Parametrization of identity interactors and the discrete-time all-pass property

Kenji Sugimoto
Okayama University

Yi Liu
Okayama University

Akira Inoue
Okayama University

Proceedings of the
American Control Conference
Seattle, Washington • June 1995

Parametrization of Identity Interactors and the Discrete-Time All-Pass Property

Kenji Sugimoto, Yi Liu, and Akira Inoue
Department of Information Technology

Faculty of Engineering, Okayama University
Tsushima-naka 3-1-1, Okayama, 700 JAPAN
Fax: +81-86-255-9136
sugimoto@suri.it.okayama-u.ac.jp

Abstract

This paper gives a concise parametrization of all identity interactors of a discrete-time multivariable square system. This is performed by means of a state-space description computed from a given particular interactor of the system. The paper then proposes a selection of the parameter which leads to an all-pass closed-loop transfer matrix. This closed-loop system turns out to be equivalent to a certain LQ (linear quadratic) optimal feedback system. A numerical example is given to illustrate the results.

1. Introduction

A special polynomial matrix called ***an interactor*** plays an essential role in various fields of multivariable control; see [2], [6], [9], just to name a few. This matrix was originally defined by [9] as follows: Let $T(z)$ be a given $m \times m$ transfer matrix. A polynomial matrix $\xi_T[z]$ is called an interactor iff

$$\lim_{z \to \infty} \xi_T[z]T(z) = M_T \tag{1}$$

for some constant nonsingular matrix M_T, where

$$\xi_T[z] = H_T[z]\mathrm{diag}(z^{f_1}, \cdots, z^{f_m}) \tag{2}$$

and

$$H_T[z] = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ h_{ij}[z] & & 1 \end{pmatrix} \tag{3}$$

with $h_{ij}[z]$ being appropriate polynomials of certain degrees.

The interactor is introduced in order to generalize the concept of the relative degree of scalar systems into the multivariable case. In this sense, however, it is enough to pose Condition (1); the lower triangular structure in (2) and (3) is required only because of algorithmic simplicity as well as guaranteeing uniqueness. Various modifications of this concept have thus been attempted in literature [3], [4], [5].

A natural definition in this context has been given by [3]; namely, a polynomial matrix $L_T[z]$ is called ***an identity interactor*** iff

$$\lim_{z \to \infty} L_T[z]T(z) = I_m \tag{4}$$

with I_m denoting the identity matrix. This appears to be more useful than the original definition (1) – (3) in the following sense. In the design of EMM (Exact Model Matching) or MRACS (Model Reference Adaptive Control Systems), the feedback system is often designed so that the closed-loop transfer matrix coincides with $L_T[z]^{-1}$, see for example [2]. Hence it is less advantageous to require additional conditions as in (2) and (3), which may reduce the degree of freedom in the design.

An important feature of the identity interactor is nonuniqueness. In this sense, it should be more desirable to parametrize the set of all possible $L_T[z]$ which are identity interactors for a given $T(z)$, rather than studying a specific one of them. It is also desirable if we can give some guidelines as to which identity interactor should be selected when designing a control system.

The objective of this paper is as follows. For a discrete-time invertible (not necessarily known) plant $T(z)$, suppose that we are given a polynomial matrix $\xi_T[z]$ and a constant matrix M_T in (1), (2), and (3). In what follows, $\xi_T[z]$ is called a ***particular interactor***, in order to distinguish it from the identity interactor. Then, we

i) parametrize the set of all identity interactors $L_T[z]$

for fixed $T(z)$;

ii) select, in this set, a parameter such that $N(z) := L_T[z]^{-1}$ satisfies

$$N^{\mathrm{T}}(z^{-1})N(z) = \Phi \qquad \forall z \in \mathbb{C} \qquad (5)$$

for some constant Φ; and

iii) show that the feedback achieving $N(z)$ as a closed-loop transfer matrix is LQ optimal for some weightings.

