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EXPANSION OF PERIODIC BOUNDARY CONDITION FOR 3-D FEM ANALYSIS USING EDGE ELEMENTS

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Abstract **— Application of periodic boundary conditions in the analysis of three-dimensional magnetic fields by finite element methods leads to a substantial reduction of computation labour and storage. In this paper, the expansion of the condition for magnetostatic curl-curl formulation with the magnetic vector potential employing edge tetrahedral elements is discussed. Differences between the definitions of the condition for nodal and edge elements are examined. Vectorial nature of edge elements is emphasized and arising from it difficulties in the formulation and application of the condition are carefully analyzed and overcome. Details for computer implementation are given and a simple test problem to verify the validity of the software is proposed. Finally, advantages gained when the condition is used for TEAM Workshop problem 13 as an example, are shown.**

I. INTRODUCTION

The problem of exploiting geometrical symmetry in 3-D eddy current analysis with edge elements was examined in detail in [1]. In this work, a broad discussion of the case in which the source field does not share the symmetry of the system of conductors is performed. Due to the symmetry of the latter, the problem is decomposed into symmetry elements which constitute subproblems then solved separately and joined together by appropriate boundary conditions.

In our case, we assume either coils and magnetic circuits to be periodically distributed in space along fixed rotation axis. Owning to the assumed geometry, the problem can be reduced to only one symmetry element with periodic boundary planes being axially symmetric with respect to a distinct rotation axis. Consequently, the flux density distribution on one periodic plane with respect to the local coordinate system assigned to that plane is identical on the other periodic plane, when referred to the coordinate system assigned to this plane. Although the case discussed in this paper is more restricted than the one considered in [1], it may be commonly encountered in rotational electrical machines and other axially symmetric devices. Therefore, its practical importance seems to be well-founded. With regard to other boundary conditions, we want to point out at the moment that Dirichlet conditions remain unchanged in the analysis and must be properly set as in usual case. Neumann conditions are natural ones in the formulation and need not to be imposed explicitly.

The choice of discrete variables associated with edges for the curl-curl formulation with the magnetic vector potential A stems from the physical interpretation of this vector, which is given in terms of a line integral. As proved in [2], edge-based interpolation contrary to node-based one, does not overconstrain A by forcing its normal component to be continuous across the interface between two different materials; the condition not required by the formulation. Furthermore, edge element codes have been found in one study [4] to provide cheaper and more accurate solutions.

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II. PERIODIC BOUNDARY CONDITION

From now on, we will assume the flux distribution to be periodically distributed in space with the z-axis as the revolution line. We shall further refer to the dihedral angle formed by two closest periodic boundary planes as a period angle α. In order to facilitate the comparison between the edge and nodal definitions of the condition, either of them will be given in the two subsequent subsections.

A. Edge Elements

One should remember that in case of tangentially-continuous edge-based methods degrees of freedom are vectors, and not scalars [2]. With this in mind, we have (see Fig. 1):

1a) Setting of edges on one of the periodic boundary planes is obtained by the revolution of the setting of edges of the other boundary plane about the z-axis through the period angle α. Then, directions of degrees of freedom along symmetrical edges on both planes are rotational symmetric;

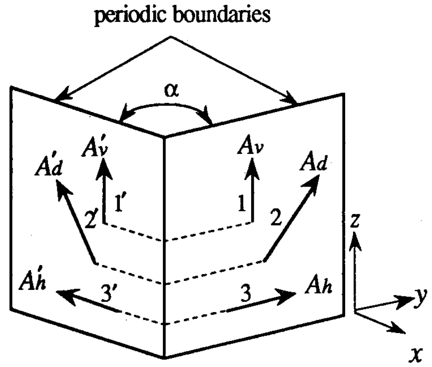

Fig. 1. Periodic boundary condition for edge elements.

