

# *Hilbert $C^*$ -bimodules and countably infinite continuous graphs*

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## Abstract

In this paper, we construct Hilbert  $C^*$ -bimodules for continuous graphs whose vertexes are countable 1-dimensional tori, and show some uniqueness property of the  $C^*$ -representation of these bimodules.

**Keywords:** Hilbert bimodule,  $C^*$ -algebras, continuous graph

## 1. Introduction

Pimsner [P] presented the canonical construction of  $C^*$ -algebra from Hilbert  $C^*$ -bimodules. We call this the bimodule algebra associated with the original Hilbert  $C^*$ -bimodule. This construction contains classical finitely generated Cuntz-Krieger algebras and  $C^*$ -crossed products by the action of integer  $\mathbf{Z}$ . In [KPW1], we investigated the simplicity and ideal structure of the bimodule algebras, and presented examples related with Jones index theory. In [KPW2] we constructed a countably generated Hilbert  $C^*$ -bimodule which bimodule algebra associates with is a countably generated Cuntz-Krieger algebra, and studied simplicity and ideal structure of them. In [KW2], we treated a class of finite continuous graph whose components are one dimensional tori originally treated by Deaconu [D]. Deaconu treated  $C^*$ -algebras associated these graphs using the method of locally compact  $r$ -discrete groupoid.

This paper is a some remark of [KPW2] and [KW2]. We study the uniqueness of the  $C^*$ -representation of bimodules associated with countably infinite continuous graphs whose component are one dimensional tori using the method of [KPW2] and [KW2], and some minor modifications.

## 2. Preliminary results

In this section we review the fundamental matters of Hilbert  $C^*$ -bimodules and associated bimodule algebras following [KPW1], [KPW2] and [KW2]. Let  $A$  be a  $\sigma$ -unital  $C^*$ -algebra and  $X$  be a full and countably generated right Hilbert  $C^*$ -module. We denote  $L_A(X_A)$  as the set of all adjointable linear operators on  $X$ , and  $K_A(X_A)$  as the set of all compact operators on  $X$ . Let  $\phi$  be a non-degenerate isometric  $*$ -homomorphism from  $A$  to  $L_A(X)$ . We call this system  $(X, \phi)$  a Hilbert  $A$ - $A$  bimodule in this paper.

We define the bimodule algebra  $O_X$  from  $X$  following [P]. Let  $F(X)$  be the Fock Hilbert  $A$ - $A$  bimodule. Let  $T_x$  be the creation operator on  $F(X)$  given by  $x \in X$ . We denote  $T_X$  as the  $C^*$ -algebra generated by  $\{T_x | x \in X\}$  and call this the Toeplitz  $C^*$ -algebra generated by  $X$ . In this paper we assume that  $\phi(A) \subset K_A(X_A)$ . In this case,  $K_A(F(X))$  is contained in  $T_X$ . We put  $O_X = T_X/K_A(F(X))$ . We denote  $S_x$  as the quotient image of  $T_x$  in  $O_X$ . We call this  $O_X$  the bimodule algebra generated by  $X$ .

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We review about representations of Hilbert  $C^*$ -bimodules. Let  $D$  be a  $C^*$ -algebra. Let  $\pi_A$  be a  $*$ -homomorphism from  $A$  to  $D$  and  $\pi_X$  be a contraction from  $X$  to  $D$  such that

$$\pi_X(xa) = \pi_X(x)\pi_A(a) \quad \pi_X((x|y)_A) = \pi_X(x)^*\pi(y) \quad (1)$$

for  $x, y \in X$  and  $a \in A$ . Then there exists a unique  $*$ -homomorphism  $\pi_K$  from  $K = K_A(X_A)$  to  $D$  such that

$$\pi_K(\theta_{y,x}) = \pi_X(y)\pi_X(y)^* \quad \pi_X(kx) = \pi_K(k)\pi_X(x)$$

for  $x, y \in X$  and  $k \in K$ . We denote  $\theta_{y,x}$  as the one rank operator on  $X$ .

**Definition 1.** A triple  $(\pi_X, \pi_A, D)$  satisfying (1) and

$$\pi_K(\phi(a)) = \pi_A(a)$$

for each  $a \in A$  is called a representation of Hilbert  $C^*$ -bimodule  $(X, \phi)$ .

Pimsner showed that for every representation of  $(X, \phi)$  there exists a  $*$ -homomorphism from  $O_X$  to  $D$  extending  $\pi_X, \pi_A$ .

