

Mathematical Journal of Okayama University

Volume 38, Issue 1

1996

Article 8

JANUARY 1996

A Remark on Torsion-Free Subgroups in Group Rings

Tôru Furukawa*

*Sanyo Gakuen University

Copyright ©1996 by the authors. *Mathematical Journal of Okayama University* is produced by
The Berkeley Electronic Press (bepress). <http://escholarship.lib.okayama-u.ac.jp/mjou>

A REMARK ON TORSION-FREE SUBGROUPS IN GROUP RINGS

TÔRU FURUKAWA

Introduction. Let RG be the group ring of a group G over a commutative ring R with identity. For any normal subgroup N of G we write $\Delta_R(G, N)$ for the kernel of the natural map $RG \rightarrow R(G/N)$. Also, for any ideal I of RG , we put $U(1 + I) = \{u \in U(RG) \mid u - 1 \in I\}$, where $U(RG)$ is the unit group of RG . Note that $U(1 + I)$ forms a normal subgroup of $U(RG)$.

We shall denote by TN the set of torsion elements in a group N . The aim of this note is to prove the following theorem which generalizes [2, Theorem B]. The theorem below was proved in [2] under the condition that N is periodic or finitely generated.

Theorem. *Let R be an integral domain of characteristic 0 in which no rational prime is invertible. Let N be a nilpotent normal subgroup of a group G and let A be an abelian normal subgroup of G with $N \supseteq A$. Assume that N/TN is finitely generated. Then $U(1 + \Delta_R(G, N)\Delta_R(G, A))$ is torsion-free.*

In what follows, unless otherwise stated, R denotes a commutative ring with identity. Also, for simplicity of notation, we omit the subscript R from $\Delta_R(G, N)$, which will be denoted by $\Delta(G, N)$.

1. Preliminaries. For the proof of our theorem we shall establish some lemmas concerning augmentation ideals. Let N be any group and let $\Delta(N)$ denote the augmentation ideal of RN . Let L be a subgroup of N , and T a right transversal for L in N with $T \ni 1$. Then each element g of N can be written uniquely in the form $g = tx$, $t \in T$, $x \in L$ and so, setting $\theta(g) = x$, we have an R -linear map $\theta : RN \rightarrow RL$. It is easy to see that θ is a right RL -homomorphism and that $\theta(\Delta(N)) = \Delta(L)$. We write $\theta = \theta(N, L, T)$. Recall that the projection map $\pi : RN \rightarrow RL$, which is defined by $\pi(\sum_{g \in N} \alpha(g)g) = \sum_{g \in L} \alpha(g)g$, is also a right RL -homomorphism.

Lemma 1.1. *Let C be a normal subgroup of N and \mathcal{A} a nonempty set of normal subgroups of N which contain C . Suppose that $L_1 \cap L_2 \in \mathcal{A}$*

whenever $L_1, L_2 \in \mathcal{A}$. Then

$$\bigcap_{L \in \mathcal{A}} (\Delta(N, L)\Delta(N, C)) = \Delta(N, \bigcap_{L \in \mathcal{A}} L)\Delta(N, C).$$

Proof. We have only to verify that the left-hand side is contained in the right-hand side. Let $\alpha \in \bigcap_{L \in \mathcal{A}} (\Delta(N, L)\Delta(N, C))$ and set $M = \bigcap_{L \in \mathcal{A}} L$. Then, choosing a transversal T for M in N , α can be written uniquely in the form $\alpha = \sum_{i=1}^n t_i \alpha_i$, $\alpha_i \in RM$, $t_i \in T$. By the property of \mathcal{A} , we may take some $L \in \mathcal{A}$ such that $\{t_i^{-1}t_i | 1 \leq i \leq n\} \cap L = \{1\}$. Then, under the projection map $\pi : RN \rightarrow RL$,

$$\alpha_1 = \pi(t_1^{-1}\alpha) \in \pi(\Delta(N, L)\Delta(N, C)) = \Delta(L)\Delta(L, C).$$

Let S be a transversal for M in L with $S \ni 1$, and consider the map $\theta = \theta(L, M, S) : RL \rightarrow RM$. Then we have $\alpha_1 = \theta(\alpha_1) \in \theta(\Delta(L)\Delta(L, C)) = \Delta(M)\Delta(M, C)$. Similarly, we see that all α_i 's are in $\Delta(M)\Delta(M, C)$ so that $\alpha \in \Delta(N, M)\Delta(N, C)$. This completes the proof.

