq 1)$, then $\{x_n\}$ has a cluster point.

(E) If $y_n \in g_n(x_n)$ $(n \geq 1)$ and $\{y_n\}$ has a cluster point, then $\{x_n\}$ has a cluster point.

(F) If $x_n \in g_n(x)$ $(n \geq 1)$, then $\{x_n\}$ has a cluster point.

(G) If $y_n \in g_n(p)$, $x_n \in g_n(y_n)$ $(n \geq 1)$, then p is a cluster point of $\{x_n\}$.

(H) If $y_n \in g_n(p)$, $x_n \in g_n(y_n)$ $(n \geq 1)$, then $\{x_n\}$ has a cluster point.

(I) If $y_n \in g_n(p)$, $x_n, p \in g_n(y_n)$ $(n \geq 1)$, then p is a cluster point of $\{x_n\}$.

(J) If $y_n \in g_n(p)$, $x_n, p \in g_n(y_n)$ $(n \geq 1)$, then $\{x_n\}$ has a cluster point.

(K) If $x_n, p \in g_n(y_n)$ $(n \geq 1)$, then p is a cluster point of $\{x_n\}$.

(L) If $x_n, p \in g_n(y_n)$ $(n \geq 1)$, then $\{x_n\}$ has a cluster point.

In the above conditions (A)-(L), we can assume that $g_{n+1}(x) \subset g_n(x)$ for every $x \in X$ and each $n \in \mathbb{N}$.

Definition 1.2. A space with a g-function satisfying (A) is called a $Nagata$ space [15] (Nagata spaces were first defined by Ceder [7]) and a space with a g-function satisfying (B) is called a wN-space [18]. In this case the g-function is called a $Nagata\text{-}function$ (a $wN\text{-}function$, respectively).

Definition 1.3. A space X is called a $semistratifiable$ (β-, *wcc* (=*weak contraconvergent*)-, q-, γ-, $w\gamma$-, θ-, $w\theta$-) space if X has a g-function satisfying (C) ((D), (E), (F), (G), (H), (I), (J), respectively). (See [17], [18] and [31])

The following result is not difficult to see.

Proposition 1.4. [31; Theorem 3.5] *A space X is wN if, and only if, it is q and wcc.*

Definition 1.5. A space X is called $stratifiable$ [3] (equivalently, M_3 [7]) if X has a g-function that satisfies (C) and if $x \notin \overline{g_m(F)}$ for some $m \in \mathbb{N}$, whenever F is closed and $x \notin F$. The class of $k\text{-}semistratifiable$ spaces introduced by Lutzer [24] can be characterized by the following conditions [12, 31]. A space X is k-semistratifiable if, and only if, X has a g-function $(\{g_n(x)\}|x \in X)$ such that $g_m(F) \cap K = \emptyset$ for some $m \in \mathbb{N}$, whenever F is closed, K is compact and $F \cap K = \emptyset$, if, and only if, in the class of T_2-spaces, X has a g-function $(\{g_n(x)\}|x \in X)$ such that whenever $y_n \in g_n(x_n)$ $(n \geq 1)$ and $\{y_n\} \longrightarrow y$, then $\{x_n\} \longrightarrow y$.

The following implications are known.
Nagata $\Longrightarrow$ stratifiable $\Longrightarrow$ k-semistratifiable $\Longrightarrow$ semistratifiable $\Longrightarrow$ β.
Also, it is known that a Nagata space is equivalent to a first countable stratifiable space and every stratifiable space is paracompact. Every semistratifiable space X is subparacompact and has a G_δ-diagonal if it is T_2 [14; Theorem 5.11].

2. c-STRATIFIABLE SPACES AND STRONGLY α-SPACES

We begin by considering the relations between c-stratifiable spaces and strongly α-spaces, and the conditions for spaces to be c-stratifiable or strongly α.

Definition 2.1. A space X is called *c-stratifiable* [22] (*c-semistratifiable* [25]) if X has a g-function such that if $x \notin K$, where K is compact, then $x \notin \overline{g_m(K)}$ ($x \notin g_m(K)$; in [25], it is assumed that K is closed compact) for some $m \in \mathbb{N}$. A space X is called $cs\text{-}stratifiable$ if X has a g-function such

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that if $x \notin C$, where C is the union of a convergent sequence and any one of its limit points, then $x \notin \overline{g_m(C)}$ for some $m \in \mathbb{N}$. A space X is called *weak c-stratifiable* if X has a g-function such that, whenever C and D are disjoint compact subsets, then $g_m(C) \cap D = \emptyset$ for some $m \in \mathbb{N}$.

Every stratifiable space or γ, T_2-space is c-stratifiable [22], but the Sorgenfrey line is a paracompact c-stratifiable space which is not semistratifiable. Also, every c-stratifiable space is cs-stratifiable and weak c-stratifiable [22; Theorem 1.3], every cs-stratifiable space is T_2 and every semistratifiable T_2-space is c-semistratifiable.

Definition 2.2. A space X is called *strongly* α [31] (α [17]) if X has a g-function such that (i) for each $n \in \mathbb{N}$, $y \in g_n(x) \Longrightarrow g_n(y) \subset g_n(x)$ and (ii) $\cap_{n\geq 1}\overline{g_n(x)} = \{x\}$ ($\cap_{n\geq 1} g_n(x) = \{x\}$).

For g-functions in Definitions 2.1 and 2.2, we can assume that $g_{n+1}(x) \subset g_n(x)$ for every $x \in X$ and each $n \in \mathbb{N}$. Every strongly α-space is also T_2.

Theorem 2.3. *Every cs-stratifiable q-space X or regular weak c-stratifiable q-space X is a first countable c-stratifiable space.*

Proof. Let g be a q and cs-stratifiable function of a space X. We show that $\{g_n(x)\}$ is an open neighbourhood base of x for every $x \in X$. Suppose that $x \in X$ and $x_n \in g_n(x) \setminus U$ $(n \geq 1)$ for some open neighbourhood U of x. Since g is a q-function, $\{x_n\}$ has a cluster point p and $p \notin \{x\}$. Since g is a cs-stratifiable function, $p \notin \overline{g_m(x)} \supset \overline{\{x_j | j \geq m\}} \ni p$ for some $m \in \mathbb{N}$. This contradiction implies that $\{g_n(x)\}$ is a neighbourhood base of x. To see that g is a c-stratifiable function, suppose that $x \notin K$, where K is compact in X, and $x \in \cap_{n\geq 1}\overline{g_n(K)}$. Then, there exist sequences $\{x_n\} \subset K$ and $\{y_n\}$ such that $y_n \in g_n(x) \cap g_n(x_n)$. Since K is sequentially compact, $\{x_{n(i)}\} \longrightarrow p$ for some point $p \in K$ and some subsequence $\{x_{n(i)}\} \subset \{x_n\}$, and $\{y_{n(i)}\} \longrightarrow x$ for the subsequence $\{y_{n(i)}\} \subset \{y_n\}$. Then $x \notin \{x_{n(i)} | i \geq 1\} \cup \{p\} = C$, and hence $x \notin \overline{g_m(C)}$ for some $m \in \mathbb{N}$. Therefore, for some $n(j) \geq m$, $y_{n(j)} \notin \overline{g_m(C)} \supset g_{n(j)}(x_{n(j)}) \ni y_{n(j)}$. This contradiction implies that g is also a c-stratifiable function.

For the second part, we can assume that g is a weak c-stratifiable q-function satisfying $\overline{g_{n+1}(x)} \subset g_n(x)$. If x and y are distinct points, then $g_m(x) \cap \{y\} = \emptyset$ for some $m \in \mathbb{N}$. Hence, $\cap_{n\geq 1}\overline{g_n(x)} = \{x\}$. Suppose that $x \in X$ and $x_n \in g_n(x) \setminus U$ $(n \geq 1)$ for some open neighbourhood U of x. Then $\{x_n\}$ has a cluster point p. Hence, $p \in \overline{g_n(x)}$ for each $n \in \mathbb{N}$. This contradiction asserts that $\{g_n(x)\}$ is a neighbourhood base of x. Therefore,

g is a c-stratifiable function by [22; Theorem 1.3].

