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## A Lower Bound for the Rational LS-category of a Coformal Elliptic Space

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# A Lower Bound for the Rational LS-category of a Coformal Elliptic Space

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## Abstract

We give a lower bound for the rational LS-category of certain spaces, including the coformal elliptic ones, in terms of the dimension of its total rational cohomology.

**KEYWORDS:** rational LS-category, elliptic space, coformal space, minimal model, Toomer invariant

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## A LOWER BOUND FOR THE RATIONAL LS-CATEGORY OF A COFORMAL ELLIPTIC SPACE

Dedicated to the memory of Professor Shiroshi SAITO

TOSHIHIRO YAMAGUCHI

ABSTRACT. We give a lower bound for the rational LS-category of certain spaces, including the coformal elliptic ones, in terms of the dimension of its total rational cohomology.

### 1. INTRODUCTION

The Lusternik Schnirelmann category of a space  $X$ ,  $\text{cat } X$ , is the least integer  $m$  such that  $X$  can be covered by  $m + 1$  open sets, each of which is contractible in  $X$ . Let  $X$  be a simply connected CW complex with rational cohomology of finite type. Denote its Sullivan minimal model [3] as  $(\Lambda V, d)$ . The rational LS category,  $\text{cat}_0(X)$ , is the least integer  $n$  such that  $X$  has the rational homotopy type of  $Y$  with  $\text{cat}(Y) = n$ . Then  $\text{cat}_0(X) \leq \text{cat}(X)$ . A minimal model  $(\Lambda V, d)$  is called elliptic if  $\dim H^*(\Lambda V, d) < \infty$  and  $\dim V < \infty$  [3]. The rational Toomer invariant  $e_0(\Lambda V, d)$  is given by the biggest integer  $s$  for which there is a non trivial cohomology class in  $H^*(\Lambda V, d)$  represented by a cycle in  $\Lambda^{\geq s} V$ . If  $(\Lambda V, d)$  is elliptic, it is proved that  $e_0(\Lambda V, d) = \text{cat}_0(X)$  [2, Theorem 3]. An elliptic model  $(\Lambda V, d)$  is called an  $F_0$ -model if its cohomology is concentrated in even degrees. It is equivalent to the condition that  $\dim V^{\text{even}} = \dim V^{\text{odd}}$  [4]. If  $(\Lambda V, d)$  is elliptic, is  $\dim H^*(\Lambda V, d) \leq 2^{e_0(\Lambda V, d)}$ ? ([7]) The answer is affirmative if the model is formal, in particular if it is an  $F_0$ -model [5]. Since  $e_0(X \times Y) = e_0(X) + e_0(Y)$  [3, p.391], the proposed inequality is closed under products, that is, if  $X$  and  $Y$  satisfy the inequality, then so too does  $X \times Y$ .

Let  $k$  be the biggest integer for which we may write  $d = \sum_{i \geq k} d_i$  with  $d_i(V) \subset \Lambda^i V$ . Thus  $d_k$  induces a differential in  $\Lambda V$ . In this paper, we prove:

**Theorem.** *If both  $(\Lambda V, d)$  and  $(\Lambda V, d_k)$  are elliptic, then*

$$\dim H^*(\Lambda V, d) \leq 2^{e_0(\Lambda V, d)}.$$

Recall that  $(\Lambda V, d)$  is coformal if and only if  $d(V) \subset \Lambda^2 V$ . We then have

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**Corollary.** *If an elliptic space  $X$  is coformal, then*

$$\dim H^*(X; \mathbb{Q}) \leq 2^{\text{cat}_0(X)}.$$

In view of [1, Proposition 10.6], we have: If an elliptic model  $(\Lambda V, d)$  is coformal,  $\dim H^*(\Lambda V, d) \leq 2^{e_0(\Lambda V, d)} = 2^{\dim V^{\text{odd}}}$ .