Lemma 1.2. *Let $L \supseteq A$ be normal subgroups of N and let \mathcal{B} be a nonempty set of normal subgroups of A . Suppose that $\bigcap_{B \in \mathcal{B}} (\Delta(A)^2 + \Delta(A, B)) = \Delta(A)^2$. Then*

$$\bigcap_{B \in \mathcal{B}} (\Delta(N, L)\Delta(N, A) + RN\Delta(B)) = \Delta(N, L)\Delta(N, A).$$

Proof. We have only to show that the left-hand side is contained in the right-hand side. Take $\alpha \in \bigcap_{B \in \mathcal{B}} (\Delta(N, L)\Delta(N, A) + RN\Delta(B))$ and let T be a transversal for L in N . Then α can be written uniquely as a finite sum of the form $\alpha = \sum_{t \in T} t\alpha_t$ with $\alpha_t \in RL$. Similarly, let us fix $t \in T$ and write $\alpha_t = \sum_{s \in S} s\beta_s$, $\beta_s \in RA$ where $S (\ni 1)$ is a transversal for A in L . Then, under the projection map $\pi : RN \rightarrow RL$,

$$\alpha_t = \pi(t^{-1}\alpha) \in \pi(\Delta(N, L)\Delta(N, A) + RN\Delta(B)) = \Delta(L)\Delta(L, A) + RL\Delta(B)$$

for any $B \in \mathcal{B}$. Therefore, considering $\theta = \theta(L, A, S) : RL \rightarrow RA$,

$$\sum_{s \in S} \beta_s = \theta(\alpha_t) \in \theta(\Delta(L)\Delta(L, A) + RL\Delta(B)) = \Delta(A)^2 + \Delta(A, B)$$

for all $B \in \mathcal{B}$, and hence $\sum_{s \in S} \beta_s \in \Delta(A)^2$ by hypothesis. Since $\alpha_t \in \Delta(L, A)$, $\beta_s \in \Delta(A)$ for all $s \in S$. Thus $\alpha_t \in \Delta(L)\Delta(A)$ because $\alpha_t = \sum_{s \in S} (s-1)\beta_s + \sum_{s \in S} \beta_s$. This is true for any $t \in T$, so $\alpha \in \Delta(N, L)\Delta(N, A)$ and the lemma is proved.

Lemma 1.3. *Let A be a nontrivial elementary abelian p -group for*

some prime p and let \mathcal{B} be the set of all maximal subgroups of A . Then

$$\bigcap_{B \in \mathcal{B}} (\Delta(A)^2 + \Delta(A, B)) = \Delta(A)^2.$$

Proof. We need only to prove the following:

$$(*) \quad \bigcap_{B \in \mathcal{B}} (\Delta(A)^2 + \Delta(A, B)) \subseteq \Delta(A)^2.$$

To do this, we first assume that A is finite and let $|A| = p^n$. We proceed by induction on n , the case $n = 1$ being trivial. Let $n \geq 2$ and assume that (*) holds for $n - 1$. Take $\alpha \in \bigcap_{B \in \mathcal{B}} (\Delta(A)^2 + \Delta(A, B))$ and let $B \in \mathcal{B}$ so that $|B| = p^{n-1}$. Then clearly there is a subgroup C of A such that $A = B \times C$. Consider here the map $\theta = \theta(A, B, C) : RA \rightarrow RB$. Then, for any maximal subgroup M of B , $MC \in \mathcal{B}$ so $\alpha \in \Delta(A)^2 + \Delta(A, MC)$, and hence $\theta(\alpha) \in \Delta(B)^2 + \Delta(B, M)$ because θ is a ring homomorphism with $\text{Ker } \theta = \Delta(A, C)$. Therefore $\theta(\alpha) \in \Delta(B)^2$ by induction hypothesis. On the other hand, since $\alpha \in \Delta(A)^2 + \Delta(A, B) = \Delta(A)^2 + \Delta(B)$, we can write $\alpha = \gamma + \beta$, $\gamma \in \Delta(A)^2$, $\beta \in \Delta(B)$, and so $\theta(\alpha) = \theta(\gamma) + \beta$. Thus we obtain $\beta \in \Delta(B)^2$ so that $\alpha \in \Delta(A)^2$ and hence (*) follows here.