The item i) enables us to design $L_T[z]$ by adjusting this parameter: for example, we can allocate closed-loop poles at specified points other than the origin. In this paper, however, we use this degree of freedom for achieving the item ii). This condition corresponds to what is called the all-pass property in the continuous-time case, and guarantees that $\|N(z)\|$ is constant at all frequencies. In the design of EMM, we usually apply a feedback so as to cancel invariant zeros inside the unit disk, and put a pre-compensator $C(z)$ so that $N(z)C(z)$ has a desirable frequency property (see Fig. 1). In such a case, we may want $N(z)$ to have a Bode gain-plot as flat as possible, although a phase lag is inevitable because of time delay.

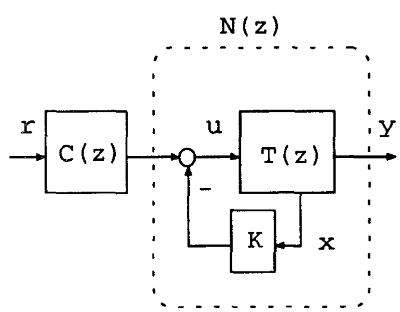

Figure 1: A block diagram of EMM

In fact, for diagonalizable systems (i.e., the case where $H_T[z] = I_m$), it is natural to adopt a diagonal interactor with $z^{f_1}, \cdots, z^{f_m}$ in diagonal entries. In this case, Condition (5) is automatically satisfied. In view of this fact, the item ii) gives a natural generalization of the diagonal interactor into the non-diagonalizable case. All-pass interactor is also found in [4], by the name of unitary interactor, in the context of explicit formula of the LQ optimal control.

The item iii) is given in order to clarify the control-theoretic meaning of the obtained interactor.

Throughout the paper we use the bracket $[\cdot]$ as in $A[z]$ for polynomial matrices, and the parenthesis $(\cdot)$ as in $A(z)$ for rational function matrices, in order to avoid confusion of these two different categories.

2. Parametrization of Interactors

Consider the discrete-time system

$$x(t+1) = Ax(t) + Bu(t), \qquad t = 0, 1, \cdots \quad (6)$$

$$y(t) = Cx(t) \quad (7)$$

$$x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad y \in \mathbb{R}^m.$$

Throughout the paper, we assume that the (square) transfer matrix $T(z) = C(zI_n - A)^{-1}B$ is nonsingular on the field of all rational functions, and that this system has no unstable invariant zeros. We will not use the explicit values of (A, B, C) but will use only the knowledge of a corresponding particular interactor $\xi_T[z]$ together with M_T, so that the result may be applied to adaptive control design in future work.

Lemma 1 Consider $\xi_T[z]$ and M_T in (1), (2), and (3) for a system (6), (7). Then, any polynomial matrix $L_T[z]$ is an identity interactor iff

$$\lim_{z\to\infty} L_T[z]\xi_T[z]^{-1} = {M_T}^{-1}. \qquad (8)$$

Proof (Necessity) If $L_T[z]$ is an identity interactor, then we have

$$\begin{aligned} & \lim_{z\to\infty} L_T[z]\xi_T[z]^{-1} \\ = & \ (\lim_{z\to\infty} L_T[z]T(z))(\lim_{z\to\infty} \xi_T[z]T(z))^{-1} \\ = & \ {M_T}^{-1} \end{aligned}$$

by definition. Hence (8) holds. Sufficiency can also be shown similarly. □

Lemma 2 Consider $\xi_T[z]$ and M_T in (1), (2), and (3). Since $\xi_T[z]^{-1}$ is strictly proper, there exists a minimal realization, denoted by $(\bar{A}, \bar{B}, \bar{C})$:

$$\xi_T[z]^{-1} = \bar{C}(zI_d - \bar{A})^{-1}\bar{B}, \qquad (9)$$

where d is the McMillan degree of $\xi_T[z]^{-1}$. Then

$$\bar{S}[z] := (zI_d - \bar{A})^{-1}\bar{B}\xi_T[z] \qquad (10)$$

is a polynomial matrix. Furthermore, any identity interactor $L_T[z]$ is parametrized as

$$L_T[z] = {M_T}^{-1}(\xi_T[z] + \bar{K}\bar{S}[z]) \qquad (11)$$

in terms of a constant parameter matrix $\bar{K}$.