We denote  $F_{s,r}$  as the norm closure of the linear span of  $\{S_{x_1} \cdots S_{x_r} S_{y_s}^* \cdots S_{y_1}^*\}$ . By the condition  $\phi(A) \subset K_A(X_A)$ , we have a filtration  $F_{r,r+k} \subset F_{r+1,r+k+1} \subset \cdots$ . We denote by  $F_{(k)}$  be the norm closure of  $\cup_{r=0}^\infty F_{r,r+k}$ . We call  $F_{(0)}$  the AT-part of  $O_X$  in this example. There exist an action  $\gamma$  of  $\mathbf{T}$  on  $O_X$  such that  $\gamma_t(S_x) = tS_x$  for  $t \in \mathbf{T}$ . We call this the gauge action on  $O_X$ . Using the gauge action  $\gamma$ , we define a conditional expectation  $E_X$  from  $O_X$  to  $F_{(0)}$ .

**3. Continuous Cuntz-Krieger algebras associated with countably infinite continuous graphs**

Let  $\Omega$  be a countable disjoint union  $\cup_{i=1}^\infty \Omega_i$  where each  $\Omega_i$  is isomorphic to a one dimensional torus  $\mathbf{T}$ . Following [KW2], we consider closed subsets  $C^{p,1} = \{(z, z^p) | z \in \mathbf{T}\} \subset \mathbf{T} \times \mathbf{T}$  and  $C^{p,-1} = \{(z^p, z) | z \in \mathbf{T}\} \subset \mathbf{T} \times \mathbf{T}$  for nonzero integer  $p$ . Let  $C$  be a closed subset of  $\Omega \times \Omega$  where  $C_{i,j} = C(\Omega_i \times \Omega_j)$  is  $C^{p,1}, C^{p,-1}$  or empty set  $\emptyset$ . We always assume that for each  $i$  (resp  $j$ ) there exists a  $j$  (resp  $i$ ) such that  $C_{i,j} \neq \emptyset$ . We also assume that for each  $i$   $\{j | C_{i,j} \neq \emptyset\}$  is a finite set and for each  $j$   $\{i | C_{i,j} \neq \emptyset\}$  is a finite set. We call this the locally finite condition. We call these  $C$  a countable circle correspondence.

We put  $A = C_0(\Omega)$ , and  $A_{00}$  be a  $*$ -subalgebra of  $A$  whose elements vanish outside finite components  $\Omega_i$ , and put  $A_i = C(\Omega_i) \simeq C(\mathbf{T})$ . We put  $X_{00}$  be a set of continuous functions on  $C$  whose elements vanish outside finite  $C_{i,j}$ . Then  $X_{00}$  is made into a pre Hilbert right  $A$  module as follows. Let  $f, f_1$  and  $f_2$  be in  $X_{00}, a \in A$ .

$$(f_1|f_2)_A(\omega) = \sum_{\omega;(\omega',\omega) \in C} \overline{f_1(\omega',\omega)} f_2(\omega',\omega)$$

$$(f \cdot a)(\omega) = f(\omega)a(\omega)$$

We denote  $X$  as the completion of  $X_{00}$  with respect to this right  $A$  inner product. For  $f \in X_{00}$  and  $a \in A$ , we define  $\phi(a)$  as follows.

$$(\phi(a)f)(\omega) = a(\omega)f(\omega)$$

Then  $\phi(a)$  is extended to an element in  $L_A(X_A)$ , and  $\phi$  is a non degenerate  $*$ -isometric homomorphism.

**Lemma 2.**  $(X, \phi)$  is a countably generated Hilbert  $A$ - $A$  bimodule and  $\phi(A) \subset K_A(X_A)$ .

This lemma follows from the locally finite condition.

We may construct a countable basis using finite basis for nonzero  $X_{i,j}$ 's. In [KW2], we have already defined a Hilbert  $C^*$ -bimodule for a finite continuous graph, and we use the basis given there.

**Definition 3.** We call this Hilbert  $A$ - $A$  bimodule  $X$  the bimodule corresponding to countable circle correspondence.

We have proved some fundamental results about countably generated Hilbert  $C^*$ -bimodules in [KPW2], and use these in this paper.

As in [D] and [KW2], we shrink each torus to a point and collapse the corresponding edge to get a discrete graph  $G^d$ . Thus we consider the countably infinite set  $\Sigma = \{1, 2, 3, \dots\}$  of discrete vertices and the set  $E = \{(i, j) \in \Sigma \times \Sigma \mid C_{i,j} \neq \emptyset\}$  of discrete edges. Discrete paths and loops are defined as paths and loops in the discrete graph  $G^d$ . We define discrete loops and simple discrete loops as in [KW2].

We define a rational number  $p(L)$  for a discrete loop  $L = (i_0, i_1, \dots, i_{s-1}, i_0)$  following [KW2]. We define the type of a discrete edge  $(i, j)$  in  $G^d$  by  $(p, t)$  if  $C_{i,j} = C^{p,t}$ , and denote this by  $(p(i, j), t(i, j))$ . We put

$$p(L) = (p(i_0, i_1))^{t(i_0, i_1)} (p(i_1, i_2))^{t(i_1, i_2)} \dots (p(i_{s-1}, i_0))^{t(i_{s-1}, i_0)}$$

**Definition 4.** [KW2] Let  $L$  be a discrete loop. We call  $L$  is periodic if  $p(L)$  is equal to 1 or  $-1$  and there exists some  $k$  such that  $p(i_{k-1}, i_k)$  is not equal to 1 or  $-1$ . A discrete loop  $L$  is expansive if  $p(L) > 1$  and contractive if  $p(L) < 1$ . We say  $L$  is trivial if  $|p(i_{k-1}, i_k)| = 1$  for any  $k$ .