Theorem 2.4. (1) *Every strongly α-space X or k-semistratifiable T_2-space X is weak c-stratifiable.*

(2) *Every strongly α, q-space X is c-stratifiable.*

Proof. (1): First, let g be a strongly α-function of X. Suppose that there are disjoint compact subsets C and D such that $x_n \in g_n(C) \cap D$ $(n \geq 1)$. Then $x_n \in g_n(y_n)$ for some sequence $\{y_n\} \subset C$. Hence $\{y_n\}$ clusters at a point $y \in C$ and contains a subsequence $\{y_{n(i)}\}$ such that $y_{n(i)} \in g_i(y)$. Then $x_{n(i)} \in g_{n(i)}(y_{n(i)}) \subset g_i(y_{n(i)}) \subset g_i(y) (i \geq 1)$, and $\{x_{n(i)}\}$ has a cluster point $x \in D$. Then for each $i \in \mathbb{N}$, $x \in \overline{\{x_{n(j)} | j \geq i\}} \subset \overline{g_i(y)}$. Therfore $x = y$, which is a contradiction. Next, that a k-semistratifiable T_2-space is weak c-stratifiable follows from the equivalent condition of a k-semistratifiable space in Definition 1.5.

(2): Let g be a q-function and h be a strongly α-function of a space X. Here, we can assume that $g_n(x) \subset h_n(x)$. For some $x \in X$ and some compact subset K, suppose that $x \notin K$ and $x \in \cap_{n \geq 1} \overline{g_n(K)}$. Then there exist sequences $\{y_n\}$ and $\{z_n\}$ such that $y_n \in K$ and $z_n \in g_n(x) \cap g_n(y_n)$. Let $y \in K$ be a cluster point of $\{y_n\}$. Then $y_{n(i)} \in g_i(y)$ for some increasing subsequence $\{n(i)\}$ of $\mathbb{N}$. Also, since $z_{n(i)} \in g_i(x) (i \geq 1)$, $\{z_{n(i)}\}$ has a cluster point z. Since $y_{n(i)} \in h_i(y) (i \geq 1)$, $z_{n(i)} \in g_{n(i)}(y_{n(i)}) \subset h_i(y_{n(i)}) \subset h_i(y)$. Therefore, $\{z_{n(j)} | j \geq i\} \subset h_i(y) (i \geq 1)$ and hence, $z \in \overline{h_i(y)} (i \geq 1)$, which implies that $y = z$. Moreover, since $z_{n(i)} \in g_i(x) \subset h_i(x) (i \geq 1)$, we have $\{z_{n(j)} | j \geq i\} \subset h_i(x)$, and hence $z \in \overline{h_i(x)}$. Consequently, $x = z$. This contradiction implies that g is a c-stratifiable function.

We now study the conditions for spaces to be c-stratifiable or strongly α.

Definition 2.5. A space X is called a $w\Delta$-space [4] if it has a sequence $\{\mathcal{G}_n\}$ of open covers such that whenever $x_n \in st(x, \mathcal{G}_n)$ $(n \geq 1)$, then $\{x_n\}$ has a cluster point. A space X is called a *developable* space if it has a sequence $\{\mathcal{G}_n\}$ of open covers such that for each $x \in X$, the sequence $\{st(x, \mathcal{G}_n)\}$ is a neighbourhood base of x. A regular developable space is called a *Moore* space. These spaces are characterized by g-functions as follows [18]: A space X is $w\Delta$ (*developable*) if and only if X has a g-function satisfying (L) ((K), respectively).

Definition 2.6. (1) For each $k \in \mathbb{N}$, a space X is said to have a $G_\delta(k)$-*diagonal* if X has a sequence $\{\mathcal{G}_n\}$ of open covers such that for any distinct points x and y, there exists $m \in \mathbb{N}$ such that $y \notin st^k(x, \mathcal{G}_m)$, where $st^{k+1}(x, \mathcal{G}_m) = st(st^k(x, \mathcal{G}_m))$.

(2) A sequence $\{\mathcal{G}_n\}$ of open covers of a space X is said to satisfy the 3-*link property* [32] (equivalently, it is a $G_\delta(3)$-diagonal sequence) if it is true that for any distinct points x and y, there exists $m \in \mathbb{N}$ such that no member of $\mathcal{G}_m$ intersects both $st(x, \mathcal{G}_m)$ and $st(y, \mathcal{G}_m)$.

(3) A space X is said to have a *regular G_δ-diagonal* [32] if there is a sequence $\{\mathcal{G}_n\}$ of open covers of X such that if x and y are distinct points of X, then there are an integer m and open neighbourhoods U and V of x and y, respectively, such that no member of $\mathcal{G}_m$ intersects both U and V.

(4) A space X is said to have a *G^*_δ-diagonal* if X has a sequence $\{\mathcal{G}_n\}$ of open covers such that whenever $x \neq y$, there exists $m \in \mathbf{N}$ that satisfies $y \notin \overline{st(x, \mathcal{G}_m)}$.

It is easily seen that for a sequence $\mathcal{G} = \{\mathcal{G}_n\}$ of open covers of a space X, $\mathcal{G}$ is $G_\delta(2)$-diagonal if, and only if, whenever $x \neq y$, there exists $m \in \mathbf{N}$ satisfying $x \notin st(p, \mathcal{G}_m)$ or $y \notin st(p, \mathcal{G}_m)$ for every $p \in X$ (this property is called *strong G_δ-diagonal* in [31]).

We note that for properties of a sequence $\{\mathcal{G}_n\}$ of open covers of a space X, the following implications hold:

3-link property $\Rightarrow$ regular G_δ-diagonal $\Rightarrow$ G^*_δ-diagonal $\Rightarrow$ G_δ-diagonal and

3-link property $\Rightarrow$ $G_\delta(2)$-diagonal = strong G_δ-diagonal $\Rightarrow$ G^*_δ-diagonal.

In the realm of paracompact spaces, these properties are all equivalent. Every Nagata space is paracompact and has a G_δ-diagonal. Every developable T_2-space has a $G_\delta(2)$-diagonal and every regular semistraifiable space has a G^*_δ-diagonal [14, 17]. On the other hand, the space Ψ in Example 4.5 is a Moore space which does not have a regular G_δ-diagonal.

Definition 2.7. (1) A space X is called *orthocompact* if every open cover of X has an open refinement $\mathcal{V}$ such that $\cap\mathcal{W} = \cap\{W | W \in \mathcal{W}\}$ is open for every $\mathcal{W} \subset \mathcal{V}$.

(2) A space X is called *submetrizable* if there is a continuous one-to-one map from X onto a metric space.

It is well known that the following implications hold:

metacompact spaces $\Longrightarrow$ orthocompact spaces, and

stratifiable spaces $\Longrightarrow$ paracompact spaces with a G_δ-diagonal [3, 29] $\Longrightarrow$ submetrizable spaces .

Theorem 2.8. (1) *Every space X with a regular G_δ-diagonal is c-stratifiable.*

(2) *Every orthocompact space X with a G_δ^*-diagonal, or orthocompact regular space X with a G_δ-diagonal is strongly α.*

(3) *Every orthocompact developable T_2-space X is strongly α and c-stratifiable.*

(4) *Every orthocompact regular c-semistratifiable β-space X is strongly α.*

(5) *Every submetrizable space X is strongly α and c-stratifiable.*

Proof. (1) is proved in [22; Proposition 3.2].