Proof First, consider the coprime factorization

$$(zI_d - \bar{A})^{-1}\bar{B} = \Gamma[z]\Xi[z]^{-1}$$

by polynomial matrices. In view of (9), it follows that

$$\Xi[z] = \xi_T[z]U[z], \quad \bar{C}\Gamma[z] = U[z]$$

for some unimodular polynomial matrix $U[z]$. Hence

$$\bar{S}[z] = \Gamma[z]U[z]^{-1},$$

which is a polynomial matrix.

The latter half is a direct consequence of the condition (8) and an elementary property of polynomial matrices [8]. □

Remark 1 Lemmas 1 and 2 are nothing but a generalization of the fact that "For a scalar system of relative degree d, any polynomial of degree d can be taken as an interactor."

Remark 2 $\bar{S}[z]$ is the numerator polynomial matrix that is known in the polynomial matrix approach [8]. This matrix, however, does not necessarily have the well-known arranged structure with monomials $1, z, z^2, \cdots$ in block-diagonal entries. This is because $\xi_T[z]$ is not column proper in general nor is $(\bar{A}, \bar{B}, \bar{C})$ in the reachability canonical form.

Remark 3 In practice, it is necessary to choose a Hurwitz polynomial matrix $L_T[z]$ (i.e., det $L_T[z] \neq 0$ if $|z| \geq 1$) among those satisfying (11), so that the closed-loop system $L_T[z]^{-1}$ is stable. Taking this into consideration, we may say that "the identity interactors are parametrized by $\bar{K}$ such that $L_T[z]$ is Hurwitz."

3. Selection of An Interactor

In this section we show how to select an interactor $L_T[z]$ such that the closed-loop transfer matrix $N(z) = L_T[z]^{-1}$ has the all-pass property (5). This is done by making full use of the degree of freedom of the identity interactors revealed in Section 2.

Theorem 1 Consider $\xi_T[z]$ and M_T in (1), (2), and (3), and the realization $(\bar{A}, \bar{B}, \bar{C})$ as in Lemma 2. Compute a positive semi-definite solution $\bar{P}$ to the discrete-time Riccati equation

$$\bar{P} = \bar{A}^{\mathrm{T}}\bar{P}\bar{A} - \bar{A}^{\mathrm{T}}\bar{P}\bar{B}(\bar{B}^{\mathrm{T}}\bar{P}\bar{B})^{-1}\bar{B}^{\mathrm{T}}\bar{P}\bar{A} + \bar{C}^{\mathrm{T}}\bar{C}, \quad (12)$$

and obtain the LQ optimal gain

$$\bar{K}_o := (\bar{B}^{\mathrm{T}}\bar{P}\bar{B})^{-1}\bar{B}^{\mathrm{T}}\bar{P}\bar{A}. \quad (13)$$

Finally, take an identity interactor

$$L_T[z] := M_T{}^{-1}(\xi_T[z] + \bar{K}_o\bar{S}[z]). \quad (14)$$

Then, the closed-loop transfer matrix $N(z) := L_T[z]^{-1}$ satisfies (5) for

$$\Phi := M_T{}^{\mathrm{T}}\bar{B}^{\mathrm{T}}\bar{P}\bar{B}M_T.$$

Proof It is well known (e.g., [1]) that if $\bar{K}_o$ is given by (12) and (13), then the Kalman equation

$$W^{\mathrm{T}}(z^{-1})\bar{B}^{\mathrm{T}}\bar{P}\bar{B}W(z) = \bar{Z}^{\mathrm{T}}(z^{-1})\bar{C}^{\mathrm{T}}\bar{C}\bar{Z}(z) \quad (15)$$

holds, where

$$W(z) := I_m + \bar{K}_o\bar{Z}(z), \quad \bar{Z}(z) := (zI_d - \bar{A})^{-1}\bar{B}.$$