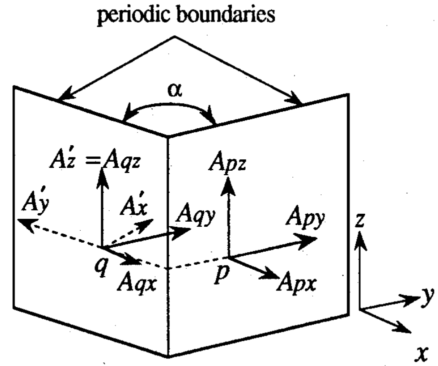

Fig. 2. Periodic boundary condition for nodal elements.

2a) Moduli of degrees of freedom along symmetrical edges are the same

$$|A_h| = |A_h'|, \quad |A_v| = |A_v'|, \quad |A_d| = |A_d'|, \qquad (1)$$

where A_h, A_h', A_v, A_v', A_d, A_d' are circulations of the vector potential A along symmetrical pairs of edges 1-1', 2-2', 3-3', as depicted in Fig. 1;

3a) Signs are assigned to each symmetrical pair of degrees of freedom in a fashion in which a degree of freedom located on one boundary points out in the direction identical to the one indicated by its symmetrical partner lying on the other boundary, while rotated about the z-axis through the period angle α.

This definition may also be concluded from [1], where it is given in the context described in the introduction.

B. Nodal Elements

In order to implement periodic conditions for nodal elements, the usual set of rules must be considered (see Fig. 2):

1b) Setting of nodes on one of the periodic boundary planes (node p) is obtained by the revolution of the setting of nodes of the other boundary plane (node q) about the z-axis through the period angle α;

2b) Directions of the components A_{qx}, A_{qy} and A_{qz} of the vector A at the node q remain parallel to the global coordinate system and their values are governed by the transformation equations of the form

$$\begin{aligned} A_x' &= A_{px}, & A_{qx} &= A_x' \cos\alpha - A_y' \sin\alpha \\ A_y' &= A_{py}, & A_{qy} &= A_x' \sin\alpha + A_y' \cos\alpha \quad (2) \\ A_z' &= A_{pz}, & A_{qz} &= A_z' \end{aligned}$$

where A_x', A_y', A_z' denote components of the vector A in the local coordinate system that is rotationally symmetric to the global one.

C. Discussion

Points 1a and 1b account for the main node-edge difference between both definitions. In case of nodal elements, node-symmetry is needed for periodic boundaries; edges require edge-symmetry. This requirement should be taken into an account while generating mesh for a problem in which we want to employ periodicity conditions. From points 2a and 3a, we immediately have the modifications for the finite element code - set signs at paired edges according to point 3a of the definition, assign to both of them the same global number of the degree of freedom and remove one of them from the system of equations. There are two issues, the explanation of which we owe to the reader. Firstly, in standard edge element codes, the vectorial character of global degrees of freedom is included by setting their directions in a specific manner in relation to the global node numbering (e.g. the degree of freedom points from the lower global node number to the higher) [3]. For the degrees of freedom located on periodic boundaries however, their resultant directions must be consistent with point 3a of the definition. Secondly, in order to assign signs properly, it is helpful to introduce a coordinate system with respect to which directions are easily identified. No matter how large the period angle α is, one of such systems is radially outwards and upwards.

III. SIMPLE TEST PROBLEM

A quarter of an air cored coil - Fig. 3 - was used in the first stage of numerical tests. In this case the period angle α is

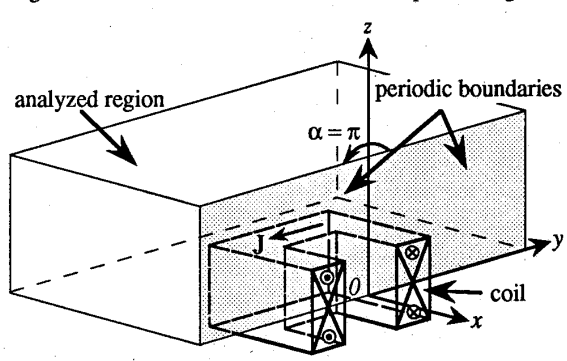

Fig. 3. Geometry of the test model.