(2): Let X be a regular space and let $\{\mathcal{G}_n\}$ be a G_δ-diagonal sequence of X. Then for each $n \in \mathbb{N}$, $\mathcal{G}_n$ has an open refinement $\mathcal{H}_n$ such that $\{\overline{H} | H \in \mathcal{H}_n\}$ is a refinement of $\mathcal{G}_n$ and $\cap\mathcal{W}$ is open for every $\mathcal{W} \subset \mathcal{H}_n$. Therefore, in both cases, we may assume that there exists a sequence $\{\mathcal{H}_n\}$ of open covers such that

(i) for each $n \in \mathbb{N}$, $\cap\mathcal{W}$ is open for every $\mathcal{W} \subset \mathcal{H}_n$, and

(ii) for distinct points x and y, there is an $m \in \mathbb{N}$ such that $x \in H \subset \overline{H}$ and $y \notin \overline{H}$ for some $H \in \mathcal{H}_m$.

Here, for any $x \in X$ and each $n \in \mathbb{N}$, we put $h_n(x) = \cap\{H \in \mathcal{H}_n | x \in H\}$. Then the g-function $(\{h_n(x)\} | x \in X)$ satisfies the conditions of Definition 2.2.

(3): Since every developable T_2-space has a G_δ^*-diagonal, X is a strongly α from (2) and it is c-stratifiable from Theorem 2.4.

(4): Since every regular c-semistratifiable β-space is semistratifiable [25; Theorem 3], it has a G_δ-diagonal. Hence X is strongly α from (2).

(5): Let $f : X \longrightarrow M$ be a continuous one-to-one onto map, where M is a metric space. By (3), M is strongly α and c-stratifiable. Therefore (5) follows from the following fact:

Let $f : X \longrightarrow Y$ be a continuous one-to-one onto map. If h is a strongly α-function (c-stratifiable function) of Y, then $(\{g_n(x)\} | x \in X)$, where $g_n(x) = f^{-1}[h_n(f(x))]$, is a strongly α-function (c-stratifiable function, respectively) of X.

We note that every metacompact regular semistratifiable q-space is strongly α and hence it is c-stratifiable. Also, every regular k-semistratifiable q-space is Nagata [31], and hence it is strongly α, c-stratifiable. On the other hand, the separable Moore space X in Example 4.6 is neither strongly α nor c-stratifiable.

The following question arises naturally from (4) of the above Theorem.

Question 2.9. Is every paracompact (or metacompact regular) c-semistratifiable q-space, c-stratifiable ?

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Theorem 2.10. *For a cs-stratifiable space X, the following implications hold:*

(1) $\beta \Longrightarrow semistratifiable$, (2) $wcc \Longrightarrow k$-$semistratifiable$, (3) $wN \Longrightarrow Nagata$, (4) $w\theta \Longrightarrow \theta$, (5) $w\gamma \Longrightarrow \gamma$ and (6) $w\Delta \Longrightarrow developable$.

Proof. (1): Let g be a cs-stratifiable and β-function of X and let $x \in g_n(x_n)$ $(n \geq 1)$. For any subsequence $\{x_{n(i)}\}$ of $\{x_n\}$, $\{x_{n(i)}\}$ has a cluster point since $x \in g_i(x_{n(i)})(i \geq 1)$. Let p be a cluster point of $\{x_n\}$ and $p \neq x$. Then $\{x_n\}$ contains a subsequence $S = \{x_{n(j)}\}$ such that $x_{n(j)} \in g_j(p) \setminus \{x\}(j \geq 1)$. Since $\{x_{n(k)} | k \geq j\} \subset g_j(p)(j \geq 1)$ and $\{p\} = \cap_{j\geq 1}\overline{g_j(p)}$, p is the only cluster point of S. Hence S converges to p. Since $x \notin C = S \cup \{p\}$, $x \notin \overline{g_m(C)}$ for some $m \in \mathbb{N}$. But, for some $n(k) \geq m$, $x \in g_{n(k)}(x_{n(k)}) \subset g_m(C)$. This contradiction implies that $x = p$ is a cluster point of $\{x_n\}$.

(2): Let g be a cs-stratifiable, wcc-function satisfying $\{x\} = \cap_{n\geq 1}\overline{g_n(x)}$ for every $x \in X$. Since X is a T_2-space, it is enough to show that g satisfies the k-semistratifiable condition of Definition 1.5. Let $y_n \in g_n(x_n)$ $(n \geq 1)$ and $\{y_n\} \longrightarrow y$. First, we show that $\{x_n\}$ contains a subsequence which converges to y. Indeed, for any subsequence $\{x_{n(i)}\}$ of $\{x_n\}$, $y_{n(i)} \in g_{n(i)}(x_{n(i)}) \subset g_i(x_{n(i)})(i \geq 1)$ and $\{y_{n(i)}\} \longrightarrow y$, hence $\{x_{n(i)}\}$ has a cluster point. Let p be a cluster point of $\{x_n\}$. It is easily seen that there exists a subsequence $S = \{x_{n(i)}\}$ of $\{x_n\}$ such that $x_{n(i)} \in g_i(p)$. Since $\overline{\{x_{n(j)} | j \geq i\}} \subset \overline{g_i(p)}$ for each $i \in \mathbb{N}$, p is a unique cluster point of S. Hence S converges to p. If $p \neq y$, then $y \notin \{x_{n(i)} | i \geq m\} \cup \{p\} = C$ for some $m \in \mathbb{N}$. Therefore $y \notin \overline{g_{n(k)}(C)}$ for some $k \geq m$. This contradiction asserts that S converges to y. Next, if $\{x_n\}$ does not converge to y, then we have an open neighbourhood W of y and a subsequence $\{x_{n(i)}\}$ such that $\{x_{n(i)}\} \cap W = \emptyset$. Then since $y_{n(i)} \in g_{n(i)}(x_{n(i)}) \subset g_i(x_{n(i)})(i \geq 1)$ and $\{y_{n(i)}\} \longrightarrow y$, $\{x_{n(i)}\}$ contains a subsequence which converges to y. This contradiction implies that $\{x_n\} \longrightarrow y$.

(3): Since X is q and wcc, by Theorem 2.3 and (2) of this theorem, there is a g-function g such that, whenever $y_n \in g_n(x_n)$ $(n \geq 1)$ and $\{y_n\} \longrightarrow y$, then $\{x_n\} \longrightarrow y$, and $\{g_n(x)\}$ is a neighbourhood base of x. To see that g is a Nagata function, let $y_n \in g_n(x) \cap g_n(x_n)$ $(n \geq 1)$. Then $\{y_n\} \longrightarrow x$. Hence $\{x_n\} \longrightarrow x$.

(4): Let g be a cs-stratifiable $w\theta$-function of X. Then g is a q-function. Indeed, let $x_n \in g_n(x)$ $(n \geq 1)$, then $\{x_n\}$ has a cluster point since $x \in g_n(x), x_n, x \in g_n(x)$ $(n \geq 1)$. Therefore, $\{g_n(x)\}$ is a neighbourhood base of x by Theorem 2.3. Now, suppose that $y_n \in g_n(p)$ and $x_n, p \in g_n(y_n)$ $(n \geq 1)$. Then $\{x_n\}$ has a cluster point x and $\{y_n\}$ converges to p. If $x \neq p$, then

$x \notin \{y_n | n \geq m\} \cup \{p\} = C$ for some $m \in \mathbb{N}$. Therefore $x \notin \overline{g_k(C)}$ for some $k \geq m$. This contradiction implies $x = p$, and hence g is a θ-function.