Also, note that in this case

$$\bar{C}\bar{Z}(z) = \xi_T[z]^{-1}$$

from (9) and the definition of $\bar{Z}(z)$. Hence the Kalman equation (15) is reduced to

$$W^{\mathrm{T}}(z^{-1})\bar{B}^{\mathrm{T}}\bar{P}\bar{B}W(z) = (\xi_T[z^{-1}]^{-1})^{\mathrm{T}}\xi_T[z]^{-1}. \quad (16)$$

On the other hand, by substituting (10) into (14) we have

$$M_T L_T[z]\xi_T[z]^{-1} = I_m + \bar{K}_o(zI_d - \bar{A})^{-1}\bar{B} = W(z).$$

Hence by pre- and post-multiplying (16) respectively by $\xi_T{}^{\mathrm{T}}[z^{-1}]$ and $\xi_T[z]$, we have

$$L_T{}^{\mathrm{T}}[z^{-1}]M_T{}^{\mathrm{T}}\bar{B}^{\mathrm{T}}\bar{P}\bar{B}M_T L_T[z] = I_m$$

Thus $L_T[z]$ satisfies the condition. □

We briefly mention a method for numerically computing $L_T[z]$ given above. First, note that $H_T[z]$ is unimodular, and hence, $H_T[z]^{-1}$ is a polynomial matrix. Then we see that

$$\xi_T[z]^{-1} = \mathrm{diag}(z^{f_1}, \cdots, z^{f_m})^{-1}H_T[z]^{-1}$$

gives a left coprime factorization by polynomial matrices. By using this factorization, a realization for $\xi_T[z]^{-1}$ can easily be derived in the observable canonical form. For example, when the particular interactor is diagonal (i.e., $H_T[z] = I_m$), then the tripplet

$$\bar{A} = \text{block-diag}\left(\begin{pmatrix} 0 & 0 \\ I_{f_1} - 1 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & 0 \\ I_{f_m} - 1 & 0 \end{pmatrix}\right),$$

$$\bar{B} = \begin{pmatrix} e_1{}^{\mathrm{T}} \\ 0 \\ \vdots \\ e_m{}^{\mathrm{T}} \\ 0 \end{pmatrix}, \qquad \bar{C} = \begin{pmatrix} 0 & e_1 & \cdots & 0 & e_m \end{pmatrix}$$

is such a realization. Here e_p is the unit m-dimensional vector with unity only in the p-th entry, and the sizes of the diagonal blocks in $\bar{A}$ are $f_1, \cdots, f_m$, respectively. If, on the other hand, the particular interactor is non-diagonal, then it suffices to construct $\bar{B}$ by putting the coefficients of $H_T[z]^{-1}$ in suitable positions.

It is fairly easy to solve the Riccati equation in this case, as will be shown in a later example. We should note here that if a particular interactor is diagonal, then the equation has a trivial solution $\bar{P} = I_d$, $\bar{K}_o = 0$, and hence

$$\xi_T[z] = \mathrm{diag}(z^{f_1}, \cdots, z^{f_m})$$

is the very $L_T[z]$ that we want. In other words, for the diagonalizable case, the particular interactor itself gives the all-pass property (5) (see Section 1).

4. LQ optimality

Now we should note that $\bar{K}_0$ introduced in Section 3 is a mere imaginary gain for $(\bar{A}, \bar{B}, \bar{C})$ and not an actual feedback law. The feedback corresponding to the resulting $L_T[z]$, however, turns out to be LQ optimal in reality. This section is devoted to showing this.