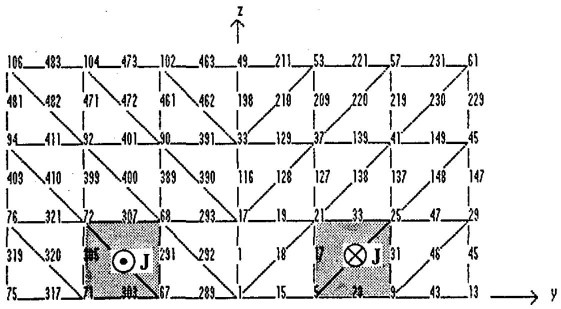

Fig. 4. Setting of edges on periodic boundaries.

equal to π. Such a model may be considered as a test problem for periodic boundary condition in general sense, although we chose it because of several reasons. First, the definition of the periodicity condition may be easily accommodated. Second, the solution with periodic conditions imposed on the boundary plane $x = 0$ on which flux density is periodic may be easily verified by comparison with the one with a zero Dirichlet constraint set at this plane. Third, it geometrically corresponds to Workshop problem 13. And at last, it is very simple and well illustrates the definition. In Fig. 4, setting of edges on periodic boundaries conformal with point 1a of the definition is shown. When we deal with the period angle α equal to π and tetrahedral elements, then the simplest way to generate mesh with rotational symmetric edges on periodic boundaries is the following - subdivide a half of the quarter ($x, y, z \geq 0$) first, and next get its second half ($x, z \geq 0, y < 0$) as a mirror image (in the plane $y = 0$ in this specific case). The inconvenience reported here is absent however, when nodal or hexahedral-edge elements are used. Fig. 5 illustrates the last point of the definition. It is not significant how the directions of a given pair of degrees of freedom are directed (e.g. radially outward or inward), but they must be arranged similarly.

IV. APPLICATION OF THE PERIODIC CONDITION TO TEAM WORKSHOP PROBLEM 13

The TEAM Workshop problem 13 [5] - Fig. 6 - is a 3-D nonlinear, magnetostatic problem with $z = 0$ symmetry plane and rotational symmetry along z-axis. Due to its geometry, one quarter ($x, z \geq 0$, y - free) of the model is sufficient to

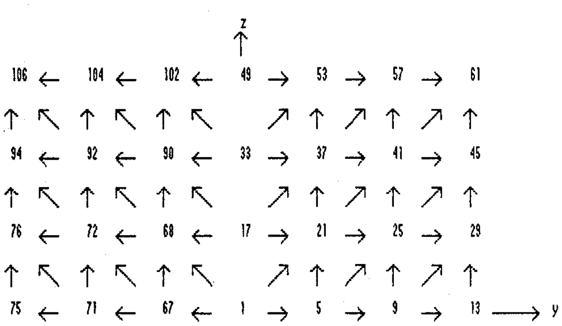

Fig. 5. Setting of signs of degrees of freedom on periodic boundaries.