(5): Let g be a cs-stratifiable, $w\gamma$-function of a space X. Suppose that $y_n \in g_n(p), x_n \in g_n(y_n)$ $(n \geq 1)$. Then $\{x_n\}$ has a cluster point x, and $\{y_n\}$ converges to p since g is a q-function. If $p \neq x$, then $x \notin \{y_n | n \geq m\} \cup \{p\} = C$ for some $m \in \mathbb{N}$. Therefore $x \notin \overline{g_k(C)}$ for some $k \geq m$, which is a contradiction.

(6): Let g be a cs-stratifiable $w\Delta$-function of X. Since g is a β-function, g is a semistratifiable function. Now, suppose that $x_n, p \in g_n(y_n)$ $(n \geq 1)$. Then $\{y_n\}$ converges to p and $\{x_n\}$ has a cluster point x. If $x \neq p$, then $x \notin \{y_n | n \geq m\} \cup \{p\} = C$ for some $m \in \mathbb{N}$. Hence $x \notin \overline{g_k(C)}$ for some $k \geq m$. This contradiction implies that the function g satisfies condition (K).

Remark 2.11. (1) In the class of strongly α-spaces, the implications (3)-(6) in Theorem 2.10 are true by Theorem 2.4, (1) follows from [17; Theorem 5.2] and (2) follows from [31; Proposition 4.7].

(2) In the class of weak c-stratifiable regular spaces, the implications (3)-(6) in Theorem 2.10 are true by Theorem 2.3. For the implications (1) and (2), let g be a g-function satisfying the respective conditions. Then, since $\cap_{n\geq 1}\overline{g_n(x)} = \{x\}$, (1) and (2) are also true by a similar argument to the proof of Theorem 2.10.

The following questions regarding c-stratifiable spaces and strongly α-spaces are natural.

Question 2.12. When are c-stratifiability and strong α-ness coincident ?

Question 2.13. Is every paracompact first coutable c-stratifiable space strongly α ?

3. K-SEMIMETRIZABLE SPACES

Definition 3.1. Let X be a space. Then a function $d : X \times X \longrightarrow \mathbb{R}$ is called a *semimetric* if (i) $d(x,y) \geq 0$, (ii) $d(x,y) = 0 \iff x = y$ and (iii) $d(x,y) = d(y,x)$. X is called a *semimetrizable* space or X has a *compatible semimetric* if there exists a semimetric d on X such that for any subset $M \subset X$, $x \in \overline{M} \iff d(x,M) = 0$, or equivalently, for any $x \in X$ and any open neighbourhood U of x, $x \in \mathrm{int}B(x;\epsilon) \subset B(x;\epsilon) \subset U$ for some $\epsilon>0$; where $B(A;\delta) = \{y \in X | d(A,y) = \inf\{d(a,y) | a \in A\} < \delta\}$ for each $\delta>0$ and any subset $A \subset X$ and $B(x;\delta) = B(\{x\};\delta)$. Then, for a sequence $\{x_n\}$ in a semimetrizable space (X,d), $\lim_{n\to\infty} d(x,x_n) = 0$

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$\Longleftrightarrow \{x_n\} \longrightarrow x$ in X. A semimetrizable space X with a compatible semimetric d is K-*semimetrizable* [26] if $d(H,K) = \inf\{d(x,y) | x \in H, y \in K\} > 0$ for any disjoint compact subsets H and K. In this situation, d is called a K-*semimetric* on X.

It is well known [8] that a space X is semimetrizable if, and only if, it is a first countable, semistratifiable space.

Definition 3.2. Let $(X,\ d)$ be a semimetrizable space. For each $n \in \mathbb{N}$, we put $\mathcal{G}_n = \{\mathrm{int}B(x;\epsilon) | \delta B(x;\epsilon) < 1/n\}$, where for subset A of X, $\delta A = \sup\{d(x,y) | x, y \in A\}$. The semimetric d is said to be *full* if $\mathcal{G}_n$ is a cover of X for each $n \in \mathbb{N}$, or equivalently, if d satisfies Arhangel'skiǐ's condition (AN): At each point, there is a neighbourhood of arbitrarily small diameter [1]. A space X is called *full K-semimetrizable* if X has a compatible full K-semimetric.

Zenor investigated spaces with a regular G_δ-diagonal and gave the following result.

Theorem 3.3 ([32; Theorem 2]). *For a space X, the following conditions are equivalent.*

(1) *X has a development satisfying the 3-link property.*

(2) *X is a $w\Delta$-space with a regular G_δ-diagonal.*

(3) *X has compatible semimetric d satisfying*

(I) *If $\{x_n\} \longrightarrow x$ and $\{y_n\} \longrightarrow x$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$, and*

(II) *If $\{x_n\} \longrightarrow x$, $\{y_n\} \longrightarrow y$ and $x \neq y$, then there exist $r>0$ and $m \in \mathbb{N}$ such that $d(x_n, y_n) > r$ for each $n \geq m$.*

In substance, the first part of the following theorem is proved in (1)$\Longleftrightarrow$(3) of [32; Theorem 2] or [22; Lemma 5.3].

Theorem 3.4. (1) *For a space X, the following conditions are equivalent.*

(i) *X is a developable space.*

(ii) *X has a compatible full semimetric d.*

(iii) *X has a compatible semimetric d satisfying* (I) *of* Theorem 3.3.

(2) *A space X is developable T_2 if, and only if, it is $w\theta$, β and has a G^*_δ-diagonal.*

Proof of (2). We only prove the "if" part. Every β-space with a G^*_δ-diagonal is semistratifiable [17; Theorem 5.2] and every semistratifiable $w\theta$-space is $w\Delta$ [18; Proposition 4.5]. Hence X is developable [17; Theorem 2.5].

For a semimetric space, we have the following characterization. A regular space X is semimetrizable if, and only if, it is a q, β-space with a G^*_δ-diagonal.

Indeed, let g be a q-function and $\{\mathcal{G}_n\}$ be a G^*_δ-diagonal sequence of a space X. We put $h_n(x) = g_n(x) \cap st(x, \mathcal{G}_n)$, then $\{h_n(x)\}$ is a neighbourhood base of x. Also, X is semistratifiable from the proof of Theorem 3.4(2). For the converse implication, see [17].

The following theorem improves the result [11; Proposition 2.7] or [22; Theorem 5.2] that a space X is K-semimetrizable if, and only if, it is c-stratifiable and semimetrizable.

Theorem 3.5. *For a space X, the following conditions are equivalent.*

(1) *X is a K-semimetrizable space.*

(2) *X is a c-stratifiable semimetrizable space.*

(3) *X is a cs-stratifiable q, β-space.*

(4) *X has a compatible semimetric d satisfying* (II) *of* Theorem 3.3.

(5) *X has a compatible semimetric d such that, $x \notin \overline{B(K; 1/m)}$ for some $m \in \mathbb{N}$, whenever $x \notin K$ and K is compact.*

Proof. (1)$\Longrightarrow$(2) is proved in [22; Theorem 5.2] and (2)$\Longrightarrow$(3) is evident.

(3)$\Longrightarrow$(1): Let g be a cs-stratifiable q, β-function of X. Then by Theorems 2.3 and 2.10, g is a c-stratifiable and semistratifiable function, and $\{g_n(x)\}$ is an open neighbourhood base of x for every $x \in X$. Now, we define $d(x, x) = 0$ and $d(x, y) = 1/\inf\{j | x \notin g_j(y)$ and $y \notin g_j(x)\}$ if $x \neq y$. By [22; Theorem 5.2], (X, d) is K-semimetrizable.

(1)$\Longrightarrow$(4): Let d be a compatible K-semimetric on X. Suppose that $\{x_n\} \longrightarrow x$, $\{y_n\} \longrightarrow y$ and $x \neq y$. Since X is T_2, for some $m \in \mathbb{N}$, $H = \{x_n | n \geq m\} \cup \{x\}$ and $K = \{y_n | n \geq m\} \cup \{y\}$ are disjoint compact subsets. Therefore we have that $0 < d(H, K) \leq \inf\{d(x_n, y_n) | n \geq m\}$.