Next, assume that A is infinite. Let $\alpha = \sum_{x \in A} \alpha(x)x$ be a nonzero element in $\bigcap_{B \in \mathcal{B}} (\Delta(A)^2 + \Delta(A, B))$ and let A_0 be the subgroup of A generated by $\{x \in A \mid \alpha(x) \neq 0\}$. Then, $A = A_0 \times C$ for some subgroup C of A . Taking here the map $\theta = \theta(A, A_0, C) : RA \rightarrow RA_0$, by the same argument as above, we see that $\theta(\alpha) \in \Delta(A_0)^2 + \Delta(A_0, M)$ for any maximal subgroup M of A_0 . Therefore, since A_0 is finite, the previous case ensures that $\theta(\alpha) \in \Delta(A_0)^2$. Thus $\alpha = \theta(\alpha) \in \Delta(A)^2$ and (*) is established.

Remark. The above result is trivial in case $R = \mathbf{Z}$, the ring of rational integers. For, in this case, it is known that the map $f : A \rightarrow \Delta(A)/\Delta(A)^2$ defined by $f(a) = a - 1 + \Delta(A)^2$ is an isomorphism of abelian groups. Moreover, for any subgroup C of A , $f(C) = (\Delta(A)^2 + \Delta(C))/\Delta(A)^2 = (\Delta(A)^2 + \Delta(A, C))/\Delta(A)^2$. Therefore, $\{\Delta(A)^2 + \Delta(A, B) \mid B \in \mathcal{B}\}$ is the set of all maximal subgroups of $\Delta(A)$ which contain $\Delta(A)^2$. Since the Frattini subgroup of $\Delta(A)/\Delta(A)^2$ is 0, it follows that $\bigcap_{B \in \mathcal{B}} (\Delta(A)^2 + \Delta(A, B)) = \Delta(A)^2$.

Lemma 1.4. *Let N be a nilpotent group such that N/TN is finitely generated and let C be a normal subgroup of N . If C is a p -group of bounded exponent for some prime p , then*

$$\bigcap_{n=1}^{\infty} (\Delta(N)^n \Delta(N, C)) \subseteq \Delta(N, TN) \Delta(N, C).$$

Proof. Since N/TN is a finitely generated torsion-free nilpotent group, it is residually 'a finite p -group' (see [3, Theorem 2.1]). Therefore, denoting by \mathcal{A} the set of all normal subgroups L of N such that $L \supseteq TN$ and N/L is a finite p -group, we have $\bigcap_{L \in \mathcal{A}} L = TN$. So it follows from Lemma 1.1 that

$$\bigcap_{L \in \mathcal{A}} (\Delta(N, L)\Delta(N, C)) = \Delta(N, TN)\Delta(N, C).$$

Now, the additive group $\Delta(C)/\Delta(C)^2$ is a p -group of bounded exponent, because $R \otimes_{\mathbf{Z}} (C/C') \simeq \Delta(C)/\Delta(C)^2$ (see e.g. [4, p.23]). Thus $p^l \Delta(C) \subseteq \Delta(C)^2$ for some integer $l > 0$. Set $I = \bigcap_{n=1}^{\infty} (\Delta(N)^n \Delta(N, C))$. Let $L \in \mathcal{A}$ and consider the natural map $f : RN \rightarrow S(N/L)$ where $S = R/p^l R$. Then, since the augmentation ideal of $S(N/L)$ is nilpotent, $f(\Delta(N)^n) = 0$ for some integer $n > 0$. This implies that $\Delta(N)^n \subseteq \Delta(N, L) + p^l RN$ and hence

$$I \subseteq \Delta(N, L)\Delta(N, C) + p^l \Delta(N, C).$$

Furthermore, since $p^l \Delta(N, C) \subseteq \Delta(N, C)^2$, we obtain $I \subseteq \Delta(N, L)\Delta(N, C)$. This holds for any $L \in \mathcal{A}$ and so $I \subseteq \Delta(N, TN)\Delta(N, C)$ as asserted.