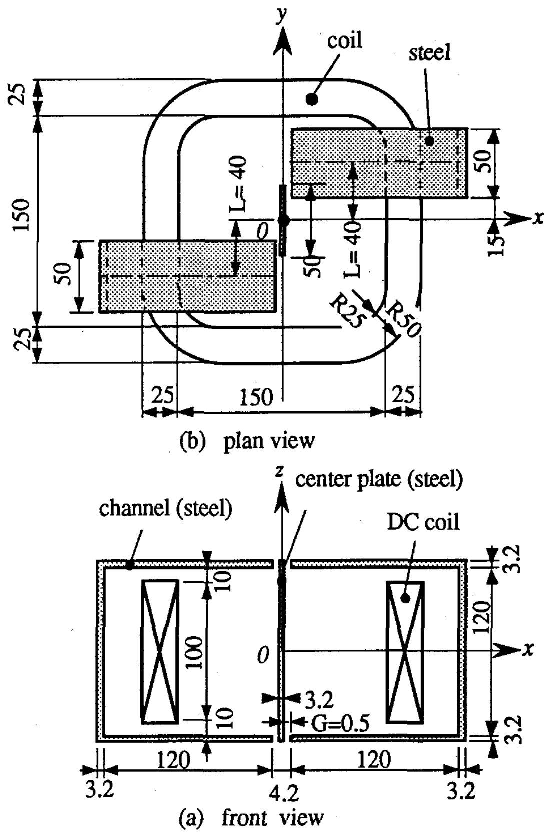

Fig. 6. Geometry of TEAM Workshop problem 13.

perform the analysis. Periodic conditions are imposed on $x = 0$ boundary. The magnetic fields calculated with and without periodic condition are naturally the same. The computer performances of both solutions are listed in Table 1. By comparing numbers in this Table, one can easily notice obtained advantages. The computer storage and CPU time are reduced approximately by a half.

TABLE I
MESH AND COMPUTATIONAL DATA OF THE SOLUTIONS WITH AND WITHOUT PERIODIC BOUNDARY CONDITION FOR TEAM WORKSHOP PROBLEM 13

method	A-method	
type of element	edge on tetrahedron	
periodic boundary condition	not imposed	imposed
number of elements	42768	21384
number of edges	53905	27565
number of unknowns	48339	24174
number of non-zero entries	399160	199087
storage (Mbytes)	15.4	7.7
number of Newton- Raphson iterations	12	12
CPU (ICCG) time (s)	1960(1760)	952(846)

In general, reduction of storage depends on to what fraction a problem may be reduced when periodic conditions are applied. Most of the CPU time, as indicated by numbers in brackets in the last raw of the Table 1, is consumed for solving finite element matrix equation. From that reason savings in computational labor are essentially dependent on the type of solver and relation how solution time varies with the number of unknowns. Our solver employs pre-conditioned conjugate gradient method. Theoretical estimate of computational labor derived in [6] for the ICCG method is of the order $O(N^{1.17})$ for N simultaneous equations. Thus, time needed for solving is almost linearly proportional to the number of degrees of freedom for this method. This estimate stands well in accord with either the results for Workshop problem 13 and the results of numerical tests shown in Fig. 7. Only a small part of the CPU time is used for the matrix assembly.

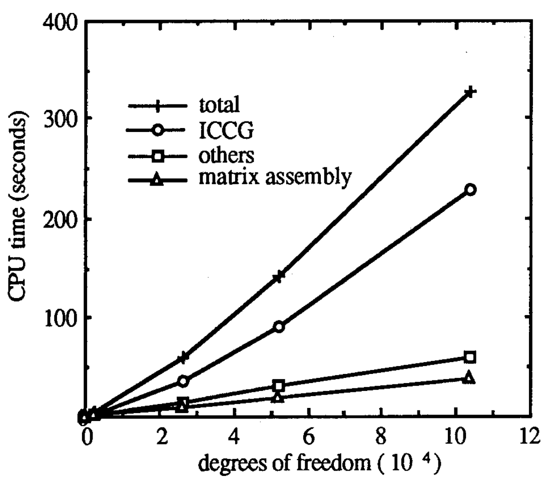

Fig. 7. CPU time as a function of the number of degrees of freedom.

Although the above relations may differ from one solution method to another, the advantage due to reduction in problem's size is evident.

V. CONCLUSIONS

Definition of the periodic boundary condition for edge elements is presented and compared with the one for nodal elements. Difficulties encountered in the application of the condition to a simple test problem are exposed and the way of resolving them is discussed in detail. Finally, benefits gained when the condition is used for Workshop problem 13 as an example, are shown and analyzed quantitatively.

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