(4)$\Longrightarrow$(5): Suppose that $x \notin K$, where K is compact, and $x \in \overline{B(K; 1/n)}$ for each $n \in \mathbb{N}$ with respect to the semimetric d satisfying the condition of (4). Then there exists a sequence $\{z_n\}$ such that

$$z_n \in B(K; 1/n) \cap \mathrm{int} B(x; 1/n).$$

Hence $\{z_n\} \longrightarrow x$. Also $d(x_n, z_n) < 1/n$ for some sequence $\{x_n\} \subset K$. Then there exist subsequences $\{x_{n(i)}\} \subset \{x_n\}$ and $\{z_{n(i)}\} \subset \{z_n\}$ such that $\{x_{n(i)}\} \longrightarrow p$ for some $p \in K$ and $\{z_{n(i)}\} \longrightarrow x$. Therefore there exist

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$j, m \in \mathbb{N}$ such that $d(x_{n(i)}, z_{n(i)}) \geq 1/m$ for each $i \geq j$. On the other hand, $d(x_{n(k)}, z_{n(k)}) < 1/n(k)$ for some $n(k) \geq \max\{n(j), m\}$, which is a contradiction.

(5)$\Longrightarrow$(1): Let d be a compatible semimetric satisfying the condition of (5). If H and K are disjoint compact subsets of X with $d(H, K) = 0$, then $\lim d(x_n, y_n) = 0$ for some sequences $\{x_n\} \subset H$, $\{y_n\} \subset K$. Since X is first countable, there exist subsequences $\{x_{n(i)}\} \subset \{x_n\}$, $\{y_{n(i)}\} \subset \{y_n\}$ and points $x \in H$, $y \in K$ satisfying $\{x_{n(i)}\} \longrightarrow x$, $\{y_{n(i)}\} \longrightarrow y$. Since $y \notin \overline{B(H; 1/m)}$ for some $m \in \mathbb{N}$, we have that $B(y; 1/k) \cap B(H; 1/k) = \emptyset$ for some $k \geq m$. This contradicts the fact that $d(x_{n(i)}, y_{n(i)}) < 1/k$ and $d(y, y_{n(i)}) < 1/k$ for some $n(i) \in \mathbb{N}$.

Remark 3.6. (1) The space Y in Example 4.9 is c-stratifiable β, but not q, and the Sorgenfrey line is c-stratifiable q, but not β.

(2) The space X in Example 4.6 is Moore (hence, X has a G^*_δ-diagonal), but not K-semimetrizable, and the Nagata space X in Example 4.9 is K-semimetrizable, but not Moore.

(3) The space Y in Example 4.9 is stratifiable (hence c-stratifiable) Fréchet as the perfect image of a Nagata space (hence, K-semimetrizable), but Y is not semimetrizable (not even q).

Proposition 3.7. *Every K-semimetrizable space has a G^*_δ-diagonal.*

Proof. By Theorems 2.3 and 3.5, let g be a cs-stratifiable q, β-function of X such that $\{g_n(x)\}$ is a neighbourhood base of x. For each $n \in \mathbb{N}$, we put $\mathcal{G}_n = \{g_n(x) | x \in X\}$. To see that the sequence $\{\mathcal{G}_n\}$ is a G^*_δ-diagonal, suppose that $x \neq y \in \cap_{n \geq 1} \overline{st(x, \mathcal{G}_n)}$. Then there exist $z_n \in g_n(y) \cap st(x, \mathcal{G}_n)$ $(n \geq 1)$. Hence $\{z_n\} \longrightarrow y$ and $x, z_n \in g_n(x_n)$ for some sequence $\{x_n\}$. Then $\{x_n\} \longrightarrow x$ and $y \notin C = \{x_n | n \geq m\} \cup \{x\}$ for some $m \in \mathbb{N}$. Hence $y \notin \overline{g_k(C)}$ for some $k \geq m$. This is a contradiction.

The following theorem gives a condition for strong α-ness and c-stratifiability to be equivalent, and follows directly from Theorems 2.4, 2.8 and 3.5 and Proposition 3.7.

Theorem 3.8. *For an orthocompact β, q-space, the following conditions are equivalent.*

(1) *X is K-semimetrizable.*
(2) *X has a G^*_δ-diagonal.*
(3) *X is strongly α.*

(4) *X is cs-stratifiable.*

An analogue to Theorem 3.5 for the class of regular spaces follows directly from Theorem 2.3.

Theorem 3.9. *For a regular space X, X is K-semimetrizable if, and only if, it is weak c-stratifiable q, β.*

We next give some partial answers to the question of Burke [6; Question 2] on what minimal topological condition on a Moore space (or semimetric space) will ensure that the space is K-semimetrizable.

Theorem 3.10. (1) *Every T_2, orthocompact developable space X is K-semimetrizable.*

(2) *Every regular orthocompact semistratifiable q-space (hence, regular orthocompact semimetrizable space) X is K-semimetrizable.*

(3) *Every regular orthocompact c-semistratifiable q, β-space X is K-semimetrizable.*

(4) *Every regular k-semistratifiable q-space X is K-semimetrizable.*

Proof. Since a developable T_2-space has a $G_\delta(2)$-diagonal, (1) follows from Theorems 2.8 and 3.5. Since every semistratifiable T_2-space has a G_δ-diagonal, (2) follows from Theorems 2.8 and 3.5. For (3), since X is semistratifiable, (3) follows from (2). (4) follows from Theorems 2.3, 2.4 and 3.5.

Remark 3.11. (1) With regards to (2) of Theorem 3.10, it is already known [1; page 133] or [22; page 441], that every paracompact semimetrizable space is K-semimetrizable.

(2) In (2) and (3) of Theorem 3.10, we can not change orthocompactness to subparacompactness by Example 4.6.

(3) In (4) of Theorem 3.10, we already know that a space is regular k-semistratifiable q if, and only if, it is Nagata [31; Theorem 2.1]. But, we do not know whether every T_2, k-semistratifiable q-space is c-stratifiable. (If this answer is affirmative, then every T_2, k-semistratifiable q-space is first countable and Nagata.) The converse of (4) does not hold, because the space Ψ in Example 4.5 is not k-semistratifiable.

In the following theorem, the equivalence of (1) and (4) is proved in [22; Theorem 5.4].

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Theorem 3.12. *For a space X, conditions* (1)-(5) *are all equivalent and* (5) $\Longrightarrow$ (6) *holds.*

(1) *X is a full K-semimetrizable space.*

(2) *X has a development $\{\mathcal{G}_n\}$ such that if K_1 and K_2 are disjoint compact subsets, then $st(K_1, \mathcal{G}_m) \cap K_2 = \emptyset$ for some $m \in \mathbb{N}$.*

(3) *X has a development $\{\mathcal{G}_n\}$ such that if $p \notin C$, where C is the union of a convergent sequence and any one point of its limit points, then $p \notin \overline{st(C, \mathcal{G}_m)}$ for some $m \in \mathbb{N}$.*

(4) *X satisfies one of the equivalent conditions in* Theorem 3.3.