Remark. The above result does not hold in general without the condition that N/TN is finitely generated. For example, take the direct product $N = \mathbf{Q}/\mathbf{Z} \times \mathbf{Q}$, where \mathbf{Q} is the field of rational numbers. Then N is a divisible abelian group such that N/TN is not finitely generated. Let $C = \langle c \rangle$ be a cyclic subgroup of prime order p in N and consider the case when $R = \mathbf{Z}$. Then, for any $g \in N$ there exists $x \in N$ such that $g = x^p$, and thus $g - 1 \equiv (x - 1)^p \pmod{p\mathbf{Z}N}$. So it follows that $\Delta(N) \subseteq \Delta(N)^2 + p\Delta(N)$, and hence $\Delta(N)\Delta(N, C) \subseteq \Delta(N)^2\Delta(N, C)$, since $p\Delta(C) \subseteq \Delta(C)^2$. Consequently, $\Delta(N)\Delta(N, C) = \Delta(N)^n\Delta(N, C)$ for all $n \geq 1$. Therefore, considering an element $\alpha = (x - 1)(c - 1)$ with $x \in N \setminus TN$, it is sure that $\alpha \in \bigcap_{n=1}^{\infty} (\Delta(N)^n \Delta(N, C))$. However $\alpha \notin \Delta(N, TN)\Delta(N, C)$. For, if $\alpha \in \Delta(N, TN)\Delta(N, C)$, then $\alpha = \beta(c - 1)$ for some $\beta \in \Delta(N, TN)$, and so we can write $x - 1 - \beta = \gamma \hat{c}$, $\gamma \in RN$, where $\hat{c} = 1 + c + \dots + c^{p-1}$. Then under the natural map $\bar{} : \mathbf{Z}N \rightarrow \mathbf{Z}(N/TN)$, we have $\bar{x} - \bar{1} = p\bar{\gamma}$ and hence $\bar{x} = \bar{1}$ i.e. $x \in TN$, contrary to our choice of x .

Lemma 1.5. *Let N be a nilpotent group such that N/TN is finitely generated and let A be a central subgroup of N . Suppose that A is an elementary abelian p -group for some prime p and that p is not a zero divisor*

in R . Then, for an additive subgroup I of RN ,

$$I \subseteq \Delta(N)\Delta(N, A), \quad pI \subseteq I^p \Rightarrow I \subseteq \Delta(N, TN)\Delta(N, A).$$

Proof. The case $A = \{1\}$ being trivial, so let $A \neq \{1\}$. In case $|A| = p$, we have $\Delta(N, A)^p = p\Delta(N, A)$ (see e.g. [1, Lemma 3.4]). Therefore, if $I \subseteq \Delta(N)^n\Delta(N, A)$ then

$$pI \subseteq I^p \subseteq \Delta(N)^{np}\Delta(N, A)^p = p\Delta(N)^{np}\Delta(N, A)$$

and so $I \subseteq \Delta(N)^{np}\Delta(N, A)$, since p is not a zero divisor in R . Thus $I \subseteq \Delta(N)^n\Delta(N, A)$ for all $n \geq 1$ and hence by Lemma 1.4, $I \subseteq \Delta(N, TN)\Delta(N, A)$.

For the general case, let \mathcal{B} be the set of all maximal subgroups of A and take $B \in \mathcal{B}$. Then, under the natural map $\bar{} : RN \rightarrow R(N/B)$, $|\bar{A}| = p$ and so the previous case shows that $I \subseteq \Delta(N, TN)\Delta(N, A) + \Delta(N, B)$. Thus

$$I \subseteq \bigcap_{B \in \mathcal{B}} (\Delta(N, TN)\Delta(N, A) + \Delta(N, B)).$$

On the other hand, Lemma 1.3 and Lemma 1.2 ensure that $\bigcap_{B \in \mathcal{B}} (\Delta(N, TN)\Delta(N, A) + \Delta(N, B)) = \Delta(N, TN)\Delta(N, A)$ and hence the lemma is proved.