Theorem 2 Let the system (6) be reachable. Apply a stabilizing feedback $u = -Kx$ to this system, and define

$$N(z) := C(zI_n - A + BK)^{-1}B.$$

Then, if $N(z)$ satisfies

$$N^{\mathrm{T}}(z^{-1})N(z) = \Phi \qquad \forall z \in \mathbb{C} \tag{17}$$

for some positive definite constant matrix Φ, then K is expressed as

$$K = (B^{\mathrm{T}}PB)^{-1}B^{\mathrm{T}}PA, \tag{18}$$

where P is a positive semi-definite solution of the Riccati equation

$$P = A^{\mathrm{T}}PA - A^{\mathrm{T}}PB(B^{\mathrm{T}}PB)^{-1}B^{\mathrm{T}}PA + C^{\mathrm{T}}C. \tag{19}$$

Proof This can be regarded as a special case of the inverse problem of optimal control. We can not apply, however, an existing solution technique given in [7], since the control weighting R was assumed to be positive definite there, while $R = 0$ in the present problem. Instead, we will show it in the following way. Define

$$A_K := A - BK.$$

Then A_K is stable, and hence the Lyapunov equation

$$P = A_K{}^{\mathrm{T}}PA_K + Q \tag{20}$$

admits a positive definite solution P, where $Q := C^{\mathrm{T}}C$. Using this P, we will show (18). To this end, we expand

$$N(z) = CBz^{-1} + CA_KBz^{-2} + CA_K{}^2Bz^{-3} + \cdots$$

Then we substitute it into (17), and compare the both hand sides with respect to z:

$$\begin{array}{ll} \text{constant term} & B^{\mathrm{T}}QB + B^{\mathrm{T}}A_K{}^{\mathrm{T}}QA_KB \\ & +B^{\mathrm{T}}A_K{}^{2\mathrm{T}}QA_K{}^2B + \cdots = \Phi, \\ z^{-1} \text{ term} & B^{\mathrm{T}}QA_KB + B^{\mathrm{T}}A_K{}^{\mathrm{T}}QA_K{}^2B \\ & +B^{\mathrm{T}}A_K{}^{2\mathrm{T}}QA_K{}^3B + \cdots = 0, \\ z^{-2} \text{ term} & B^{\mathrm{T}}QA_K{}^2B + B^{\mathrm{T}}A_K{}^{\mathrm{T}}QA_K{}^3B \\ & +B^{\mathrm{T}}A_K{}^{2\mathrm{T}}QA_K{}^4B + \cdots = 0, \\ \cdots & \cdots \end{array} \tag{21}$$

where $A_K{}^{2\mathrm{T}} = (A_K{}^2)^{\mathrm{T}}$. Now observe that, from (20), we have

$$P = \sum_{i=0}^{\infty}(A_K{}^{\mathrm{T}})^iQA_K{}^i.$$

Hence we have

$$\begin{aligned} &B^{\mathrm{T}}PB = \Phi \\ &B^{\mathrm{T}}PA_K[B,\ A_KB,\ A_K{}^2B, \cdots] = 0 \end{aligned} \tag{22}$$

from (21). By the first equality $B^{\mathrm{T}}PB$ is positive definite. By the second one, together with reachability of the pair (A_K, B),

$$0 = B^{\mathrm{T}}PA_K = B^{\mathrm{T}}P(A - BK) = B^{\mathrm{T}}PA - B^{\mathrm{T}}PBK$$

holds. Pre-multiplying the both hand-sides in this equality by $(B^{\mathrm{T}}PB)^{-1}$, we obtain (18). Finally, substituting (18) into (20), we see that P is a solution to the Riccati equation (19). □

Corollary In particular, if we construct EMM by means of the identity interactor given in Theorem 1, then the closed-loop transfer matrix coincides with the one by an optimal regulator.