(5) *X is a $w\theta$, β-space with a regular G_δ-diagonal.*

(6) *X is a developable c-stratifiable space.*

Proof. (1)$\Longrightarrow$(2): Let d be a compatible full K-semimetric on X. For each $n \in \mathbb{N}$, we put $\mathcal{G}_n = \{\mathrm{int}B(x;\epsilon) | \delta B(x;\epsilon) < 1/n\}$. Then $\{\mathcal{G}_n\}$ is a development of X since d is a full semimetric. For, suppose that $x \in X$ and $x_n \in st(x, \mathcal{G}_n) \setminus U$ $(n \geq 1)$ for some open neighbourhood U of x. Then $x, x_n \in G_n$ and $\delta G_n < 1/n$ for some $G_n \in \mathcal{G}_n$, which is a contradiction. Now, suppose that K_1 and K_2 are disjoint compact subsets and $x_n \in st(K_1, \mathcal{G}_n) \cap K_2$ for each $n \in \mathbb{N}$. Then $y_n \in G_n \cap K_1$ and $x_n \in G_n$ for some $G_n \in \mathcal{G}_n$. Since $\delta G_n < 1/n$ $(n \geq 1)$, $\lim_{n\to\infty} d(x_n, y_n) = 0$. This contradicts $d(K_1, K_2) > 0$.

(2)$\Longrightarrow$(3): Let $\{\mathcal{G}_n\}$ be a development of X satisfying (2). To see that X is T_2. let $x \neq y$ and $x_n \in st(x, \mathcal{G}_n) \cap st(y, \mathcal{G}_n)$ for each $n \in \mathbb{N}$. Then $\{x_n\} \longrightarrow x$ and $\{x_n\} \longrightarrow y$. Given any open neighbourhood U of x with $y \notin U$, $S = \{x_n | n \geq m\} \cup \{x\} \subset U$ for some $m \in \mathbb{N}$. Then $st(y, \mathcal{G}_k) \cap S = \emptyset$ for some $k \geq m$. This contradicts $\{x_n\} \longrightarrow y$. Next, suppose that $p \notin K$, where K is compact, and $p \in \cap_{n\geq 1} \overline{st(K, \mathcal{G}_n)}$. Then $a_n \in st(p, \mathcal{G}_n) \cap st(K, \mathcal{G}_n) (n \geq 1)$. Hence $a_n \in st(x_n, \mathcal{G}_n)$ for some sequence $\{x_n\}$ in a sequentially compact K, and $\{x_n\}$ contains a subsequence $\{x_{n(i)}\}$ converging to some point $x \in K$. Since X is T_2, $L = \{x_{n(i)} | n(i) \geq m\} \cup \{x\}$ and $H = \{a_{n(i)} | n(i) \geq m\} \cup \{p\}$ are disjoint for some $m \in \mathbb{N}$. Therefore, $a_{n(k)} \in st(L, \mathcal{G}_{n(k)}) \cap H = \emptyset$ for some $n(k) \geq m$, which leads to a contradiction.

(3)$\Longrightarrow$(4): Let $\{\mathcal{G}_n\}$ be a development of X such that $\mathcal{G}_{n+1}$ is a refinement of $\mathcal{G}_n$ and satisfies (3). We now show that $\{\mathcal{G}_n\}$ satisfies the 3-link property. Suppose that $x \neq y$ and for each $n \in \mathbb{N}$, there exists $G_n \in \mathcal{G}_n$ such that $x_n \in G_n \cap st(x, \mathcal{G}_n)$ and $y_n \in G_n \cap st(y, \mathcal{G}_n)$. Since $\{x_n\} \longrightarrow x$, $\{y_n\} \longrightarrow y$ and X is T_2, $y \notin C = \{x_n | n \geq m\} \cup \{x\}$ for some $m \in \mathbb{N}$. Hence $y \notin \overline{st(C, \mathcal{G}_k)}$ for some $k \geq m$. Then $y_l \in X \setminus \overline{st(C, \mathcal{G}_k)}$ for some $l \geq k$ and $x_l \in C$. Therefore, $y_l \in G_l \subset st(x_l, \mathcal{G}_l) \subset st(C, \mathcal{G}_k)$, which is a contradiction.

(4)$\Longrightarrow$(5): Let X be a $w\Delta$-space with a regular G_δ-diagonal. Then X satisfies condition (5).

(5)$\Longrightarrow$(1): By Theorem 3.4, X is a developable space with a regular G_δ-diagonal. Hence there exists a compatible semimetric d on X satisfying (I) and (II) of (3) in Theorem 3.3. Then d is full by (3)$\Longrightarrow$(1) of [32; Theorem 2]. To see that d is K-semimetric, suppose that $d(K,H)=0$ for some disjoint compact subsets K and H. Then there are sequences $\{x_n\} \subset K$ and $\{y_n\} \subset H$ such that $\lim_{n\to\infty} d(x_n, y_n) = 0$. On the other hand, since X is a q-space with a G^*_δ-diagonal, X is first countable. Hence $\{x_n\}$ ($\{y_n\}$) contains a subsequence $\{x_{n(i)}\}$ ($\{y_{n(i)}\}$) converging to a point $x \in K$ ($y \in H$, respectively). Hence there are $k, m \in \mathbb{N}$ such that $d(x_{n(i)}, y_{n(i)}) \geq 1/m$ for each $i \geq k$ by (II). This is a contradiction. Finally, (5)$\Longrightarrow$(6) follows from Theorems 2.8 and 3.4.

Remark 3.13. (1) The space Ψ in Example 4.5 is Moore and K-semi -metrizable, but not full K-semimetrizable.

(2) Every $w\Delta$-space is $w\theta$ and β. Although the converse is an open problem [18; Problem 4.10], (4) $\Longleftrightarrow$ (5) of Theorem 3.12 (or (2) of Theorem 3.4) may be a slight progress to [32; Theorem 2] ([17; Theorem 2.5], respectively).

(3) The space X in Example 4.8 is T_2 metacompact, full K-semi -metrizable, but not regular.

Question 3.14. Is every normal metacompact, full K-semimetrizable space, metrizable?

We next investigate conditions for spaces to be developable and K-semimetrizable.

Theorem 3.15. *Consider the following conditions for a space X.*

(1) *X is developable and K-semimetrizable.*

(2) *X is K-semimetrizable $w\theta$.*

(3) *X is cs-stratifiable $w\theta$ and β.*

(4) *X is strongly α, $w\theta$ and β.*

(5) *X is developable T_2.*

Then, (1), (2) *and* (3) *are equivalent.*

Moreover, if X is orthocompact, then all conditions are equivalent.

Proof: (1)$\Rightarrow$(2) $\Rightarrow$(3) are evident. For (3)$\Longrightarrow$(1), X is K-semimetrizable by Theorem 3.5. Since X is semistratifiable and θ by Theorem 2.10, X is developable [18; Remark 4.8]. (4)$\Longrightarrow$(3) follows from Theorem 2.4, and (3)$\Longrightarrow$(5) is evident. Moreover, if X is orthocompact, (5)$\Longrightarrow$(4) follows from Theorem 2.8.

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Martin [26] showed that a locally connected rim-compact T_2-space X is K-semimetrizable if, and only if, it is developable γ.

Definition 3.16. A space X is said to be *rim-compact* if each point of X has a neighbourhood base consisting of open subsets with compact boundaries. A space X is *locally connected* if each point of X has a neighbourhood base consisting of connected open subsets.

We need the following lemma.