2. PROOF

We first observe that the two numerical invariants of a model “ $\dim H^*(\Lambda V, d)$ ” and “ $e_0(\Lambda V, d)$ ” do not depend on the gradation of  $V = \bigoplus_i V^i$ :

**Lemma 1.** *If  $(\Lambda V, d_V) \cong (\Lambda W, d_W)$  as differential (generally not graded) algebras, then  $\dim H^*(\Lambda V, d_V) = \dim H^*(\Lambda W, d_W)$  and  $e_0(\Lambda V, d_V) = e_0(\Lambda W, d_W)$ .*

**Lemma 2.** [5] *If the cohomology of an  $F_0$ -space  $X$  is given by  $H^*(X; \mathbb{Q}) \cong \mathbb{Q}[x_1, \dots, x_p]/(f_1, \dots, f_p)$ , then*

$$\dim H^*(X; \mathbb{Q}) \leq 2^{\deg f_1 + \dots + \deg f_p - p}.$$

Here  $\deg f$  means the degree of a polynomial  $f$  in  $\mathbb{Q}[x_1, \dots, x_p]$ , not the degree defined by each one of  $x_i$  in the graded algebra  $\mathbb{Q}[x_1, \dots, x_p]$ . For a given model  $(\Lambda V, d)$ , we denote by  $(\Lambda V, d_\sigma)$  its associated pure model, i.e.,  $d_\sigma$  is the component of  $d$  which satisfies  $d_\sigma V^{\text{even}} = 0$  and  $d_\sigma V^{\text{odd}} \subset \Lambda V^{\text{even}}$  [4, page 181, the first paragraph].

*Proof of Theorem.* Put

$$(\Lambda V, d) = (\Lambda(x_1, \dots, x_n, y_1, \dots, y_m), d)$$

with  $|x_i|$  even and  $|y_i|$  odd. Then  $n \leq m$  [4, Theorem 1’]. For the associated pure model  $(\Lambda V, d_{k\sigma})$  of  $(\Lambda V, d_k)$ ,  $\dim H^*(\Lambda V, d_{k\sigma}) < \infty$  from our assumption and [4, Proposition 1]. This is isomorphic as differential algebras to

$$(\Lambda W, d_W) = (\Lambda(u_1, \dots, u_n, v_1, \dots, v_m), d_W)$$

with  $|u_i| = 2$ ,  $|v_i| = 2k - 1$  and  $d_W = d_{k\sigma}$ , i.e.,  $d_W$  is given by  $\phi d_{k\sigma} \phi^{-1}$  for the isomorphism  $\phi : \Lambda V \rightarrow \Lambda W$  with  $\phi(x_i) = u_i$  and  $\phi(y_i) = v_i$ . From Lemma 1,  $\dim H^*(\Lambda W, d_W) < \infty$ . By [4, Lemma 8], there is a subspace  $W'$  of  $W$  for which

$$(\Lambda W', d_W) = (\Lambda(u_1, \dots, u_n, v'_1, \dots, v'_n), d_W)$$

and such that  $f_1 = d_W(v'_1), \dots, f_n = d_W(v'_n)$  is a regular sequence in  $\mathbb{Q}[u_1, \dots, u_n]$ , i.e.,  $(\Lambda W', d_W)$  is an  $F_0$ -model. Thus, there is a KS extension

$$(1) \quad (\Lambda W', d_W) \rightarrow (\Lambda W, d_W) \rightarrow (\Lambda(v'_{n+1}, \dots, v'_m), 0)$$

where  $v'_{n+1}, \dots, v'_m$  is a basis of a complement of  $W'^{2k-1}$  in  $\mathbb{Q}\langle v_1, \dots, v_m \rangle$ . Then the following inequalities are combined to establish our statement.

$$\begin{aligned}
 \dim H^*(\Lambda V, d) &\leq_{(a)} \dim H^*(\Lambda V, d_k) \\
 &\leq_{(b)} \dim H^*(\Lambda V, d_{k\sigma}) \\
 &=_{(c)} \dim H^*(\Lambda W, d_W) \\
 &\leq_{(d)} 2^{m-n} \dim H^*(\Lambda W', d_W) \\
 &\leq_{(e)} 2^{m-n} \cdot 2^{\deg f_1 + \dots + \deg f_n - n} \\
 &= 2^{m-n} \cdot 2^{kn-n} \\
 &= 2^{n(k-2)+m} \\
 &=_{(f)} 2^{e_0(\Lambda V, d)},
 \end{aligned}$$

in which we have (a) from the Milnor-Moore spectral sequence, (b) from the odd spectral sequence [4, Section 5], (c) from Lemma 1, (d) from the Serre spectral sequence applied to (1), (e) from Lemma 2 and (f) from [6, Theorem 1.1].  $\square$

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