2. Proof of Theorem. We see that for any normal subgroup L of G , consider $\bar{G} = G/L$, and then $\bar{N}/T\bar{N}$ is finitely generated. Therefore, as in the proof of [2, Theorem B], we may harmlessly assume that A is central in N .

Now, let $u \in TU(1 + \Delta(G, N)\Delta(G, A))$. Then, to show that $u = 1$ we may assume that $u^p = 1$ for some prime p . Let $T_p(A)$ be the set of p -elements in A and consider the natural map $\bar{} : RG \rightarrow R(G/T_p(A))$. Then, since $T_p(\bar{A}) = \{1\}$, we know from [2, Lemma 2.3] that $T_p(U(1 + \Delta(\bar{G}, \bar{A}))) = \{1\}$. Thus $\bar{u} = 1$ and so it follows from [2, Lemma 1.3] that

$$u - 1 \in \Delta(G, N)\Delta(G, A) \cap \Delta(G, T_p(A)) = \Delta(G, N)\Delta(G, T_p(A)).$$

Therefore $u - 1$ can be written in the form

$$u - 1 = \sum_{i=1}^n \lambda_i g_i (x_i - 1)(a_i - 1) \quad (\lambda_i \in R, g_i \in G, x_i \in N, a_i \in T_p(A)).$$

Set $B = \langle a_1, \dots, a_n \rangle$ so that $u - 1 \in \Delta(G, N)\Delta(G, B^G)$, where B^G is the normal closure of B in G . Note here that B^G is of bounded exponent (see [2, Lemma 2.1]). We may therefore assume that A is a p -group of bounded

exponent and hence that $A^{p^n} = \{1\}$ for some $n \geq 0$. We proceed by induction on n . The case $n = 0$ is trivial, so let $n \geq 1$ and put $C = A^{p^{n-1}}$. Then, under the natural map $\bar{} : RG \rightarrow R(G/C)$ we have $\overline{A^{p^{n-1}}} = \{1\}$ and hence $\bar{u} = 1$ by induction. Thus $u - 1 \in \Delta(G, N)\Delta(G, C)$ and consequently we may assume that A is an elementary abelian p -group.

Set $I = RG(u - 1)RG$ and let $\pi : RG \rightarrow RN$ be the projection map. Then, since $pI \subseteq I^p$, we obtain $p\pi(I) \subseteq \pi(I)^p$. Furthermore, $\pi(I) \subseteq \Delta(N)\Delta(N, A)$ and so, by virtue of Lemma 1.5, $\pi(I) \subseteq \Delta(N, TN)\Delta(N, A)$. Thus $I \subseteq RG\pi(I) \subseteq \Delta(G, TN)\Delta(G, A)$ and hence we see that $u \in U(1 + \Delta(G, TN)\Delta(G, A))$. So we conclude from [2, Theorem B] that $u = 1$, which completes the proof of the theorem.

REFERENCES

- [1] G. H. CLIFF, S. K. SEHGAL and A. R. WEISS: Units of integral group rings of metabelian groups, *J. Algebra* **73**(1981), 167–185.
- [2] T. FURUKAWA: Some torsion-free subgroups in group rings, *Math. J. Okayama Univ.* **35**(1993), 17–28.
- [3] K. W. GRUENBERG: Residual properties of infinite soluble groups, *Proc. London Math. Soc.* (3) **7**(1957), 29–62.
- [4] G. KARPILOVSKY: Unit groups of group rings, *Pitman Monographs and Surveys in Pure and Applied Mathematics*, Vol. **47**(1989), Longman.

T. FURUKAWA

THE FACULTY OF INTERCULTURAL STUDIES
 SANYO GAKUEN UNIVERSITY
 OKAYAMA 703-8282
 JAPAN

(Received January 31, 1997)