Lemma 3.17. (1) *Every locally connected rim-compact weak c-stratifiable* (*or, cs-stratifiable*) *space X is a c-stratifiable γ-space.*

(2) *Every pseudocompact Tychonoff weak c-stratifiable* (*or, cs-stratifiable*) *space X is a c-stratifiable γ-space.*

Proof. (1): First, let g be a weak c-stratifiable function of X. Then, we can assume that $g_n(x)$ is connected for every $x \in X$ and each $n \in \mathbb{N}$. To see that X is a γ-space, we use the same method given in the proof of [26; Theorem 4]. Suppose that $K \subset W$, where K is non-empty compact and W is open. Then there is an open subset G such that $K \subset G \subset W$ and the boundary ∂G of G is compact. Since $K \cap \partial G = \emptyset$, $g_m(K) \cap \partial G = \emptyset$ for some $m \in \mathbb{N}$. Let $K = \cup\{K_\alpha | \alpha \in A\}$, where K_α is a connected component of K. Since $g_m(K_\alpha)$ is connected for each $\alpha \in A$, $g_m(K) = \cup_{\alpha\in A} g_m(K_\alpha) \subset G$. Hence g is a γ-function by [23; Theorem 2.1]. Since X is first countable, g is a c-stratifiable function by [22; Theorem 1.3]. Next, let g be a cs-stratifiable function of X. To see that $\{g_n(x)\}$ is a neighbourhood base of x for every $x \in X$, in the above proof, let K be a single point x. Since $\{x\} \cap \partial G = \emptyset$ and ∂G is compact, we have that $\overline{g_m(x)} \cap \partial G = \emptyset$ for some $m \in \mathbb{N}$. This asserts that $\overline{g_m(x)} \subset G$, which implies that X is first countable and regular. Therefore X is c-stratifiable by Theorem 2.3, and hence X is a γ-space.

(2): Let g be a weak c-stratifiable function or a cs-stratifiable function of X. By regularity of X, we assume that $\overline{g_{n+1}(x)} \subset g_n(x)$. Since $\cap_{n\geq 1}\overline{g_n(x)} = \{x\}$, X is first countable by [27; Lemma 2.3]. Hence X is c-stratifiable by Theorem 2.3 and hence, X is γ by [22; Theorem 4.2].

Theorem 3.18. *Let X be a locally connected rim-compact space or a pseudocompact Tychonoff space. Then the following conditions are equivalent.*

(1) *X is developable and K-semimetrizable.*

(2) *X is K-semimetrizable.*

(3) *X is T_2, developable and γ.*

(4) *X is weak c-stratifiable and β.*

(5) *X is cs-stratifiable and β.*
(6) *X is T_2, γ and β.*

Proof. First, we note that every γ, β-space is developable [18; Proposition 4.2]. (1)$\Longleftrightarrow$(4) and (1)$\Longleftrightarrow$(5) follow from Theorem 3.5 and Lemma 3.17.
(1)$\Longrightarrow$(2)$\Longrightarrow$(4) and (1)$\Longrightarrow$(3)$\Longrightarrow$(6) is evident. Since every T_2, γ-space is c-stratifiable, (6)$\Longrightarrow$(5) is true.

By the proof of the above theorem and Theorem 3.5, we have that in the class of T_2, γ-spaces, the following properties are coincident: (1) developable and K-semimetrizable, (2) K-semimetrizable, (3) developable and (4) β.

The next theorem follows from Theorem 3.10.

Theorem 3.19. *For an orthocompact T_2-space X, X is developable and K-semimetrizable if, and only if, it is developable*

A Tychonoff space X is called a *p-space* [2] if in the Stone-Čech compactification βX, there is a sequence $\{\mathcal{G}_n\}$ of open covers of X such that $\cap_{n\geq 1} st(x, \mathcal{G}_n) \subset X$ for every $x \in X$. Every locally compact T_2-space is a p-space.
Burke [5] showed that there is a locally compact T_2-space with a G_δ-diagonal, which is not $w\Delta$. But, it is known that every locally compact semistratifiable T_2-space or every θ-refinable p-space with a G_δ-diagonal is Moore [8, 21]. Then we have the following result by Theorem 3.10.

Theorem 3.20. *For a metacompact p-space X, X is Moore and K-semi -metrizable if, and only if, it has a G_δ-diagonal.*

The next result was studied by Kotake [20] in the class of regular spaces.

Theorem 3.21. *For a space X, the following conditions are equivalent.*
(1) *X is Nagata.*
(2) *X is K-semimetrizable wcc.*
(3) *X is cs-stratifiable wN.*
(4) *X is strongly α, wN.*
(5) *X is a wN-space with a G^*_δ-diagonal.*
(6) *X is regular semimetrizable wcc.*

Proof. Every Nagata space is stratifiable and first countable, hence it is c-stratifiable q and β. Therefore (1)$\Longrightarrow$(2) and (2)$\Longrightarrow$(3) follow from

Proposition 1.4 and Theorem 3.5, and (3)$\Longrightarrow$(1) follows from Theorem 2.10. (1)$\Longrightarrow$(4) and (4)$\Longrightarrow$(3) follow from Theorems 2.4 and 2.8. Also, (1)$\Longrightarrow$(5) is evident. To prove (5)$\Longrightarrow$(4), let g be a wN-function and $\{\mathcal{G}_n\}$ be a G^*_δ-diagonal sequence. Since regularity is not used to show that every β-space with a G^*_δ-diagonal is semistratifiable [17; Theorem 5.2], X is a subparacompact wN-space. Then X is metacompact by [18; Corollary 3.5]. Hence X is strongly α by Theorem 2.8. (1)$\Longrightarrow$(6) is evident. Finally, since every regular semistratifiable space has a G^*_δ-diagonal [14; Theorem 5.11], (6)$\Longrightarrow$(5) follows from Proposition 1.4.

Regarding Question 2.12, we have the following corollary which follows from the fact that every wcc-space is β.

Corollary 3.22. *For a wN-space, the classes of the following spaces are all coincident.*

(1) Nagata spaces, (2) strongly α-spaces, (3) c-stratifiable spaces, (4) K-semimetrizable spaces and (5) spaces with a G^*_δ-diagonal.

Remark 3.23. Ceder [7; page 114] asked whether every paracompact semimetrizable space must be a Nagata space. Heath [16] showed that there is a paracompact K-semimetrizable cosmic (the continuous image of a separable metric space) space which is not a stratifiable space (hence, neither k-semistratifiable [24; Example 4.2] nor wcc). He also posed the following problem: What topological condition is necessary for a paracompact semimetrizable (= K-semimetrizable) space to be an M_3-space? As a remark to this problem, one can note that in the class of regular semimetrizable spaces, Nagata spaces, k-semistratifiable spaces and wcc-spaces are coincident.

4. Metrizabilities and examples

We begin this section with metrizations of wM-spaces. The concept of wM-spaces was given by Ishii [19]. Here we define a wM-space by an equivalent condition given by Hodel.

Definition 4.1 [18; Theorem 5.2]. A space X is wM if, and only if, it is $w\gamma$ and wN.

The following implications are well known.

An M-space (in the sense of Morita) $\Longrightarrow$ a wM-space $\Longrightarrow$ a $w\Delta$-space.

The class of wM-spaces is contained in the class of $w\theta$, wcc-spaces. Therefore, we consider metrizations for the class of $w\theta$, wcc-spaces. Metrizations for this class was studied in [28]. For metrizations of wM-spaces, Martin [25] proved that every regular c-semistratifiable wM-space is metrizable, and Ishii [19] proved that every normal wM-space with a G^*_δ-diagonal is metrizable. On the other hand, the space Ψ in Example 4.4 is a c-stratifiable Moore γ-space which is not metrizable.

Theorem 4.2. *Let X be a $w\theta$, wcc-space. Then X is metrizable if X satisfies any one of the following statements.*

(1) *X is K-semimetrizable.*
(2) *X is strongly α.*
(3) *X is cs-stratifiable.*
(4) *X has a G^*_δ-diagonal.*
(5) *X is regular c-semistratifiable.*

Proof. For all conditions (1)-(5), X is a wN-space by Proposition 1.4. Hence for (1)-(4), X is a $w\theta$, Nagata space by Theorem 3.21. Therefore, X is metrizable [30; Theorem 5]. For (5), since every wcc-space is β, X is regular c-semistratifiable β, hence X is semistratifiable. Then X is wcc Moore [18; Corollary 4.6], which implies that X is metrizable [31; Corollary 3.6].