Proof Since we have assumed that the plant $T(z)$ is nonsingular, so is $N(z)$ and hence Φ cannot be singular. This means that all assumptions in Theorem 2 are satisfied. $\square$

5. An Example

Consider the 2×2 transfer matrix of a plant

$$T(z) := \begin{pmatrix} t_{11}(z) & t_{12}(z) \\ t_{21}(z) & t_{22}(z) \end{pmatrix},$$

and assume that the relative degrees of t_{11} and t_{21} are unity, while those of t_{12} and t_{22} are two. Then this system does not have a diagonal interactor, but has a particular interactor of the form, say

$$\xi_T[z] = \begin{pmatrix} 1 & 0 \\ 2z+5 & 1 \end{pmatrix} \mathrm{diag}(z, z^2). \qquad (23)$$

Supose that, for simplicity, $M_T = I_2$ in this case.

We now apply the algorithm given in Sections 2 and 3. To begin with, a minimal realization of $\xi_T[z]^{-1}$ is given by

$$\bar{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} 1 & 0 \\ -5 & 1 \\ -2 & 0 \end{pmatrix}, \qquad (24)$$

$$\bar{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad (25)$$

Next we solve the discrete-time Riccati equation. This is rather easy to solve since $\bar{A}$ is a nilpotent matrix. Take $\bar{P}_0 = I_d$ and set

$$\bar{P}_{k+1} = \bar{A}^{\mathrm{T}}\bar{P}_k\bar{A} - \bar{A}^{\mathrm{T}}\bar{P}_k\bar{B}(\bar{B}^{\mathrm{T}}\bar{P}_k\bar{B})^{-1}\bar{B}^{\mathrm{T}}\bar{P}_k\bar{A} + \bar{C}^{\mathrm{T}}\bar{C}$$
$$k = 0, 1, \ldots \qquad (26)$$

recursively. Then the sequence converges to $\bar{P}$ in finite times of recursion (in this case, twice). Hence we have

$$\bar{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \bar{K}_o = \begin{pmatrix} 0 & -0.4 & 0 \\ 0 & -2 & 0 \end{pmatrix}. \qquad (27)$$

On the ohter hand, from (10) it follows that

$$\bar{S}[z] = \begin{pmatrix} 1 & 0 \\ 2z & z \\ 0 & 1 \end{pmatrix}, \qquad (28)$$

and hence

$$L_T[z] = \begin{pmatrix} 0.2z & -0.4z \\ 2z^2 + z & z^2 - 2z \end{pmatrix} \qquad (29)$$

is the desired interactor.

Acknowledgments:

The authors wish to thank Lorant, Co. Ltd, Osaka, Japan, for their finantial support.

References

[1] B. D. O. Anderson and J. B. Moore, *Linear Optimal Control,* 2nd ed., Englewood Cliffs, NJ: Prentice-Hall, 1990.

[2] H. Elliott and W. A. Wolovich, "A parameter adaptive control structure for linear multivariable systems," IEEE Trans. Automat. Contr., vol. AC-27, no. 2, pp. 340-352, 1982.

[3] H. Elliott and W. A. Wolovich, "Parametrization issues in multivariable adaptive control," Automatica, vol. 20, pp. 533-545, 1984.

[4] Y. Peng and M. Kinnaert, "Explicit solution to the singular LQ regulation problem," IEEE Trans. Automat. Contr., vol. AC-37, no. 5, pp. 633-636, 1992.

[5] M. W. Rogozinski, et. al, "An algorithm for the calculation of a nilpotent interactor matrix for linear multivariable systems," IEEE Trans. Automat. Contr., vol. AC-32, no. 3, pp. 234-237, 1987.

[6] S. L. Shar et. al, "Multivariable adaptive control without a prior knowledge of the delay matrix," Systems & Control Let., vol. 9, pp. 295-306, 1987.

[7] K. Sugimoto and Y. Yamamoto, "Solution to the inverse regulator problem for discrete-time systems," Int. J. of Contr., vol. 48, no. 3, pp. 1285-1300, 1988.

[8] W. A. Wolovich, *Linear Multivariable Systems,* Springer-Verlag, 1974.

[9] W. A. Wolovich and P. L. Falb, "Invariants and canonical forms under dynamic compensation," SIAM J. Contr., & Opt., vol. 14, no. 6, pp. 996-1008, 1976.