We consider the uniqueness property of representations of bimodules in  $C^*$ -algebras and present a sufficient condition of simplicity of bimodule algebra associated with these countable circle correspondences.

**Definition 5.** A countable circle correspondence  $C$  is called to satisfy the condition "countable circle free" if every simple discrete loop without exit is not trivial.

We refer the (I)-free condition for bimodules from [KW2].

**Definition 6.** A countably generated Hilbert  $A$ - $A$  bimodule  $(X, \phi)$  is called to be (I)-free if there exists a dense subset  $D \subset O_X^0$  (algebraic elements) such that for each  $B \in D$  with  $B = \sum_{j=-m}^m B_j$  ( $B_j \in K_A(X_A^{\otimes p+j}, X_A^{\otimes p})$ ), for every  $\varepsilon > 0$ , there exists a contraction  $P \in O_X$  in a spectral subspace under the gauge action of  $\mathbf{T}$  on  $O_X$  satisfying the followings:

1. For each  $j$  such that  $j \neq 0$ ,  $\|PB_jP^*\| \leq \varepsilon$  holds.
2.  $\|PB_0P^*\| \geq \|B_0\| - \varepsilon$  holds.

**Proposition 7.** If a countable circle correspondence  $C$  satisfies the countable circle (I) condition, then the bimodule associated with  $C$  is (I)-free.

This is proved by showing that  $m$ -aperiodic points are dense for every  $m$  using Lemma 23 [KW2], and some minor modification.

The following Proposition holds for general Hilbert  $C^*$ -bimodule such that  $\phi(A) \subset K_A(X_A)$ , and proved in [KPW2] and [KW2].

**Proposition 8.** We assume that  $(X, \phi)$  is (I)-free. Let  $\varphi$  be a  $*$ -homomorphism from  $O_X$  to a  $C^*$  algebra  $D$  such that the restriction of  $\varphi$  to  $A$  is faithful. Then  $\varphi$  is also faithful.

This proposition means representation of  $(X, \phi)$  is unique in some sense. The proof is carried out by showing that there exists a conditional expectation  $E^\varphi$  from  $\varphi(O_X)$  to  $\varphi(F_{(0)})$  which is compatible with  $E^X$  and  $\varphi$  ([KPW1]).

Let  $C$  be a countable circle correspondence. A subset  $U$  in  $C$  is called hereditary if  $(\omega, \omega') \in C$  and  $\omega \in U$ , then  $\omega' \in U$ . A subset  $U$  is called saturated if  $(\omega, \omega') \in U$  for all  $\omega' \in U$  for all  $\omega$  then  $\omega$  is contained in  $U$ .

A closed ideal  $J$  in  $A$  is called  $X$ -invariant if  $(x|\phi(a)y)_A \in J$  for every  $x, y \in X$  and  $a \in A$ .  $J$  is called  $X$ -saturated if  $(x|\phi(a)y)_A$  for all  $x, y \in X$  implies  $a \in J$ .

As [KW2], we have the following.

**Lemma 9.** [KW2] Let  $J$  be closed two sided ideal in  $A$  and  $U$  be the open subset of  $\Omega$  corresponding  $J$ . Then  $J$  is  $X$ -invariant if and only if  $U$  is hereditary, and  $J$  is  $X$ -saturated if and only if  $U$  is saturated.

From these, we have the following Theorem.

**Theorem.** Assume that a countable circle correspondence  $C$  satisfies countable circle (I)-condition. Let  $X$  be a Hilbert  $C^*$ -bimodule associated with  $C$ . Then every representation  $(\pi_X, \pi_A)$  of  $(X, \phi)$  to a  $C^*$ -algebra  $D$  such that  $\pi_A$  is faithful is extended to a faithful  $*$ -homomorphism from  $O_X$  to  $D$ . Moreover, the bimodule algebra  $O_X$  is simple if and only if every hereditary and saturated open subset of  $\Omega$  is  $\Omega$  or  $\emptyset$ .

**Remark 10.** It is possible as in [KW2] to define countable circle (II)-condition and to classify all ideals in the bimodule algebra corresponding to countable circle correspondences. But the circle (II) condition is complicated because if the circle correspondence is finite and discrete graph associated with this satisfies the (II)-free condition in the sense of Cuntz-Krieger, the original countable circle correspondence need not to satisfy the (II)-free condition in bimodule sense.

**Remark 11.** It is not so difficult to generalize to  $n$  dimensional product type circle correspondences.

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