Remark 4.3. In Theorem 4.2, the condition $w\theta$ (wcc) can not be weakened to q (β, respectively). Indeed, the Nagata-space X in Example 4.9 is a q, wcc-space which satisfies all of the conditions (1)-(5) in Theorem 4.2, but is not metrizable. Also, the space Ψ in Example 4.5 is a γ, β-space which satisfies all of the conditions (1)-(5) in Theorem 4.2, but is not metrizable.

The second part (2) of the next theorem is a generarization of Lee's result [22] that every pseudocompact Tychonoff stratifiable space is metrizable.

Theorem 4.4. (1) *Every locally connected rim-compact k-semistratifiable space X is metrizable.*

(2) *Every pseudocompact Tychonoff k-semistratifiable space X is metrizable.*

Proof. First, we show that if X satisfies the conditions of (1), then X is a first countable T_2-space. Let g be a k-semistratifiable function such that $g_n(x)$ is connected. To see that $\{g_n(x)\}$ is a neighbourhood base of x for every $x \in X$, suppose that $x \in U$ and $g_n(x) \setminus U \neq \emptyset$ $(n \geq 1)$, where U is open. Then there is an open neighbourhood W of x such that $W \subset U$ and the boundary ∂W is compact. Since $g_m(x) \cap \partial W = \emptyset$ for some $m \in \mathbb{N}$,

$g_m(x) = (g_m(x) \cap W) \cup (g_m(x) \setminus \overline{W})$ is not connected. This contradiction implies that $\{g_n(x)\}$ is a neighbourhood base of x. To see that X is Hausdorff, let $x \neq y$ and $x_n \in g_n(x) \cap g_n(y)$ $(n \geq 1)$. Then for any open neighbourhood U of x with $y \notin U$, $K = \{x_n | n \geq m\} \cup \{x\} \subset U$ for some $m \in \mathbb{N}$. Hence $g_l(y) \cap K = \emptyset$ for some $l \geq m$, which is a contradiction. Next, in both cases, X is a γ-space by Theorem 2.4 and Lemma 3.17. Also, X is a *wcc*-space. Indeed, let g be a k-semistratifiable function such that, whenever $b_n \in g_n(a_n)$ $(n \geq 1)$ and $\{b_n\} \longrightarrow b$, then $\{a_n\} \longrightarrow b$. Now, suppose that $y_n \in g_n(x_n)$ $(n \geq 1)$ and $\{y_n\}$ has a cluster point y. Since X is first countable, there exists a subsequence $\{y_{n(i)}\}$ of $\{y_n\}$ converging to y and $y_{n(i)} \in g_i(x_{n(i)})$ $(n \geq 1)$. Hence $\{x_{n(i)}\}$ converges to y, which implies that g is a *wcc*-function. Finally, every γ, *wcc* T_2-space is metrizable [31; Corollary 3.6].

We note that Martin [26; Example 3] showed that there exists a locally connected locally compact K-semimetrizable Moore space X which is not normal. This space is not *wcc* by Theorem 3.21.

As regards to Theorem 4.4, (2) is proved in [30; Corollary 4] in a different way, and as for (1), every locally compact T_2 (even sieve-complete regular) k-semistratifiable is metrizable [30; Theorem 18].

Example 4.5. [22; Example 6.6] The space Ψ in [13; 5I] is Moore and K-semimetrizable that is not full K-semimetrizable. First, it is known that Ψ is a locally compact pseudocompact separable Moore c-stratifiable space that is not metacompact. To see that Ψ is orthocompact, for any $E = \{x_k^E | k \in \mathbb{N}\} \in \mathcal{E}$, where $\{x_k^E | k \in \mathbb{N}\}$ is an infinite subsequence of $\mathbb{N}$, we put $B(\omega_E, n) = \{\omega_E\} \cup \{x_n^E, x_{n+1}^E, ...\}(n \in \mathbb{N})$. Then any open cover $\mathcal{G}$ of Ψ has the refinement $\mathcal{H} = \{\{n\} | n \in \mathbb{N}\} \cup \{B(\omega_E, n(E)) | E \in \mathcal{E}\}$, where for any $E \in \mathcal{E}$, $B(\omega_E, n(E)) \subset G$ for some $G \in \mathcal{G}$ and some $n(E) \in \mathbb{N}$. And $\cap \mathcal{W}$ is open for any $\mathcal{W} \subset \mathcal{H}$. Therefore, Ψ is strongly α by Theorem 2.8. Then Ψ is K-semimetrizable and γ by Theorem 3.5 and Lemma 3.17. But Ψ does not have a regular G_δ-diagonal [27; Theorem 2.6], and not *wcc* from Theorem 3.21. Hence it is not full K-semimetrizable by Theorem 3.12 and not k-semistratifiable since every first countable k-semistratifiable space is *wcc*.

Example 4.6. [6] Burke constructed the separable Moore (hence, semimetrizable) space X which is not K-semimetrizable. Hence, X is a c-semistratifiable α-space which is neither strongly α nor cs-stratifiable by Theorems 2.4 and 3.5. Also, X is not metacompact by Theorem 2.8 and not

γ.

Example 4.7. [18; Example 4.14]. The Sorgenfrey line K is a paracompact γ-space with a G_δ-diagonal. Hence K is strongly α and c-stratifiable, but not semistratifiable (not even β [18; Proposition 4.2]).

Example 4.8. [9: Example 5.3.4] There exists a metacompact full K-semimetrizable space which is neither *wcc* nor regular.

Indeed, let X be the space of real numbers with the topology generated by the neighbourhood system $\{\mathcal{U}(x)|x \in X\}$, where $\mathcal{U}(x) = \{U_n(x)|n \in \mathbb{N}\}$ and

$$U_n(x) = \begin{cases} (x - 1/n,\ x + 1/n) & \text{if } x \neq 0, \\ (x - 1/n,\ x + 1/n) \setminus \{1/k|k \in \mathbb{Z} \setminus \{0\}\} & \text{if } x = 0, \end{cases}$$

where $\mathbb{Z}$ denotes the set of integers. It is well known that X is a metacompact T_2-space which is not regular. For each $x \in X$, we put

$$W_n(x) = \begin{cases} U_n(x) \setminus \{0\} & \text{if } x \neq 0, \\ U_n(x) & \text{if } x = 0. \end{cases}$$

Let $\mathcal{W}_n = \{W_n(x)|x \in X\}$ for each $n \in \mathbb{N}$. Then it is easily seen that the sequence $\{\mathcal{W}_n\}$ is a development satisfying the 3-link property. Therefore, X is full K-semimetrizable. Then X is strongly α and c-stratifiable by Theorem 2.8. Also, if X is *wcc*, then it is metrizable by Theorem 4.2, which is a contradiction.

Example 4.9. [24; Example 4.3] There exist a first countable stratifiable space X and a perfect map f from X onto a non-q-space Y. Then X is a Nagata space (hence, X is K-semimetrizable) which is not $w\theta$ [30; Theorem 5] and Y is a stratifiable space which is not q. Then, Y is strongly α and c-stratifiable but not semimetrizable.

Example 4.10. [10; Example 4.2] A regular full K-semimetrizable space that is not orthocompact. Let $R = \{(x, y)|x, y$ are rational and $y>0\}$. Let J be the set of irrational numbers and let $X = R \cup (J \times \{0\})$. We give R the usual subspace topology $\mathcal{T}^*$. For each $x \in J$ and each $\epsilon>0$, let $B(x, \epsilon) = \{(x, 0)\} \cup \{(x + k, h)||k|<h<\epsilon\}$. Then $\mathcal{T}^* \cup \{B(x, \epsilon)|x \in J, \epsilon>0\}$ is a basis for a topology on X. Then X is a separable Moore space that is not orthocompact. Also, X has a development satisfying the 3-link property, hence full K-semimetrizable and c-stratifiable.

But I don't know whether this space is strongly α.

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